Lecture Notes on The Algebra of Logic

Tommaso Moraschini

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CHAPTER 1

Preliminaries

1.1 Algebras and equations

We begin by reviewing some fundamentals of general algebraic systems. For more information, the reader may consult [12, 30].

Definition 1.1.

(i) A *type* is a map $\rho\colon \mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case, $\rho(f)$ is said to be the *arity* of the function symbol f, for every $f \in \mathcal{F}$. Function symbols of arity zero are called *constants*.

(ii) An *algebra* of type ρ is a pair $\boldsymbol{A} = \langle A; F\rangle$ where A is a nonempty set and $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ, in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set A is called the *universe* of $\boldsymbol{A}$.

When $\mathcal{F} = \{f_1, \dots, f_n\}$, we shall write $\langle A; f_1^{\boldsymbol{A}}, \dots, f_n^{\boldsymbol{A}}\rangle$ instead of $\langle A; F\rangle$. In this case, we often drop the superscripts, and write simply $\langle A; f_1, \dots, f_n\rangle$.

Classical examples of algebras are groups and rings. For instance, the type of groups ρ_G consists of a binary symbol $+$, a unary symbol $-$ and a constant symbol 0. Then a group is an algebra $\langle G; +, -, 0\rangle$ of type ρ_G in which $+$ is associative, 0 is a neutral element for $+$ and $-$ produces inverses.

Lattices, Heyting algebras and modal algebras are also algebras in the above sense. For instance, the type of lattices ρ_L consists of two binary symbols $\wedge$ and $\vee$ and a lattice is an algebra $\langle A; \wedge, \vee\rangle$ of type ρ_L that satisfies the idempotent, commutative, associative and absorption laws. Similarly, the type of Heyting algebras ρ_H consists of three binary operations symbols $\wedge, \vee$ and $\to$ and of two constant symbols 0 and 1. Then a Heyting algebra is an algebra $\langle A; \wedge, \vee, \to, 0, 1\rangle$ such that $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice and, for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c. \qquad (\textit{residuation law})$$

Boolean algebras can be viewed as the Heyting algebras that satisfy the following equational version of the *excluded middle law*:

$$x \vee (x \to 0) \approx 1.$$

In this case, the complement operation $\neg x$ can be defined as $x \to 0$.

Perhaps less obviously, even algebraic structures whose operations are apparently *external* can be viewed as algebras in the sense of the above definition. For instance, modules over a ring $\boldsymbol{R}$ can be viewed as algebras whose type ρ_R extends that of groups with the unary symbols $\{\lambda_r : r \in R\}$. From this point of view, a module over $\boldsymbol{R}$ is an algebra $\langle G; +, -, 0, \{\lambda_r : r \in R\}\rangle$ of type ρ_R such that $\langle G; +, -, 0\rangle$ is an abelian group and, for every $r, s \in R$ and $a, c \in G$,

$$\begin{aligned} \lambda_r(a + c) &= \lambda_r(a) + \lambda_r(c) \\ \lambda_{r+s}(a) &= \lambda_r(a) + \lambda_s(a) \\ \lambda_r(\lambda_s(a)) &= \lambda_{r \cdot s}(a) \\ \lambda_1(a) &= a. \end{aligned}$$

Given a type $\rho \colon \mathcal{F} \to \mathbb{N}$ and a set of variables X disjoint from $\mathcal{F}$, the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;

(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$; and

(iii) if $\varphi_1, \dots, \varphi_{\rho(f)} \in T_\rho(X)$ and $f \in \mathcal{F}$, then $f\varphi_1 \dots \varphi_{\rho(f)} \in T_\rho(X)$.

For the sake of readability, we shall often write $f(\varphi_1, \dots, \varphi_{\rho(f)})$ instead of $f\varphi_1 \dots \varphi_{\rho(f)}$. Similarly, if f is a binary operation $+$, we often write $\varphi_1 + \varphi_2$ instead of $f(\varphi_1, \varphi_2)$.

Definition 1.2. Let $\rho \colon \mathcal{F} \to \mathbb{N}$ be a type and X a set of variables disjoint from $\mathcal{F}$. The *term algebra* $\boldsymbol{T}_\rho(X)$ of type ρ over X is the unique algebra of type ρ whose universe is $T_\rho(X)$ and with basic n-ary operations f defined, for every $\varphi_1, \dots, \varphi_n \in T_\rho(X)$, as

$$f^{\boldsymbol{T}_\rho(X)}(\varphi_1, \dots, \varphi_n) \coloneqq f(\varphi_1, \dots, \varphi_n).$$

When no confusion might arise, we drop the subscript and write $T(X)$ instead of $T_\rho(X)$. Given a term $\varphi \in T_\rho(X)$, we write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in φ are among $x_1, \dots, x_n$. Furthermore, given an algebra $\boldsymbol{A}$ of type ρ and elements $a_1, \dots, a_n \in A$, we define an element

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$$

of A, by recursion on the construction of φ, as follows:

(i) if φ is a variable x_i, then $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) \coloneqq a_i$;

(ii) if φ is a constant c, then $c^{\boldsymbol{A}}$ is the interpretation of c in $\boldsymbol{A}$;

(iii) if $\varphi = f(\psi_1, \dots, \psi_m)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) := f^{\boldsymbol{A}}(\psi_1^{\boldsymbol{A}}(a_1, \dots, a_n), \dots, \psi_m^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

An *equation of type* ρ *over* X is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in T_\rho(X)$. We denote by $E_\rho(X)$ the set of equations of type ρ over X. Such an equation $\varphi \approx \psi$ is *valid* in an algebra $\boldsymbol{A}$ of type ρ, if

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = \psi^{\boldsymbol{A}}(a_1, \dots, a_n), \text{ for every } a_1, \dots, a_n \in A,$$

in which case we say that $\boldsymbol{A}$ *satisfies* $\varphi \approx \psi$.

For instance, groups are precisely the algebras of type ρ_G that satisfy the equations

$$x + (y + z) \approx (x + y) + z \quad x + 0 \approx x \quad 0 + x \approx x \quad x + -x \approx 0 \quad -x + x \approx 0.$$

Similarly, lattices are the algebras of type ρ_L that satisfy the equations

$$\begin{array}{lll} x \wedge x \approx x & x \vee x \approx x & (\textit{idempotent laws}) \\ x \wedge y \approx y \wedge x & x \vee y \approx y \vee x & (\textit{commutative laws}) \\ x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \vee (y \vee z) \approx (x \vee y) \vee z & (\textit{associative laws}) \\ x \wedge (y \vee x) \approx x & x \vee (y \wedge x) \approx x. & (\textit{absorption laws}) \end{array}$$

From now on, we will work with a fixed denumerable set of variables

$$Var = \{x_n : n \in \mathbb{N}\}.$$

Accordingly, when we write $x, y, z \dots$ for variables, it should be understood that these are variables in Var.

1.2 Basic constructions

Algebras of the same type are called *similar* and can be compared by means of maps that preserve their structure.

Definition 1.3. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f \colon A \to B$ such that, for every n-ary operation g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

An injective homomorphism is called an *embedding* and, if there exists an embedding from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$. Lastly, a surjective embedding is called an *isomorphism*. Accordingly, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic* if there exists an isomorphism between them, in which case we write $\boldsymbol{A} \cong \boldsymbol{B}$.

A simple induction on the construction of terms shows that, for every pair of algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of type ρ and every term $\varphi(x_1, \dots, x_n)$ of ρ, if f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$f(\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)) = \varphi^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)),$$

for every $a_1, \ldots, a_n \in A$. Therefore homomorphisms preserve not only basic operations, but also arbitrary terms.

In the particular case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are lattices, a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f\colon A \to B$ such that, for every $a, c \in A$,

$$f(a \wedge^{\boldsymbol{A}} c) = f(a) \wedge^{\boldsymbol{B}} f(c) \text{ and } f(a \vee^{\boldsymbol{A}} c) = f(a) \vee^{\boldsymbol{B}} f(c).$$

For instance, the inclusion map from the lattice $\langle \mathbb{N}; \leqslant \rangle$ into the lattice $\langle \mathbb{Z}; \leqslant \rangle$ is an injective homomorphism, that is, an embedding. Similarly, given two sets $Y \subseteq X$, the inclusion map from the powerset lattice $\langle \mathcal{P}(Y); \subseteq \rangle$ to the powerset lattice $\langle \mathcal{P}(X); \subseteq \rangle$ is also an embedding. On the other hand, if $Y \subsetneq X$, the map

$$(-) \cap Y\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

that sends every $Z \subseteq X$ to $Z \cap Y$ is a noninjective homomorphism from $\langle \mathcal{P}(X); \subseteq \rangle$ to $\langle \mathcal{P}(Y); \subseteq \rangle$.

Proposition 1.4. *Let $\boldsymbol{A}$ be an algebra of type ρ and X a set of variables. Every function $f\colon X \to A$ extends uniquely to a homomorphism $f^*\colon \boldsymbol{T}_\rho(X) \to \boldsymbol{A}$.*

Proof. The unique extension f^* is defined, for every $\varphi(x_{\alpha_1}, \ldots x_{\alpha_n}) \in T_\rho(X)$, as

$$f^*(\varphi) = \varphi^{\boldsymbol{A}}(f(x_{\alpha_1}), \ldots, f(x_{\alpha_n})). \qquad \boxtimes$$

Exercise 1.5. Prove the above proposition. ⊠

Corollary 1.6. *If $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ and $g\colon \boldsymbol{B} \to \boldsymbol{A}$ are homomorphisms and g is surjective, there exists a homomorphism $h\colon \boldsymbol{T}(Var) \to \boldsymbol{B}$ such that $f = g \circ h$.*

Proof. As $g\colon \boldsymbol{B} \to \boldsymbol{A}$ is surjective, for every $n \in \mathbb{N}$ there exists some $b_n \in B$ such that $f(x_n) = g(b_n)$. By Proposition 1.4, there exists a homomorphism $h\colon \boldsymbol{T}(Var) \to \boldsymbol{B}$ such that $h(x_n) = b_n$, for all $n \in \mathbb{N}$. As a consequence,

$$g \circ h(x_n) = g(b_n) = f(x_n), \text{ for all } n \in \mathbb{N}.$$

Since Var is a set of generators for $\boldsymbol{T}(Var)$ and both $g \circ h$ and f are homomorphisms, we conclude that $f = g \circ h$. ⊠

Definition 1.7. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type $\rho\colon \mathcal{F} \to \mathbb{N}$. Then $\boldsymbol{A}$ is said to be a *subalgebra* of $\boldsymbol{B}$ if $A \subseteq B$ and $f^{\boldsymbol{A}}$ is the restriction of $f^{\boldsymbol{B}}$ to A, for every $f \in \mathcal{F}$. In this case, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$.

Given a class of algebras K, let

$$\mathbb{I}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \cong \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}$$
$$\mathbb{S}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \leqslant \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathsf{K}\}.$$

When $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{I}(\boldsymbol{A})$ and $\mathbb{S}(\boldsymbol{A})$ as a shorthand for $\mathbb{I}(\{\boldsymbol{A}\})$ and $\mathbb{S}(\{\boldsymbol{A}\})$, respectively. The following observation is an immediate consequence of the definitions.

Proposition 1.8. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type. Then $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ if and only if there exists an embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$. In this case, $\boldsymbol{A}$ is isomorphic to the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.*

As we mentioned, homomorphisms can be used to compare similar algebras.

Definition 1.9. Given two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we say that $\boldsymbol{A}$ is a *homomorphic image* of $\boldsymbol{B}$ if there exists a surjective homomorphism $f\colon \boldsymbol{B} \to \boldsymbol{A}$.

Accordingly, given a class of algebras K, we set

$$\mathbb{H}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a homomorphic image of some } \boldsymbol{B} \in \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{H}(\boldsymbol{A})$ as a shorthand for $\mathbb{H}(\{\boldsymbol{A}\})$.

Observe that every (not necessarily surjective) homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ induces a homomorphic image of $\boldsymbol{A}$.

Proposition 1.10. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[A]$ is the universe of a subalgebra of $\boldsymbol{B}$ that, moreover, is a homomorphic image of $\boldsymbol{A}$.*

Proof. Observe that $f[A]$ is nonempty, because A is. Then consider an n-ary function symbol g of the common type of $\boldsymbol{A}$ and $\boldsymbol{B}$ and $b_1, \dots, b_n \in f[A]$. Clearly, there are $a_1, \dots, a_n \in A$ such that $f(a_i) = b_i$, for every $i \leqslant n$. Since f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we obtain

$$g^{\boldsymbol{B}}(b_1, \dots, b_n) = g^{\boldsymbol{B}}(f(a_1), \dots, g(a_n)) = f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) \in f[A].$$

Hence, we conclude that $f[A]$ is the universe of a subalgebra $f[\boldsymbol{A}]$ of $\boldsymbol{B}$.

Furthermore, $f\colon \boldsymbol{A} \to f[\boldsymbol{A}]$ is a homomorphism, because for every basic n-ary function symbol g of the common type and $a_1, \dots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{f[\boldsymbol{A}]}(f(a_1), \dots, f(a_n)),$$

where the first equality follows from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Since the map $f\colon A \to f[A]$ is surjective, we conclude that $f[\boldsymbol{A}] \in \mathbb{H}(\boldsymbol{A})$. ⊠

In view of the above result, when $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we denote by $f[\boldsymbol{A}]$ the unique subalgebra of $\boldsymbol{B}$ with universe $f[A]$.

For instance, let $f\colon \mathbb{Z} \to \mathbb{R}$ be the absolute value map, that is, the function defined by the rule

$$f(n) := \text{the absolute value of } n.$$

Observe that f is a nonsurjective homomorphism from the lattice of integers to that of reals. Furthermore, the homomorphic image $f[\langle \mathbb{Z}; \leqslant \rangle]$ of $\langle \mathbb{Z}; \leqslant \rangle$ is the lattice of natural numbers $\langle \mathbb{N}; \leqslant \rangle$, which, in turn, is a subalgebra of lattice of reals.

Notably, the homomorphic images of an algebra $\boldsymbol{A}$ can be "internalized" as special equivalence relations on $\boldsymbol{A}$ as follows.

Definition 1.11. A *congruence* of an algebra $\boldsymbol{A}$ is an equivalence relation θ on A such that, for every basic n-ary opearation f of $\boldsymbol{A}$ and $a_1, \ldots, a_n, c_1, \ldots, c_n \in A$,

$$\text{if } \langle a_1, c_1 \rangle, \ldots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle f^{\boldsymbol{A}}(a_1, \ldots, a_n), f^{\boldsymbol{A}}(c_1, \ldots, c_n) \rangle \in \theta. \tag{1.1}$$

In this case, we often write $a \equiv_\theta c$ as a shorthand for $\langle a, c \rangle \in \theta$. The poset of congruences of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Con}(\boldsymbol{A})$.

A simple induction on the construction of terms shows that, for every congruence θ of $\boldsymbol{A}$ and every term $\varphi(x_1, \ldots, x_n)$,

$$\text{if } \langle a_1, c_1 \rangle, \ldots, \langle a_n, c_n \rangle \in \theta, \text{ then } \langle \varphi^{\boldsymbol{A}}(a_1, \ldots, a_n), \varphi^{\boldsymbol{A}}(c_1, \ldots, c_n) \rangle \in \theta,$$

for every $a_1, \ldots, a_n \in A$. Therefore, congruences preserve not only basic operations, but also arbitrary terms. Furthermore, a simple argument shows that $\mathsf{Con}(\boldsymbol{A})$ is a complete (indeed algebraic) lattice whose maximum is the total relation $A \times A$ and whose minimum is the identity relation $\mathrm{id}_A \coloneqq \{\langle a, a \rangle : a \in A\}$.

Example 1.12 (Heyting algebras)**.** Recall that a *filter* of a Heyting algebra $\boldsymbol{A}$ is a nonempty upset $F \subseteq A$ closed under binary meets. We denote by $\mathsf{Fi}(\boldsymbol{A})$ the poset of filters of $\boldsymbol{A}$ ordered under the inclusion relation. It is easy to see $\mathsf{Fi}(\boldsymbol{A})$ is a complete lattice. Furthermore, the lattices $\mathsf{Fi}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(-) \colon \mathsf{Fi}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A}) \quad \text{and} \quad \boldsymbol{\tau}^{\boldsymbol{A}}(-) \colon \mathsf{Con}(\boldsymbol{A}) \to \mathsf{Fi}(\boldsymbol{A})$$

defined by the rules

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \coloneqq \{\langle a, c \rangle \in A \times A : a \to c, c \to a \in F\}$$
$$\boldsymbol{\tau}^{\boldsymbol{A}}(\theta) \coloneqq \{a \in A : \langle a, 1 \rangle \in \theta\}.$$

Because of this, every congruence θ of a Heyting algebra $\boldsymbol{A}$ is induced by some filter F, in the sense that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} F$. ⊠

Example 1.13 (Modal algebras)**.** A *modal algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, \Box, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra and $\Box$ is a unary operation such that

$$\Box(a \wedge c) = \Box a \wedge \Box c \quad \text{and} \quad \Box 1 = 1,$$

for every $a, c \in A$. An *open filter* of a modal algebra $\boldsymbol{A}$ is a filter of the Boolean reduct of $\boldsymbol{A}$ that, moreover, is closed under the operation $\Box$. The poset of open filters of $\boldsymbol{A}$ ordered under the inclusion relation will be denoted by $\mathsf{Op}(\boldsymbol{A})$. It forms a complete lattice. Furthermore, the lattices $\mathsf{Op}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ are isomorphic via the inverse isomorphisms described in Example 1.12. Because of this, every congruence of a modal algebra $\boldsymbol{A}$ has the form $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} F$, for some open filter F. ⊠

Example 1.14 (Groups)**.** Similarly, it is well known that the lattice of congruences of a group is isomorphic to that of its normal subgroups. Because of this, every congruence of a group is induced by some normal subgroup. ⊠

As we mentioned, there is a tight correspondence between the homomorphic images and the congruences of an algebra $\boldsymbol{A}$. On the one hand, every congruence θ of $\boldsymbol{A}$ gives rise to a homomorphic image $\boldsymbol{A}/\theta$ of $\boldsymbol{A}$. Let $\mathcal{F}$ be the set of function symbols of $\boldsymbol{A}$. Given $\theta \in \mathsf{Con}(\boldsymbol{A})$ and a basic n-ary function symbol $f \in \mathcal{F}$, let $f^{\boldsymbol{A}/\theta}$ be the n-ary operation on A/θ defined by the rule

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \coloneqq f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta.$$

Notice that $f^{\boldsymbol{A}/\theta}$ is well-defined, by condition (1.1). As a consequence, the structure

$$\boldsymbol{A}/\theta \coloneqq \langle A/\theta; \{f^{\boldsymbol{A}/\theta} : f \in \mathcal{F}\}\rangle$$

is a well-defined algebra of the type as $\boldsymbol{A}$. Furthermore, $\boldsymbol{A}/\theta \in \mathbb{H}(\boldsymbol{A})$, because the map $\pi_\theta \colon A \to A/\theta$, defined, for every $a \in A$, as $\pi_\theta(a) \coloneqq a/\theta$, is a surjective homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/\theta$. To prove this, consider $a_1, \dots, a_n \in A$. We have

$$\begin{aligned}
\pi_\theta(f^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta \\
&= f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \\
&= f^{\boldsymbol{A}/\theta}(\pi_\theta(a_1), \dots, \pi_\theta(a_n)),
\end{aligned}$$

where the second equality follows from the definition of the operation $f^{\boldsymbol{A}/\theta}$.

Corollary 1.15. *If θ is a congruence of an algebra $\boldsymbol{A}$, then $\boldsymbol{A}/\theta$ is a well-defined homomorphic image of $\boldsymbol{A}$.*

In view of the above result, every congruence θ of an algebra $\boldsymbol{A}$ induces a homomorphic image of $\boldsymbol{A}$, namely $\boldsymbol{A}/\theta$. The converse is also true, as we proceed to explain.

Definition 1.16. The *kernel* of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is the binary relation

$$\mathsf{Ker}(f) \coloneqq \{\langle a, c\rangle \in A \times A : f(a) = f(c)\}.$$

Proposition 1.17. *The kernel of a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a congruence of $\boldsymbol{A}$.*

Proof. It is obvious that $\mathsf{Ker}(f)$ is an equivalence relation on A. Therefore, to prove that $\mathsf{Ker}(f)$ is a congruence of $\boldsymbol{A}$, it suffices to show that it preserves the basic operations of $\boldsymbol{A}$. Consider a basic n-ary operation g of $\boldsymbol{A}$ and $a_1, \dots, a_n, c_1, \dots, c_n \in A$ such that $\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle \in \mathsf{Ker}(f)$. By the definition of $\mathsf{Ker}(f)$,

$$f(a_i) = f(c_i), \text{ for every } i \leqslant n.$$

It follows that $g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n))$. Since $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, this yields

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) = g^{\boldsymbol{B}}(f(c_1), \dots, f(c_n)) = f(g^{\boldsymbol{A}}(c_1, \dots, c_n)).$$

Hence, we conclude that $\langle g^{\boldsymbol{A}}(a_1, \dots, a_n), g^{\boldsymbol{A}}(c_1, \dots, c_n)\rangle \in \mathsf{Ker}(f)$, as desired. ⊠

The behaviour of kernels is governed by the next principle.

Fundamental Homomorphism Theorem 1.18. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism with kernel θ, then there exists a unique embedding $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $f = g \circ \pi_\theta$.*

Proof. We begin by proving the existence of g. Let $g\colon A/\theta \to B$ be the map defined as $g(a/\theta) \coloneqq f(a)$, for every $a \in A$. To show that g is well-defined, consider $a, c \in A$ such that $a/\theta = c/\theta$. Since $\theta = \mathsf{Ker}(f)$, this means that $f(a) = f(c)$, as desired. Furthermore, the definition of g guarantees that $f = g \circ \pi_\theta$.

Now, observe g is injective, because, for every $a, c \in A$ such that $g(a/\theta) = g(c/\theta)$, we have $f(a) = f(c)$, that is, $\langle a, c \rangle \in \mathsf{Ker}(f) = \theta$ and, therefore, $a/\theta = c/\theta$. Moreover, for every basic n-ary operation p of $\boldsymbol{A}$ and $a_1, \dots, a_n \in A$, we have

$$\begin{aligned} g(p^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta)) &= g(p^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta) \\ &= f(p^{\boldsymbol{A}}(a_1, \dots, a_n)) \\ &= p^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)) \\ &= p^{\boldsymbol{B}}(g(a_1/\theta), \dots, g(a_n/\theta)). \end{aligned}$$

The first equality above follows from the definition of $\boldsymbol{A}/\theta$, the second and the last from the definition of g and the third from the assumption that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism. Hence, we conclude that $g\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ is a homomorphism and, therefore, an embedding, as desired.

The uniqueness of g follows from the fact that, if a map g^* satisfies the condition in the statement of the theorem, then, for every $a \in A$,

$$f(a) = g^* \circ \pi_\theta(a) = g^*(a/\theta),$$

that is, g^* coincides with g. ⊠

Corollary 1.19. *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. In particular, if f is surjective, $\boldsymbol{B} \cong \boldsymbol{A}/\mathsf{Ker}(f)$.*

Proof. In the proof of the Fundamental Homomorphism Theorem we showed that the map $g\colon A/\mathsf{Ker}(f) \to B$, defined by the rule $g(a/\mathsf{Ker}(f)) \coloneqq f(a)$, is an embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $\boldsymbol{B}$. As g can be viewed as a surjective embedding of $\boldsymbol{A}/\mathsf{Ker}(f)$ into $f[\boldsymbol{A}]$, we conclude that $f[\boldsymbol{A}] \cong \boldsymbol{A}/\mathsf{Ker}(f)$. ⊠

At this stage, it should be clear that if θ is a congruence on an algebra $\boldsymbol{A}$, then $\pi_\theta\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a surjective homomorphism whose kernel is θ. Similarly, if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then $\boldsymbol{A}/\mathsf{Ker}(f) \cong \boldsymbol{B}$, by Corollary 1.19. As a consequence, for every class of algebras K,

$$\mathbb{H}(\mathsf{K}) = \mathbb{I}\{\boldsymbol{A}/\theta : \boldsymbol{A} \in \mathsf{K} \text{ and } \theta \in \mathsf{Con}(\boldsymbol{A})\}. \tag{1.2}$$

Now, recall that the Cartesian product of a family of sets $\{A_i : i \in I\}$ is the set

$$\prod_{i \in I} A_i \coloneqq \{f\colon I \to \bigcup_{i \in I} A_i \ : \ f(i) \in A_i, \text{for all } i \in I\}.$$

In particular, if I is empty, then $\prod_{i \in I} A_i$ is the singleton containing only the empty map.

Definition 1.20. The *direct product* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is the unique algebra of the common type whose universe is the Cartesian product $\prod_{i \in I} A_i$ and such that, for every basic n-ary operation symbol f and every $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i \in I} A_i$,

$$f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n)(i) = f^{\boldsymbol{A}_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)), \text{ for every } i \in I.$$

We denote this algebra by $\prod_{i \in I} \boldsymbol{A}_i$.

In this case, for every $j \in I$, the projection map on the j-th component $p_j \colon \prod_{i \in I} A_i \to A_j$, defined by the rule $p_j(\vec{a}) := \vec{a}(j)$, is a surjective homomorphism from $\prod_{i \in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_j$.

Given a class of similar algebras K, we set

$$\mathbb{P}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}(\{\boldsymbol{A}\})$.

Notice that up to isomorphism, there exists a unique one-element algebra of a given type. Because of this, one-element algebras are called *trivial*. Accordingly, when the set of indexes I is empty, the direct product $\prod_{i \in I} \boldsymbol{A}_i$ is the trivial algebra of the given type. It follows that $\mathbb{P}(\mathsf{K})$ contains always a trivial algebra, for every class of similar algebras K.

Example 1.21 (Powerset algebras)**.** Boolean algebras of the form $\langle \mathcal{P}(X); \cap, \cup, -, \emptyset, X\rangle$ are called *powerset Boolean algebras*. Let $\boldsymbol{B}$ be the two-element Boolean algebra and observe that $\mathbb{IP}(\boldsymbol{B})$ is the class of algebras isomorphic to some powerset Boolean algebra. To prove this, observe that every powerset Boolean algebra $\mathcal{P}(X)$ is isomorphic to a direct product of $\boldsymbol{B}$ via the *characteristic function* $f_X \colon \mathcal{P}(X) \to \prod_{x \in X} \boldsymbol{B}_x$, defined by the rule

$$f(Y)(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y, \end{cases}$$

where $Y \in \mathcal{P}(X)$ and $x \in X$. By the same token, every direct product $\prod_{i \in I} \boldsymbol{B}_i$ of $\boldsymbol{B}$ is isomorphic to the powerset Boolean algebra $\mathcal{P}(I)$ via the isomorphism f_I. ⊠

We close this section by reviewing the subdirect product construction.

Definition 1.22. A subalgebra $\boldsymbol{B}$ of a direct product $\prod_{i \in I} \boldsymbol{A}_i$ is said to be a *subdirect product* of $\{\boldsymbol{A}_i : i \in I\}$ if the projection map p_i is surjective, for every $i \in I$. Similarly, an embedding $f \colon \boldsymbol{B} \to \prod_{i \in I} \boldsymbol{A}_i$ is said to be *subdirect* when $f[\boldsymbol{B}]$ is a subdirect product of the family $\{\boldsymbol{A}_i : i \in I\}$.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\text{SD}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is a subdirect direct product of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

As usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\text{SD}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\text{SD}}(\{\boldsymbol{A}\})$. Clearly, $\mathbb{P}_{\text{SD}}(\mathsf{K}) \subseteq \mathbb{SP}(\mathsf{K})$. Furthermore, $\mathbb{P}_{\text{SD}}(\mathsf{K})$ contains always a trivial algebra.

Example 1.23 (Distributive lattices)**.** Let DL be the class of distributive lattices and $\boldsymbol{B}$ be the two-element distributive lattice. Birkhoff's Representation Theorem states that $\mathsf{DL} = \mathbb{IP}_{\text{SD}}(\boldsymbol{B})$. The inclusion $\mathbb{IP}_{\text{SD}}(\boldsymbol{B}) \subseteq \mathsf{DL}$ follows from the fact that DL is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. For the other inclusion, consider a distributive lattice $\boldsymbol{A}$ and let I be the set of its prime filters. By Birkhoff's Representation Theorem, the map

$$\gamma\colon \boldsymbol{A} \to \prod_{F \in I} \boldsymbol{B}_F,$$

defined, for every $a \in A$ and $F \in I$, by the rule

$$\gamma(a)(F) \coloneqq \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F, \end{cases}$$

is a well-defined subdirect embedding. ⊠

Example 1.24 (Boolean algebras)**.** Similarly, Stone's Representation Theorem states that the class of Boolean algebras coincides with $\mathbb{IP}_{\text{SD}}(\boldsymbol{B})$, where $\boldsymbol{B}$ the two-element Boolean algebra. ⊠

The next result provides a general recipe to construct subdirect products.

Proposition 1.25. *Let $\boldsymbol{A}$ be an algebra and $\{\theta_i : i \in I\} \subseteq \mathsf{Con}(\boldsymbol{A})$. Then the map*

$$f\colon \boldsymbol{A}/\bigcap_{i \in I}\theta_i \to \prod_{i \in I} \boldsymbol{A}/\theta_i,$$

defined, for every $a \in A$ and $j \in I$, as

$$f(a/\bigcap_{i \in I}\theta_i)(j) \coloneqq a/\theta_j,$$

is a subdirect embedding.

Proof. For the sake of readability, set $\boldsymbol{B} \coloneqq \boldsymbol{A}/\bigcap_{i \in I}\theta_i$. To prove that f is injective, consider $a, c \in A$ such that $\langle a, c\rangle \notin \bigcap_{i \in I}\theta_i$. Then there exists $j \in I$ such that $\langle a, c\rangle \notin \theta_j$ and, therefore,

$$f(a/\bigcap_{i \in I}\theta_i)(j) \coloneqq a/\theta_j \neq c/\theta_j = f(c/\bigcap_{i \in I}\theta_i)(j).$$

It follows that $f(a/\bigcap_{i \in I}\theta_i) \neq f(c/\bigcap_{i \in I}\theta_i)$. Thus, f is injective. Moreover, by the definition of f, the composition $p_i \circ f\colon \boldsymbol{B} \to \boldsymbol{A}/\theta_i$ is surjective, for every $i \in I$.

It only remains to prove that f is a homomorphism. Consider an n-ary basic operation g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have

$$\begin{aligned} f(g^{\boldsymbol{B}}(a_1/\bigcap_{i \in I}\theta_i, \dots, a_n/\bigcap_{i \in I}\theta_i))(j) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n)/\bigcap_{i \in I}\theta_i)(j) \\ &= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta_j \\ &= g^{\boldsymbol{A}/\theta_j}(a_1/\theta_j, \dots, a_n/\theta_j) \\ &= g^{\boldsymbol{A}/\theta_j}(f(a_1/\bigcap_{i \in I}\theta_i)(j), \dots, f(a_n/\bigcap_{i \in I}\theta_i)(j)) \\ &= g^{\prod_{i \in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i \in I}\theta_i), \dots, f(a_n/\bigcap_{i \in I}\theta_i))(j). \end{aligned}$$

It follows that

$$f(g^{\boldsymbol{B}}(a_1/\bigcap_{i\in I}\theta_i,\ldots,a_n/\bigcap_{i\in I}\theta_i)) = g^{\prod_{i\in I}\boldsymbol{A}/\theta_i}(f(a_1/\bigcap_{i\in I}\theta_i),\ldots,f(a_n/\bigcap_{i\in I}\theta_i)). \qquad \boxtimes$$

CHAPTER 2

Completeness theorems

2.1 Propositional logics

For general information on propositional logics we refer the reader to [20, 35, 42, 55, 56, 57, 115, 119]. Recall that a *closure operator* on a set A is a map $C \colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that, for every $X \subseteq Y \subseteq A$,

$$X \subseteq C(X) = C(C(X)) \text{ and } C(X) \subseteq C(Y).$$

Given a closure operator C on A, a subset $X \subseteq A$ is said to be *closed* if $X = C(X)$. A *closure system* on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ that contains A and such that $\bigcap \mathcal{F}$, for every nonempty $\mathcal{F} \subseteq \mathcal{C}$. Closure operators and systems on A are two faces of the same coin. More precisely, the family of closed sets of a closure operator on A is a closure system on A. On the other hand, if $\mathcal{C}$ is a closure system on A, then the map $C \colon \mathcal{P}(A) \to \mathcal{P}(A)$, defined by the rule

$$C(X) := \bigcap \{Y \in \mathcal{C} : X \subseteq Y\},$$

is a closure operator on A. These transformations between closure operators and systems on A are one inverse to the other.

Exercise 2.1. Prove that these transformations are well-defined and one inverse to the other. ⊠

Another way of presenting closure operators or systems is by means of the following concept.

Definition 2.2. A *consequence relation* on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$ such that, for every $X \cup Y \cup \{a\} \subseteq A$,

(i) if $a \in X$, then $X \vdash a$; and

(ii) if $X \vdash y$ for all $y \in Y$ and $Y \vdash a$, then $X \vdash a$.

Furthermore, $\vdash$ is said to be *finitary* when, for every $X \cup \{a\} \subseteq A$,

$$\text{if } X \vdash a \text{, there exists a finite } Y \subseteq X \text{ such that } Y \vdash a.$$

Remark 2.3. The relation $X \vdash a$ should be read, intuitively, as "X proves a" or "a follows from X". In this reading, the demand expressed by condition (i) is rather natural, while (ii) is an abstract of the Cut rule. ⊠

Formally speaking, a consequence relation on a set A is a binary relation $\vdash \subseteq \mathcal{P}(A) \times A$. However, to simplify the notation, we will often write $a_1, \dots, a_n \vdash c$ as a shorthand for $\{a_1, \dots, a_n\} \vdash c$. Similarly, we will use $X, a \vdash c$ as a shorthand for $X \cup \{a\} \vdash c$. Lastly, for every set of formulas $X \cup Y \cup \{a, c\}$, we write

(i) $X \vdash Y$, when $X \vdash y$ for every $y \in Y$;

(ii) $a \dashv\vdash c$, when $a \vdash c$ and $c \vdash a$; and

(iii) $X \dashv\vdash Y$, when $X \vdash Y$ and $Y \vdash X$.

Definition 2.4. Let $\vdash$ be a consequence relation on a set A. A *theory* of $\vdash$ is a subset $X \subseteq A$ such that, for every $a \in A$, if $X \vdash a$, then $a \in X$. The set of theories of A will be denoted by $\mathcal{T}h(\vdash)$.

It is easy to see that $\mathcal{T}h(\vdash)$ is a closure system on A. Moreover, given a closure operator C on A, the following is a consequence relation on A:

$$\{\langle X, a\rangle \in \mathcal{P}(A) \times A : a \in C(X)\}.$$

Together with the correspondence between closure systems and operators, these transformations induce a one-to-one correspondence between consequence relations, closure operators and closure systems on A.

Exercise 2.5. Prove these facts. ⊠

In the context of logic, the term algebra $\boldsymbol{T}_\rho(Var)$ is often called the *algebra of formulas* (of type ρ) and its elements are referred to as *formulas*. An *endomorphism* of an algebra $\boldsymbol{A}$ is a homomorphism whose domain and codomain is $\boldsymbol{A}$. Endomorphisms of the algebra of formulas play a fundamental role in logic.

Definition 2.6. A *substitution* of type ρ is an endomorphism σ of $\boldsymbol{T}_\rho(Var)$.

When the type ρ is clear from the context, we will simply say that σ is a substitution.

In view of Proposition 1.4 and of the fact that Var is a set of generators for $\boldsymbol{T}_\rho(Var)$, every function $\sigma\colon Var \to T_\rho(Var)$ can be uniquely extended to a substitution σ^+ of type ρ, namely the function defined by the rule

$$\varphi(x_1, \dots, x_n) \longmapsto \varphi(\sigma(x_1), \dots, \sigma(x_n)).$$

Because of this, substitutions can be presented by exhibiting functions $\sigma\colon Var \to T_\rho(Var)$.

Definition 2.7. A *logic* of type ρ is a consequence relation $\vdash$ on the set of formulas $T_\rho(Var)$ that, moreover, is *substitution invariant* in the sense that for every substitution σ of type ρ and every set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Remark 2.8. As mentioned above, $\Gamma \vdash \varphi$ should be read as "Γ proves φ" or "φ follows from Γ". The requirement that $\vdash$ is substitution invariant, instead, is intended to capture the idea that logical inferences are true only in virtue of their form (as opposed to their content). ⊠

Example 2.9 (Hilbert calculi)**.** We work within a fixed, but arbitrary, type ρ. A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. In this case, Γ is said to be the set of *premises* of the rule and φ the *conclusion*. When $\Gamma = \emptyset$, the rule $\Gamma \rhd \varphi$ is sometimes called an *axiom*. A rule $\Gamma \rhd \varphi$ is said to be *valid* in a logic $\vdash$, when $\Gamma \vdash \varphi$. A *Hilbert calculus* is a set of rules.

Every Hilbert calculus H induces a logic, as we proceed to explain. Consider a set of formulas $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$. A *proof of φ from Γ in* H is a well-ordered sequence $\langle \psi_\alpha : \alpha \leqslant \gamma \rangle$ of formulas $\psi_\alpha \in T_\rho(Var)$ whose last element ψ_γ is φ and such that, for every $\alpha \leqslant \gamma$, either $\psi_\alpha \in \Gamma$ or there exist a substitution σ and a rule $\Delta \rhd \delta$ in H such that the formulas in $\sigma[\Delta]$ occur in the initial segment $\langle \psi_\beta : \beta < \alpha \rangle$ and $\psi_\alpha = \sigma(\delta)$.

The logic $\vdash_{\mathsf{H}}$ induced by H is defined, for every $\Gamma \cup \{\varphi\} \subseteq T_\rho(Var)$, as

$$\Gamma \vdash_{\mathsf{H}} \varphi \iff \text{there exists a proof of } \varphi \text{ from } \Gamma \text{ in } \mathsf{H}.$$

As expected, $\vdash_{\mathsf{H}}$ is a logic in the sense of Definition 2.7. Furthermore, it is the least logic $\vdash$ such that $\Gamma \vdash \varphi$, for every rule $\Gamma \rhd \varphi$ in H.

A logic $\vdash$ is said to be *axiomatized* by a Hilbert calculus H when it coincides with $\vdash_{\mathsf{H}}$. Notice that every logic $\vdash$ is vacuously axiomatized by the Hilbert calculus

$$\{\Gamma \rhd \varphi : \Gamma \vdash \varphi\}.$$

Because of this, axiomatizations in terms of Hilbert calculi H acquire special interest when H is finite or, at least, recursive. ⊠

When no confusion shall arise, given a sequence $\vec{a}$ and a set A, we write $\vec{a} \in A$ to indicate that the elements of the sequence $\vec{a}$ belong to A. The following concept is instrumental to exhibit further examples of logics.

Definition 2.10. Let K be a class of similar algebras We define a binary relation $\vDash_{\mathsf{K}} \subseteq \mathcal{P}(E_\rho(Var)) \times E_\rho(Var)$ as follows:

$$\begin{aligned}\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff & \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } \vec{a} \in A,\\ & \text{if } \varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}) \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).\end{aligned}$$

The relation $\vDash_{\mathsf{K}}$ is known as the *equational consequence relative to* K.

Example 2.11 (Equationally defined logics)**.** We work within a fixed, but arbitrary, type ρ. Given a set of equations $\boldsymbol{\tau}(x)$ in a single variable x and a set of formulas $\Gamma \cup \{\varphi\} \subseteq T(Var)$, we abbreviate

$$\{\varepsilon(\varphi) \approx \delta(\varphi) \colon \varepsilon \approx \delta \in \boldsymbol{\tau}\} \text{ as } \boldsymbol{\tau}(\varphi), \text{ and } \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma) \text{ as } \boldsymbol{\tau}[\Gamma].$$

Given a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$, we define a logic $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash_{\mathsf{K},\boldsymbol{\tau}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

It is easy to prove that $\vdash_{\mathsf{K},\tau}$ is indeed a logic in the sense of Definition 2.7. Notice that, in this case, $\vdash$ is related to K by a *completeness theorem* witnessed by the set of equations $\boldsymbol{\tau}(x)$ that allows to translate formulas into equations and, therefore, to interpret $\vdash_{\mathsf{K},\tau}$ into $\vDash_{\mathsf{K}}$.

For instance, the completeness theorem of classical propositional logic **CPC** with respect to the class of Boolean algebras BA states precisely that **CPC** coincides with $\vdash_{\mathsf{BA},\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = \{x \approx 1\}$. Similarly, the completeness theorem of intuitionistic propositional logic **IPC** with respect to the class of Heyting algebras HA states precisely that **IPC** coincides with $\vdash_{\mathsf{HA},\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = \{x \approx 1\}$. Because of this, **CPC** and **IPC** can be defined as follows: for every set of formulas $\Gamma \cup \{\varphi\}$ of the appropriate type,

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi)$$
$$\Gamma \vdash_{\mathbf{IPC}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi),$$

where $\boldsymbol{\tau} = \{x \approx 1\}$. ⊠

2.2 Algebraic semantics

The relation between logic and algebra is often explained in terms of the existence of equational completeness theorems. The following definition makes this concept precise. As we will see, however, equational completeness theorems alone are not sufficient to account for the relation between logic and algebra.

Definition 2.12. A logic $\vdash$ is said to admit an *equational completeness theorem* if there are a set of equations $\boldsymbol{\tau}(x)$ and a class K of algebras such that for all $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

In this case, $\vdash$ coincides with $\vdash_{\mathsf{K},\boldsymbol{\tau}}$ and K is said to be a $\boldsymbol{\tau}$*-algebraic semantics* (or simply an *algebraic semantics*) for $\vdash$.

This notion was introduced in [21] and studied in depth in [28, 104, 113]. For instance, the classes of Boolean and Heyting algebras are, respectively, $\boldsymbol{\tau}$-algebraic semantics for **CPC** and **IPC** where $\boldsymbol{\tau} = \{x \approx 1\}$.

Another familiar example of equational completeness theorem arises from the field of modal logic. Let Fr be the class of all Kripke frames. We can associate two distinct logics with Fr, see for instance [89, 90]. The *global consequence* $\mathbf{K}_g$ of the modal system $\mathbf{K}$ is the logic defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned}\Gamma \vdash_{\mathbf{K}_g} \varphi \iff & \text{for every } \langle W, R\rangle \in \mathsf{Fr} \text{ and evaluation } v \text{ in } \langle W, R\rangle,\\ & \text{if } w, v \Vdash \Gamma \text{ for all } w \in W, \text{ then } w, v \Vdash \varphi \text{ for all } w \in W.\end{aligned}$$

On the other hand, the *local consequence* $\mathbf{K}_\ell$ of the modal system $\mathbf{K}$ is defined, for every set of modal formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\begin{aligned}\Gamma \vdash_{\mathbf{K}_\ell} \varphi \iff & \text{for every } \langle W, R\rangle \in \mathsf{Fr}, w \in W, \text{ and evaluation } v \text{ in } \langle W, R\rangle,\\ & \text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.\end{aligned}$$

It is easy to see that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ are logics. Moreover, they are distinct, because

$$x \vdash_{\mathbf{K}_g} \Box x \text{ and } x \nvdash_{\mathbf{K}_\ell} \Box x. \tag{2.1}$$

Exercise 2.13. Prove that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ are logics. Notice also that the modal system $\mathbf{K}$ is not a logic itself, because it is not a consequence relation. Indeed, there are two ways to turn $\mathbf{K}$ into a logic, namely, $\mathbf{K}_g$ and $\mathbf{K}_\ell$. ⊠

Exercise 2.14. Prove that $\mathbf{K}_g$ and $\mathbf{K}_\ell$ have the same *theorems*, i.e., formulas provable from the empty set. Prove also that the set of their theorems is the modal system $\mathbf{K}$. This indicates that, even in the modal setting, logics should not be identified with their sets of theorems. ⊠

The global consequence $\mathbf{K}_g$ is related to the class MA of modal algebras by the following equational completeness theorem.

Theorem 2.15. *For every set $\Gamma \cup \{\varphi\}$ of modal formulas,*

$$\Gamma \vdash_{\mathbf{K}_g} \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi),$$

where $\boldsymbol{\tau} = \{x \approx 1\}$. Consequently, the class of modal algebras is a $\boldsymbol{\tau}$-algebraic semantics for $\mathbf{K}_g$.

In order to prove it, recall that a filter on a Boolean algebra $\boldsymbol{A}$ is said to be *proper* when it differs from A. Moreover, a proper filter U of $\boldsymbol{A}$ is said to be a *ultrafilter* of $\boldsymbol{A}$ if it is maximal among the proper filters of $\boldsymbol{A}$ or, equivalently, if

$$a \in U \text{ or } \neg a \in U, \text{ for every } a \in A.$$

While the following result holds in ZFC, it cannot be proved in ZF (although it is strictly weaker then the axiom of choice).

Ultrafilter Lemma 2.16. *Every proper filter on a Boolean algebra can be extended to a ultrafilter.*

We are now ready to prove Theorem 2.15.

Proof sketch. It suffices to prove that

$$\Gamma \nvdash_{\mathbf{K}_g} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi).$$

Suppose first that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. Then there are a Kripke frame $\langle W, R\rangle$, an evaluation v in it and a world u such that

$$w, v \Vdash \Gamma \text{ for all } w \in W \quad \text{and} \quad u, v \nVdash \varphi. \tag{2.2}$$

Then consider the complex algebra of $\langle W, R\rangle$, that is, the structure

$$\boldsymbol{A} := \langle \mathcal{P}(W); \cap, \cup, -, \Box, \emptyset, W\rangle,$$

where $-$ is set theoretic complement and, for every $V \subseteq W$,

$$\Box V := \{w \in W : \text{if } \langle w, t\rangle \in R, \text{ then } t \in V\}.$$

It is easy to prove that $\boldsymbol{A}$ is a modal algebra. Then consider the unique homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f(x) = \{w \in W : w, v \Vdash x\},$$

for every $x \in Var$. A simple induction of the construction of terms shows that, for every formula ψ,

$$f(\psi) = \{w \in W : w, v \Vdash \psi\}.$$

Together with (2.2), this yields

$$f[\Gamma] \subseteq \{W\} \quad \text{and} \quad f(\varphi) \neq W.$$

Hence, we conclude that $\boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi)$.

To prove the converse, suppose that $\boldsymbol{\tau}[\Gamma] \nvDash_{\mathsf{MA}} \boldsymbol{\tau}(\varphi)$. Then there are a modal algebra $\boldsymbol{A}$ and a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that

$$f[\Gamma] \subseteq \{1\} \quad \text{and} \quad f(\varphi) \neq 1.$$

Then consider the Kripke frame dual to $\boldsymbol{A}$, that is, the structure $\langle W, R\rangle$, where W is the set of ultrafilters of $\boldsymbol{A}$ and R the binary relation on W defined as follows:

$$R := \{\langle U, V\rangle \in W \times W : \{a \in A : \Box a \in U\} \subseteq V\}.$$

Let then $v\colon Var \to \mathcal{P}(W)$ be the evaluation in $\langle W, R\rangle$ defined by the rule

$$v(x) := \{U \in W : f(x) \in U\}.$$

An easy induction on the construction of terms shows that, for every formula ψ,

$$\{U \in W : f(\psi) \in U\} = \{U \in W : U, v \Vdash \psi\}. \tag{2.3}$$

Now, since $f(\varphi) \neq 1$, the Ultrafilter Lemma guarantees the existence of an ultrafilter F such that $f(\varphi) \notin F$. Furthermore, as every ultrafilter contain 1, from $f[\Gamma] \subseteq \{1\}$ it follows that $f[\Gamma] \subseteq U$, for all $U \in W$. In short,

$$f[\Gamma] \subseteq U \text{ for all } U \in W \quad \text{and} \quad f(\varphi) \notin F.$$

Together with (2.3), this yields

$$U, v \Vdash \Gamma \text{ for all } U \in W \quad \text{and} \quad F, v \nVdash \varphi.$$

Hence, we conclude that $\Gamma \nvdash_{\mathbf{K}_g} \varphi$. ⊠

At this stage, it is tempting to conjecture that the relation between logic and algebra can be explained in terms of equational completeness theorems only. As we anticipated, however, this is not the case. For instance, the relation between **CPC** and BA cannot be explained in terms of completeness theorems only, because the class of Heyting algebras HA is also an algebraic semantics for **CPC**. To explain why, it is convenient to recall the following classical result relating **CPC** and **IPC** [67].

Givenko's Theorem 2.17. *For every set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma \colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

As a consequence, we obtain the desired result.

Corollary 2.18. *The class of Heyting algebras is an algebraic semantics for* **CPC***.*

Proof. For every set of formulas $\Gamma \cup \{\varphi\}$, we have

$$\begin{aligned} \Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma \colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1 \colon \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1. \end{aligned}$$

The first equivalent above is Glivenko's Theorem, while the second is a consequence of the completeness theorem of **IPC** with respect to HA. As a consequence, the class of Heyting algebras is a $\boldsymbol{\tau}$-algebraic semantics for **IPC**, where $\boldsymbol{\tau} = \{\neg\neg x \approx 1\}$. ⊠

This means that the univocal relation between **CPC** and the class of Boolean algebras cannot be explained in terms of the existence of completeness theorems only. As we shall see, this relation arises from a deeper phenomenon, known as *algebraizability* [21].

Exercise 2.19. One may wonder whether the fact that **CPC** has many distinct algebraic semantics cannot be amended by restricting our attention to $\boldsymbol{\tau}$-algebraic semantics where $\boldsymbol{\tau} = \{x \approx 1\}$. This is not the case, as this exercise asks you to check. Let $\boldsymbol{A}$ be the three-element algebra $\langle \{0, 1, a\}; \wedge, \vee, \neg, 0, 1 \rangle$ where $\langle A; \wedge, \vee \rangle$ is the lattice with order $0 < a < 1$ and $\neg \colon A \to A$ is the map described by the rule

$$\neg 0 = \neg a = 1 \text{ and } \neg 1 = 0.$$

Clearly, $\boldsymbol{A}$ is not a Boolean algebra (as there is no three-element Boolean algebra). Prove that $\{\boldsymbol{A}\}$ is $\boldsymbol{\tau}$-algebraic semantics for **CPC** where $\boldsymbol{\tau} = \{x \approx 1\}$. Hint: use the fact that the two-element Boolean algebra is a homomorphic image of $\boldsymbol{A}$. ⊠

Indeed the existence of equational completeness theorems between a logic and a class of algebras turns out to be a very weak relation, as shown in [28, 104]. For instance, while many interesting logics lack a natural equational completeness theorem, they still admit a nonstandard one. This is the case of $\mathbf{K}_\ell$, as we proceed to explain.

A logic $\vdash$ is said to be *protoalgebraic* if there exists a set $\Delta(x, y)$ of formulas such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Notice that all logics $\vdash$ with a binary connective $\to$ such that $\emptyset \vdash x \to x$ and $x, x \to y \vdash y$ are protoalgebraic, as witnessed by the set $\Delta := \{x \to y\}$. Furthermore, a logic $\vdash$ is said to be *nontrivial* if $x \nvdash y$.

Theorem 2.20 (M. [104, Thm. 9.3])**.** *A nontrivial protoalgebraic logic. $\vdash$ has an algebraic semantics if and only it there are two distinct formulas φ and ψ that are logically equivalent in the sense that*

$$\delta(\varphi, \vec{z}) \dashv\vdash \delta(\psi, \vec{z}), \text{ for all } \delta(x, \vec{z}) \in T(Var).$$

As a consequence, we obtain the following.

Corollary 2.21. *The logic $\mathbf{K}_\ell$ has an algebraic semantics.*

Proof. Clearly, $\mathbf{K}_\ell$ is nontrivial and protoalgebraic. Furthermore, the formulas x and $x \wedge x$ are distinct, but logical equivalent in $\mathbf{K}_\ell$. Therefore, $\mathbf{K}_\ell$ has an algebraic semantics in view of Theorem 2.20. ⊠

On the other hand, $\mathbf{K}_\ell$ lacks any natural equational completeness theorem.

Theorem 2.22 (M. [104, Cor. 9.7])**.** *No class of modal algebras is an algebraic semantics for $\mathbf{K}_\ell$.*

Proof. We begin by proving that, for all $\varphi, \psi \in T(Var)$,

$$\mathsf{MA} \vDash \varphi \approx \psi \iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi. \tag{2.4}$$

To this end, observe that

$$\begin{aligned} \mathsf{MA} \vDash \varphi \approx \psi &\iff \mathsf{MA} \vDash \varphi \leftrightarrow \psi \approx 1 \\ &\iff \emptyset \vdash_{\mathbf{K}_g} \varphi \leftrightarrow \psi \\ &\iff \emptyset \vdash_{\mathbf{K}_\ell} \varphi \leftrightarrow \psi \\ &\iff \varphi \dashv\vdash_{\mathbf{K}_\ell} \psi. \end{aligned}$$

The above equivalence are justified as follows. The first is an easy property of Boolean algebras, the second is a consequence of Theorem 2.15, the third holds because $\mathbf{K}_g$ and $\mathbf{K}_\ell$ have the same theorems (see Exercise 2.14) and the last one because $\mathbf{K}_\ell$ has a standard deduction theorem.

Now, suppose, with a view to contradiction, that $\mathbf{K}_\ell$ has a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K} \subseteq \mathsf{MA}$. This implies that there exists an equation $\varepsilon \approx \delta \in \boldsymbol{\tau}$ such that $\mathsf{MA} \nvDash \varepsilon \approx \delta$. Thus, in view of the above display, we can assume, by symmetry, that $\varepsilon \nvdash_{\mathbf{K}_\ell} \delta$. This means that there are a Kripke frame $\mathbb{X} = \langle X, R \rangle$, an element $w \in X$ and a valuation v in $\mathbb{X}$ such that $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$.

Let $\mathbb{X}^+ = \langle X^+; R^+ \rangle$ be the Kripke frame obtained by adding a new point w^+ to $\mathbb{X}$ and defining the relation R^+ as follows:

$$\langle p, q \rangle \in R^+ \iff p = w^+ \text{ or } \langle p, q \rangle \in R.$$

Let also v^+ be the unique evaluation in $\mathbb{X}^+$ such that for every $y \in Var$ and $q \in \mathbb{X}^+$:

$$q, v^+ \Vdash y \iff \text{ either } (q \in X \text{ and } q, v \Vdash y) \text{ or } q = w^+.$$

From the definition of $\mathbb{X}^+$ and v^+ it follows that

$$q, v^+ \Vdash \varphi \iff q, v \Vdash \varphi$$

for all $\varphi \in T(Var)$ and $q \in \mathbb{X}$. Consequently, as $w, v \Vdash \varepsilon$ and $w, v \nVdash \delta$,

$$w^+, v^+ \Vdash x \text{ and } w^+, v^+ \nVdash \Box(\varepsilon \to \delta).$$

This implies

$$x \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta).$$

On the other hand, clearly $\emptyset \vdash_{\mathbf{K}_\ell} \Box(\delta \to \delta)$. Consequently,

$$x, \Box(\delta \to \delta) \nvdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta). \tag{2.5}$$

Now, observe that, for every $\varphi, \psi \in T(Var)$,

$$\varepsilon(x) \approx \delta(x), \varphi(\Box(\delta \to \delta)) \approx \psi(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \varphi(\Box(\varepsilon \to \delta)) \approx \psi(\Box(\varepsilon \to \delta)).$$

Since $\varepsilon \approx \delta \in \boldsymbol{\tau}(x)$, this implies

$$\boldsymbol{\tau}(x), \boldsymbol{\tau}(\Box(\delta \to \delta)) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\Box(\varepsilon \to \delta)).$$

Since K is a $\boldsymbol{\tau}$-algebraic semantics for $\mathbf{K}_\ell$, this yields $x, \Box(\delta \to \delta) \vdash_{\mathbf{K}_\ell} \Box(\varepsilon \to \delta)$, a contradiction with (2.5). ⊠

While, in view of Theorem 2.22, most logics have an algebraic semantics, examples of logics lacking any algebraic semantics are known since [17].

Exercise 2.23. Prove that $\mathbf{K}_\ell$ has a standard deduction theorem, i.e., that for every set of formulas $\Gamma \cup \{\psi, \varphi\}$,

$$\Gamma, \psi \vdash_{\mathbf{K}_\ell} \varphi \iff \Gamma \vdash_{\mathbf{K}_\ell} \psi \to \varphi.$$

Prove that this is not the case for $\mathbf{K}_g$. ⊠

Exercise 2.24. Prove that no class of distributive lattices is an algebraic semantics for the $\langle \wedge, \vee \rangle$-fragment $\mathbf{CPC}_{\wedge\vee}$ of $\mathbf{CPC}$. Hint: use the fact that every equation in a single variable holds in the class of distributive lattices.

Furthermore, prove that $\mathbf{CPC}_{\wedge\vee}$ has a nonstandard algebraic semantics. To this end, consider the three-element algebra $\boldsymbol{A} = \langle \{0^+, 0^-, 1\}; \wedge, \vee \rangle$ whose binary commutative operations are defined by the following tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

and prove that $\{\boldsymbol{A}\}$ is a $\boldsymbol{\tau}$-algebraic semantics for $\boldsymbol{\tau} = \{x \approx x \wedge x\}$. Conclude that $\mathbf{CPC}_{\wedge\vee}$ is another example of logic that admits a nonstandard algebraic semantics, but lacks a standard one. ⊠

CHAPTER 3

Universal algebra

3.1 Ultraproducts

In order to understand the relation between logic and algebra, we need to take a short detour in universal algebra and the theory of quasi-varieties. We begin by reviewing a product-like construction known as *ultraproduct* [10, 34]. First, recall that ultrafilters on powerset Boolean algebras $\mathcal{P}(X)$ are also called *ultrafilters on* X. Then let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras. The *equalizer* $[\![\vec{a} = \vec{c}]\!]$ of a pair of elements $\vec{a}, \vec{c} \in \prod_{i \in I} A_i$ is the set of indexes on which the sequences $\vec{a}$ and $\vec{c}$ agree, that is,

$$[\![\vec{a} = \vec{c}]\!] := \{i \in I : \vec{a}(i) = \vec{c}(i)\}.$$

Moreover, given an ultrafilter U on the index set I, let θ_U be the binary relation on the Cartesian product $\prod_{i \in I} A_i$ defined as

$$\theta_U := \{\langle \vec{a}, \vec{c} \rangle : [\![\vec{a} = \vec{c}]\!] \in U\}.$$

Proposition 3.1. *If $\{\boldsymbol{A}_i : i \in I\}$ is a family of similar algebras and U an ultrafilter on I, then θ_U is a congruence of $\prod_{i \in I} \boldsymbol{A}_i$.*

Proof. We begin by proving that θ_U is an equivalence relation on $\prod_{i \in I} A_i$. To this end, consider $\vec{a}, \vec{b}, \vec{c} \in \prod_{i \in I} A_i$. We have

$$[\![\vec{a} = \vec{a}]\!] = \{i \in I : \vec{a}(i) = \vec{a}(i)\} = I.$$

Observe that $I \in U$, since U is a nonempty upset of $\mathcal{P}(I)$. Together with the above display, this yields $[\![\vec{a} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{a}, \vec{a} \rangle \in \theta_U$. It follows that θ_U is reflexive. To prove that it is symmetric, suppose that $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. Then $[\![\vec{a} = \vec{c}]\!] \in U$. Since $[\![\vec{a} = \vec{c}]\!] = [\![\vec{c} = \vec{a}]\!]$, this implies $[\![\vec{c} = \vec{a}]\!] \in U$ and, therefore, $\langle \vec{c}, \vec{a} \rangle \in \theta_U$. Lastly, to prove that θ_U is transitive, suppose that $\langle \vec{a}, \vec{b} \rangle, \langle \vec{b}, \vec{c} \rangle \in \theta_U$, that is, $[\![\vec{a} = \vec{b}]\!], [\![\vec{b} = \vec{c}]\!] \in U$. Since U is closed under binary meets,

$$[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \in U$$

Clearly, $[\![\vec{a} = \vec{b}]\!] \cap [\![\vec{b} = \vec{c}]\!] \subseteq [\![\vec{a} = \vec{c}]\!]$. Since U is an upset of $\mathcal{P}(I)$, we obtain that $[\![\vec{a} = \vec{c}]\!] \in U$, whence $\langle \vec{a}, \vec{c} \rangle \in \theta_U$. We conclude that θ_U is an equivalence relation.

To prove that θ_U is a congruence, it only remains to show that it preserves the basic operations. Accordingly, let f be a basic n-ary operation and $\vec{a}_1, \dots, \vec{a}_n, \vec{c}_1, \dots, \vec{c}_n \in \prod_{i \in I} A_i$ such that

$$\langle \vec{a}_1, \vec{c}_1 \rangle, \dots, \langle \vec{a}_n, \vec{c}_n \rangle \in \theta_U.$$

By definition of θ_U, this amounts to $[\![\vec{a}_1 = \vec{c}_1]\!], \dots, [\![\vec{a}_n = \vec{c}_n]\!] \in U$. Since U is a filter, it is closed under finite meets, whence

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \in U. \tag{3.1}$$

We will show that

$$[\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!] \subseteq [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]. \tag{3.2}$$

To this end, consider $j \in [\![\vec{a}_1 = \vec{c}_1]\!] \cap \dots \cap [\![\vec{a}_n = \vec{c}_n]\!]$. We have

$$\vec{a}_1(j) = \vec{c}_1(j), \dots, \vec{a}_n(j) = \vec{c}_n(j).$$

Consequently,

$$\begin{aligned} f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a})(j) &= f^{\boldsymbol{A}_j}(\vec{a}_1(j), \dots, \vec{a}_n(j)) \\ &= f^{\boldsymbol{A}_j}(\vec{c}_1(j), \dots, \vec{c}_n(j)) \\ &= f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c})(j), \end{aligned}$$

that is, $j \in [\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!]$. This establishes (3.2). Since U is an upset of $\mathcal{P}(I)$, from (3.1) and (3.2) it follows

$$[\![f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n)]\!] \in U.$$

Hence, we conclude that $\langle f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{a}_1, \dots, \vec{a}_n), f^{\prod_{i \in I} \boldsymbol{A}_i}(\vec{c}_1, \dots, \vec{c}_n) \rangle \in \theta_U$, as desired. ⊠

In view of the above result, we can make the following definition.

Definition 3.2. An *ultraproduct* of a family of similar algebras $\{\boldsymbol{A}_i : i \in I\}$ is an algebra of the form $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$, for some ultrafilter U on I.

Given a class of similar algebras K, we set

$$\mathbb{P}_{\text{U}}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is an ultraproduct of a family } \{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}\}.$$

Notice that $\mathbb{P}_{\text{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$. Furthermore, as usual, when $\mathsf{K} = \{\boldsymbol{A}\}$, we write $\mathbb{P}_{\text{U}}(\boldsymbol{A})$ as a shorthand for $\mathbb{P}_{\text{U}}(\{\boldsymbol{A}\})$.

Exercise 3.3. Prove that if U is not free (that is, it is principal), then $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$ is isomorphic to some $\boldsymbol{A}_i$. Conclude that if I is finite, then $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$ belongs to $\mathbb{I}\{\boldsymbol{A}_i : i \in I\}$. Because of this, interesting ultraproducts arise from free ultrafilters only. ⊠

Exercise 3.4. Prove that K is a finite set of finite algebras, $\mathbb{P}_{\text{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$. ⊠

The importance of ultraproducts is largely due to the following result [30, Thm. V.2.9].

Łoś' Theorem 3.5. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I and $\phi(x_1, \ldots, x_n)$ a first order formula. For every $\vec{a}_1, \ldots, \vec{a}_n \in \prod_{i \in I} A_i$,*

$$\prod_{i \in I} \boldsymbol{A}_i / \theta_U \vDash \phi(\vec{a}_1/\theta_U, \ldots, \vec{a}_n/\theta_U) \iff \{i \in I : \boldsymbol{A}_i \vDash \phi(\vec{a}_1(i), \ldots, \vec{a}_n(i))\} \in U.$$

Corollary 3.6. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of similar algebras, U an ultrafilter on I and ϕ a sentence. If ϕ is valid in all the $\boldsymbol{A}_i$, then it is valid in $\prod_{i \in I} \boldsymbol{A}_i / \theta_U$.*

In view Łos' Theorem, ultraproducts are instrumental to construct nonstandard models of first order theories. For instance, let $\mathbb{N} = \langle \mathbb{N}; s, +, \cdot, 0 \rangle$ be the standard model of Peano Arithmetic. If U is an ultrafilter on $\mathbb{N}$, the ultraproduct $\prod_{n \in \mathbb{N}} \mathbb{N}_n / U$ is *elementarily equivalent* to $\mathbb{N}$, that is, it satisfies the same sentences as $\mathbb{N}$. On the other hand, it is not hard to see that if U is free, $\prod_{n \in \mathbb{N}} \mathbb{N}_n / U$ is uncountable and, therefore, contains many "infinite" (or nonstandard) natural numbers.

For the present purpose, however, we will not need the full strength of Łoś' Theorem and, therefore, we shall omit its proof. Instead, we shall focus on a particular embedding theorem for ultraproducts that depends on the following notion.

Definition 3.7. A *local subgraph* $\mathbb{X}$ of an algebra $\boldsymbol{A}$ is a finite subset $X \subseteq A$ endowed with the restriction of finitely many basic operations of $\boldsymbol{A}$ to X.

In this case, $\mathbb{X}$ is a finite *partial* algebra of finite type (even when the type of $\boldsymbol{A}$ is infinite).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras and $\mathbb{X}$ a local subgraph of $\boldsymbol{A}$. A map $f \colon X \to B$ is said to be an *embedding* of $\mathbb{X}$ into $\boldsymbol{B}$ if it is injective and, for every basic n-ary operation g of the type of $\mathbb{X}$ and $a_1, \ldots, a_n \in X$ such that $g^{\boldsymbol{A}}(a_1, \ldots, a_n) \in X$,

$$f(g^{\boldsymbol{A}}(a_1, \ldots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \ldots, f(a_n)).$$

Theorem 3.8. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If every local subgraph of $\boldsymbol{A}$ can be embedded into some member of K, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Proof. Let I be the set of local subgraphs of $\boldsymbol{A}$. By assumption, for every $\mathbb{X} \in I$ there are an algebra $\boldsymbol{B}_{\mathbb{X}} \in \mathsf{K}$ and an embedding $h_{\mathbb{X}} \colon \mathbb{X} \to \boldsymbol{B}_{\mathbb{X}}$. We define a partial order $\sqsubseteq$ on I as follows:

$$\mathbb{X} \sqsubseteq \mathbb{Y} \iff X \subseteq Y \text{ and the type of } \mathbb{Y} \text{ extends that of } \mathbb{X}.$$

Then, for every $\mathbb{X} \in I$, define

$$J_{\mathbb{X}} := \{\mathbb{Y} \in I \colon \mathbb{X} \sqsubseteq \mathbb{Y}\}.$$

Moreover, let $\mathcal{F}$ be the filter of $\mathcal{P}(I)$ generated by $\{J_{\mathbb{X}} \colon \mathbb{X} \in I\}$. Recall that

$$\mathcal{F} = \{Y \subseteq I : J_{\mathbb{X}_1} \cap \cdots \cap J_{\mathbb{X}_n} \subseteq Y, \text{ for some } \mathbb{X}_1, \ldots, \mathbb{X}_n \in I\}.$$

We will prove that $\mathcal{F}$ is proper. To this end, consider $\mathbb{X}_1, \ldots, \mathbb{X}_n \in I$. Then let $\mathbb{Y}$ be the local subgraph of $\boldsymbol{A}$ with universe $Y := X_1 \cup \cdots \cup X_n$ and whose type in the union of the types of the various $\mathbb{X}_i$. Then

$$\mathbb{X}_i \sqsubseteq \mathbb{Y}, \text{ for every } i \leqslant n,$$

that is, $\mathbb{Y} \in J_{\mathbb{X}_1} \cap \cdots \cap J_{\mathbb{X}_n}$. It follows that $\emptyset \notin \mathcal{F}$ and, therefore, that $\mathcal{F}$ is proper. As $\mathcal{F}$ is a proper filter, by the Ultrafilter Lemma, it can be extended to an ultrafilter U on I.

Now, consider a map

$$f \colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}}$$

such that $f(a)(\mathbb{X}) = h_{\mathbb{X}}(a)$, for every $a \in A$ and $\mathbb{X} \in I$ such that $a \in \mathbb{X}$. Moreover, let

$$f^* \colon A \to \prod_{\mathbb{X} \in I} B_{\mathbb{X}}/\theta_U$$

be the map defined by the rule

$$f^*(a) := f(a)/\theta_U.$$

We will show f^* is an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}/\theta_U$.

In order to prove that f^* is injective, consider a pair of distinct elements $a, c \in A$. Consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ containing a and c. We will show that

$$J_{\mathbb{Y}} \subseteq \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \tag{3.3}$$

Consider $\mathbb{X} \in J_{\mathbb{Y}}$. Then $\mathbb{Y} \sqsubseteq \mathbb{X}$ and, therefore, $a, c \in Y \subseteq X$. Since $a, c \in X$, we have

$$f(a)(\mathbb{X}) = h_{\mathbb{X}}(a) \ \text{ and } \ f(c)(\mathbb{X}) = h_{\mathbb{X}}(c).$$

Furthermore, $h_{\mathbb{X}}(a) \neq h_{\mathbb{X}}(c)$, because $h_{\mathbb{X}}$ is injective and $a \neq c$. This yields $f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})$, establishing (3.3).

Recall that the definition of U guarantees that $J_{\mathbb{Y}} \in \mathcal{F} \subseteq U$. Therefore, since U is an upset of $\mathcal{P}(I)$, we can apply (3.3) obtaining

$$I \smallsetminus [\![f(a) = f(c)]\!] = \{\mathbb{X} \in I : f(a)(\mathbb{X}) \neq f(c)(\mathbb{X})\} \in U.$$

Since U is a proper filter, this implies

$$[\![f(a) = f(c)]\!] \notin U$$

and, therefore,

$$f^*(a) = f(a)/\theta_U \neq f(c)/\theta_U = f^*(c).$$

Hence, we conclude that f^* is injective.

To prove that it is a homomorphism, consider a basic n-ary operation g and $a_1, \dots, a_n \in A$. Then consider a local subgraph $\mathbb{Y}$ of $\boldsymbol{A}$ whose universe contains $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n)$ and whose type contains g. We will prove that

$$J_{\mathbb{Y}} \subseteq [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X} \in I} \boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]. \tag{3.4}$$

Consider $\mathbb{V} \in J_{\mathbb{Y}}$. Since $\mathbb{Y} \sqsubseteq \mathbb{V}$, the type of $\mathbb{V}$ contains g and $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$. Since $a_1, \dots, a_n, g^{\boldsymbol{A}}(a_1, \dots, a_n) \in V$, we have

$$\begin{aligned} f(a_1)(\mathbb{V}) &= h_{\mathbb{V}}(a_1) \\ &\vdots \\ f(a_n)(\mathbb{V}) &= h_{\mathbb{V}}(a_n) \\ f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) &= h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)). \end{aligned}$$

Furthermore, as the type of $\mathbb{V}$ contains g,

$$h_{\mathbb{V}}(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\boldsymbol{B}_{\mathbb{V}}}(h_{\mathbb{V}}(a_1), \dots, h_{\mathbb{V}}(a_n)).$$

From the above displays it follows

$$f(g^{\boldsymbol{A}}(a_1, \dots, a_n))(\mathbb{V}) = g^{\boldsymbol{B}_{\mathbb{V}}}(f(a_1)(\mathbb{V}), \dots, f(a_n)(\mathbb{V})) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))(\mathbb{V}),$$

that is, $\mathbb{V} \in [\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!]$. This establishes (3.4). Lastly, as $J_{\mathbb{Y}} \in U$ and U is an upset of $\mathcal{P}(I)$, condition (3.4) implies

$$[\![f(g^{\boldsymbol{A}}(a_1, \dots, a_n)) = g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))]\!] \in U,$$

and, therefore,

$$\begin{aligned} f^*(g^{\boldsymbol{A}}(a_1, \dots, a_n)) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}}(f(a_1), \dots, f(a_n))/\theta_U \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}/\theta_U}(f(a_1)/\theta_U, \dots, f(a_n)/\theta_U) \\ &= g^{\prod_{\mathbb{X}\in I}\boldsymbol{B}_{\mathbb{X}}/\theta_U}(f^*(a_1), \dots, f^*(a_n)). \end{aligned}$$

Hence, we conclude that f^* is a homomorphism and, therefore, an embedding of $\boldsymbol{A}$ into $\prod_{\mathbb{Y}\in I}\boldsymbol{B}_{\mathbb{Y}}/\theta_U$. As a consequence,

$$\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\{\boldsymbol{B}_{\mathbb{X}} : \mathbb{X} \in I\}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}).$$ ⊠

Corollary 3.9. *Every algebra embeds into an ultraproduct of its finitely generated subalgebras.*

Exercise 3.10. Consider the consequence relation $\vdash$ on the set of sentences of a given algebraic language defined as follows:

$$\Gamma \vdash \varphi \iff \text{for every algebra } \boldsymbol{A} \text{ of the appropriate type, if } \boldsymbol{A} \vDash \Gamma, \text{ then } \boldsymbol{A} \vDash \varphi.$$

Notice that, in view of the Completeness Theorem, $\vdash$ is classical first order logic for algebraic languages. This exercise asks you to prove the Compactness Theorem for $\vdash$ using the ultraproduct construction (the same proof works for relational languages too). To this end, consider a set of sentences $\Gamma \cup \{\varphi\}$ such that $\Delta \nvdash \varphi$, for every finite $\Delta \subseteq \Gamma$. Then we can associate with every finite $\Delta \subseteq \Gamma$ an algebra $\boldsymbol{A}_\Delta$ such that

$$\boldsymbol{A}_\Delta \vDash \Delta \text{ and } \boldsymbol{A} \nvDash \varphi.$$

Prove that there exists an ultraproduct $\boldsymbol{B}$ of the family $\{\boldsymbol{A}_\Delta : \Delta \text{ is a finite subset of } \Gamma\}$ such that $\boldsymbol{B} \vDash \Gamma$ and $\boldsymbol{B} \nvDash \varphi$. Then conclude that $\Gamma \nvdash \varphi$, as desired. Hint: use Łoś' Theorem. ⊠

3.2 Universal classes

Definition 3.11. A sentence is said to be *universal* if it is of the form $\forall x_1, \ldots, x_n \varphi$ for some quantifier free formula φ. Accordingly, a class of similar algebras is said to be *universal* if it can be axiomatized by a set of universal sentences.

The following concept is instrumental for describing universal classes.

Definition 3.12. Let $\mathbb{X}$ be a local subgraph of an algebra $\boldsymbol{A}$ and assume that the universe and the type of $\mathbb{X}$ are, respectively, $\{a_1, \ldots, a_n\}$ and $f_1, \ldots, f_t$.

(i) The *positive atomic diagram* of $\mathbb{X}$ is the set of equations

$$\mathcal{D}^+(\mathbb{X}) := \{f_i(x_{m_1}, \ldots, x_{m_k}) \approx x_j : m_1, \ldots m_k, j \leqslant n \text{ and } i \leqslant t \text{ and } f_i^{\boldsymbol{A}}(a_{m_1}, \ldots, a_{m_k}) = a_j\}.$$

(ii) The *negative atomic diagram* of $\mathbb{X}$ is the set of negated equations

$$\mathcal{D}^-(\mathbb{X}) := \{x_m \not\approx x_k : m, k \leqslant n \text{ and } a_m \neq a_k\}.$$

Theorem 3.13 (Łoś). *A class of similar algebras is universal if and only if it is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$.*

Proof. The implication from left to right follows from the easy observation that the validity of universal sentences is preserved by $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, consider a class K of similar algebras closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$. Moreover, let $\mathrm{Th}_\forall(\mathsf{K})$ be the set of universal sentences valid in K and let $\mathsf{K}_\forall$ be the class of algebras satisfying $\mathrm{Th}_\forall(\mathsf{K})$. In order to conclude the proof, it suffices to show that $\mathsf{K} = \mathsf{K}_\forall$.

The inclusion $\mathsf{K} \subseteq \mathsf{K}_\forall$ follows immediately from the fact that $\mathsf{K} \vDash \mathrm{Th}_\forall(\mathsf{K})$. To prove the other inclusion, consider $\boldsymbol{A} \in \mathsf{K}_\forall$. We will show that every local subgraph of $\boldsymbol{A}$ can be embedded into an element of K. To this end, consider a local subgraph $\mathbb{X}$ of $\boldsymbol{A}$ with universe $\{a_1, \ldots, a_n\}$ and take the sentence

$$\Phi := \exists x_1, \ldots, x_n \Big(\bigwith \mathcal{D}^+(\mathbb{X}) \mathbin{\&} \bigwith \mathcal{D}^-(\mathbb{X}) \Big).$$

Now, suppose, with a view to contradiction, that $\mathsf{K} \vDash \neg\Phi$. Since $\neg\Phi$ is equivalent to a universal sentence and $\boldsymbol{A} \vDash \mathrm{Th}_\forall(\mathsf{K})$, we obtain $\boldsymbol{A} \vDash \neg\Phi$. But this is false, as witnessed by the assignment $x_1 \longmapsto a_1, \ldots, x_n \longmapsto a_n$ in $\boldsymbol{A}$. Thus, we conclude that $\mathsf{K} \nvDash \neg\Phi$. Consequently, there exists an element $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{B} \vDash \Phi$. Let then $b_1, \ldots, b_n$ be the elements that witness the validity of the existential part of Φ in $\boldsymbol{B}$. The map $f \colon X \to B$ defined by the rule $a_i \longmapsto b_i$ is an embedding of $\mathbb{X}$ into $\boldsymbol{B}$, as desired.

Since every local subgraph of $\boldsymbol{A}$ can be embedded into a member of K, we can apply Theorem 3.8 obtaining that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$. As, by assumption, K is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$, this yields $\boldsymbol{A} \in \mathsf{K}$. Hence, we conclude that $\mathsf{K}_\forall \subseteq \mathsf{K}$. ⊠

Given a class of similar algebras K, the least universal class extending K exists and will be denoted by $\mathbb{U}(\mathsf{K})$ and called the universal class *generated* by K.

Corollary 3.14. *If K is a class of similar algebras, $\mathbb{U}(\mathsf{K}) = \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.*

Proof. From Theorem 3.13 and the fact that $\mathbb{U}(\mathsf{K})$ is a universal class it follows that it is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$, whence $\mathbb{ISP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{U}(\mathsf{K})$. To prove the reverse inclusion, observe that $\mathbb{U}(\mathsf{K})$ is the class of all algebras satisfying all the universal sentences valid in K. In the proof of Theorem 3.13, we showed that this guarantees the inclusion $\mathbb{U}(\mathsf{K}) \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$. ⊠

Corollary 3.15. *Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. If $\boldsymbol{A} \in \mathbb{U}(\mathsf{K})$, then every local subgraph of $\boldsymbol{A}$ embeds into some member of K.*

Proof. This is established in the second part of the proof of Theorem 3.13. ⊠

The following provides an algebraic path to the strong finite model property in logic.

Definition 3.16. A class of similar algebras K is said to have the *finite embeddability property* (FEP, for short) if every local subgraph of a member of K can be embedded into a finite member of K.

Given a class of algebras K, we denote by $\mathsf{K}^{<\omega}$ the class of its finite members.

Proposition 3.17. *A universal class K has the FEP if and only if $\mathsf{K} = \mathbb{U}(\mathsf{K}^{<\omega})$.*

Proof. Suppose first that K has the FEP. Then

$$\mathsf{K} \subseteq \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}^{<\omega}) = \mathbb{U}(\mathsf{K}^{<\omega}).$$

The first inclusion in the above display follows from Theorem 3.8 and the second from Corollary 3.14. Furthermore, since K is a universal class, $\mathbb{U}(\mathsf{K}^{<\omega}) \subseteq \mathsf{K}$. Therefore, we conclude that $\mathsf{K} = \mathbb{U}(\mathsf{K}^{<\omega})$. To prove the converse, suppose that $\mathsf{K} = \mathbb{U}(\mathsf{K}^{<\omega})$. By Corollary 3.15, every local subgraph of a member of K can be embedded into some element of $\mathsf{K}^{<\omega}$, that is, K has the FEP. ⊠

The *universal theory* $\mathrm{Th}_{\forall}(\mathsf{K})$ of a class of algebras K is the set of universal sentences valid in K. $\mathrm{Th}_{\forall}(\mathsf{K})$ is said to be *decidable* when so is the problem of determining whether a universal sentence is valid in K. Furthermore, we say that a class of algebras is *finitely axiomatizable* if it can be axiomatized by finitely many sentences. The following result can be traced back at least to [97].

Proposition 3.18. *Let K be a finitely axiomatizable class of algebras. If K has the FEP, then $\mathrm{Th}_{\forall}(\mathsf{K})$ is decidable.*

Proof. In order to prove that $\mathrm{Th}_{\forall}(\mathsf{K})$ is decidable it suffices to show that

(i) the problem of determining whether a universal sentence belongs to $\mathrm{Th}_{\forall}(\mathsf{K})$ is semidecidable; and

(ii) the problem of determining whether a universal sentence does not belong to $\mathrm{Th}_{\forall}(\mathsf{K})$ is semidecidable.

Condition (i) holds, because K is finitely axiomatizable. Therefore, it only remains to prove (ii). To this end, let Σ be a finite set of axioms for K and $\{f_1, \dots, f_n\}$ the function symbols that appear in Σ. Given a universal sentence Φ, we enumerate the finite models $\boldsymbol{A}_1, \boldsymbol{A}_2 \dots$ of Σ in the type $\{f_1, \dots, f_n, g_1, \dots, g_m\}$, where $g_1, \dots, g_m$ are the function symbols that occur in Φ. This can be done mechanically, because Σ is finite. Our algorithm tests if Φ fails in some $\boldsymbol{A}_n$. If this is the case, it stops and answers that Φ does not belong to $\mathsf{Th}_\forall(\mathsf{K})$, otherwise it runs forever.

In order to establish (ii), it suffices to prove that the algorithm stops if and only if $\Phi \notin \mathsf{Th}_\forall(\mathsf{K})$. First, it if stops, then Φ fails in some $\boldsymbol{A}_n$. Let then $\boldsymbol{B}$ be an algebra of the type of K obtained by expanding $\boldsymbol{A}_n$ with an arbitrary interpretation of the missing function symbols. From $\boldsymbol{A}_n \nvDash \Phi$ it follows $\boldsymbol{B} \nvDash \Phi$. Moreover, since $\boldsymbol{A}_n \vDash \Sigma$, we obtain $\boldsymbol{B} \vDash \Sigma$ and, therefore, $\boldsymbol{B} \in \mathsf{K}$ (as Σ axiomatizes K). Hence, we conclude that $\Phi \notin \mathsf{Th}_\forall(\mathsf{K})$.

To prove the converse, consider a universal sentence $\forall \vec{x} \varphi$ that fails in K. Then there exist $\boldsymbol{B} \in \mathsf{K}$ and $b_1, \dots, b_n \in B$ such that $\boldsymbol{B} \nvDash \varphi(b_1, \dots, b_n)$. Let $\mathbb{X}$ be the local subgraph of $\boldsymbol{B}$ whose universe is $\{b_1, \dots, b_n\}$ and whose type consists of the function symbols occurring in φ. Since K has the FEP, there is an embedding $f \colon \mathbb{X} \to \boldsymbol{C}$, for some $\boldsymbol{C} \in \mathsf{K}^{<\omega}$. It follows that $\boldsymbol{C} \nvDash \varphi(f(a_1), \dots, f(a_n))$, whence $\forall \vec{x} \varphi$. Let $\boldsymbol{C}^-$ be the reduct of $\boldsymbol{C}$ is the language $\mathscr{L}$ consisting of the function symbols occurring in $\Sigma \cup \{\varphi\}$. Clearly, $\boldsymbol{C}^- \nvDash \varphi(f(a_1), \dots, f(a_n))$ and, therefore, $\forall \vec{x} \varphi$ fails in $\boldsymbol{C}^-$. As $\boldsymbol{C}^-$ is a finite model of Σ in the language $\mathscr{L}$, there must be some $n \in \mathbb{N}$ such that $\boldsymbol{C}^- \cong \boldsymbol{A}_n$. It follows that $\forall \vec{x} \varphi$ fails in $\boldsymbol{A}_n$, whence the algorithm stops, as desired. ⊠

Example 3.19 (Lattices)**.** We will prove that the class Latt of all lattices has the FEP. For consider a lattice $\boldsymbol{A}$ and let $\mathbb{X}$ be one of its local subgraphs. Let also

$$B := \{a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n : a_1, \dots, a_n \in X \text{ and } n \in \mathbb{N}\}.$$

Since the operation $\wedge^{\boldsymbol{A}}$ is idempotent, commutative and associative, the set B is finite. furthermore, B can be viewed as a subposet of $\boldsymbol{A}$. Let $\boldsymbol{B}^+$ be the poset obtained extending $\langle B; \leqslant \rangle$ with a new top element. Since $\boldsymbol{B}^+$ is a finite meet-semilattice with maximum, it is also a lattice. Furthermore, $\mathbb{X}$ embeds into $\boldsymbol{B}^+$. Hence, Latt has the FEP. By Propositions 3.18 and 3.17, $\mathsf{Th}_\forall(\mathsf{Latt})$ is decidable [124] and $\mathsf{Latt} = \mathbb{U}(\mathsf{Latt}^{<\omega})$. On the other hand, the first order theory of distributive lattices (and, therefore, of any nontrivial equational class of lattices) is undecidable [70]. ⊠

Example 3.20 (Heyting algebras)**.** We will prove that the class HA of Heyting algebras has the FEP. For consider a Heyting algebra $\boldsymbol{A}$ and let $\mathbb{X}$ be one of its local subgraphs. Then let $\boldsymbol{B}$ be the bounded sublattice of $\boldsymbol{A}$ generated by $\mathbb{X}$. Notice that $\boldsymbol{B}$ is finite, because it is a finitely generated bounded distributive lattice. Since $\boldsymbol{B}$ is a finite distributive lattice, it can be viewed as a finite Heyting algebra $\boldsymbol{B}^+$. Furthermore, it is easy to see that the identity map is an embedding of $\mathbb{X}$ into $\boldsymbol{B}^+$. Hence, we conclude that HA has the FEP. By Propositions 3.18 and 3.17, $\mathsf{Th}_\forall(\mathsf{HA})$ is decidable and $\mathsf{HA} = \mathbb{U}(\mathsf{HA}^{<\omega})$ [98]. On the other hand, the first order theory of nontrivial equational class of Heyting algebras other than that of Boolean algebras is known to be undecidable [29].

Notably, the fact that $\mathsf{Th}_\forall(\mathsf{HA})$ is decidable implies that **IPC** is also decidable, in the sense that, given a finite set of formulas $\{\gamma_1, \dots, \gamma_n, \varphi\}$, we can decide whether

$\gamma_1, \dots, \gamma_n \vdash_{\mathbf{IPC}} \varphi$. This is because, in view of the fact that HA is a $\{x \approx 1\}$-algebraic semantics for **IPC**, we have

$$\begin{aligned}\gamma_1, \dots, \gamma_n \vdash_{\mathbf{IPC}} \varphi &\iff \gamma_1 \approx 1, \dots, \gamma_n \approx 1 \vDash_{\mathsf{HA}} \varphi \approx 1\\ &\iff \mathsf{HA} \vDash \forall \vec{x}\,((\gamma_1 \approx 1 \,\&\, \dots \,\&\, \gamma_n \approx 1) \Longrightarrow \varphi \approx 1).\end{aligned}$$

As $\forall \vec{x}\,((\gamma_1 \approx 1 \,\&\, \dots \,\&\, \gamma_n \approx 1) \Longrightarrow \varphi \approx 1)$ is a universal sentence, we can decide whether it holds in HA or not, because the universal theory of HA is decidable. ⊠

Exercise 3.21. The following proof the class MA of modal algebras has the FEP originates in [97]. Let $\mathbb{X}$ be a local subgraph of a model algebra $\boldsymbol{A}$. Moreover, let $\boldsymbol{B}$ be the Boolean subalgebra of $\boldsymbol{A}$ generated by $X \cup \{\Box 0\}$. We consider the algebra algebra $\boldsymbol{B}^+$ obtained by endowing $\boldsymbol{B}$ with a unary operation $\Box$ defined as follows:

$$\Box^{\boldsymbol{B}^+} b := \bigvee^{\boldsymbol{B}} \{\Box^{\boldsymbol{A}} a \in B : \Box^{\boldsymbol{A}} a \in B \text{ and } a \leqslant b\}.$$

Prove that $\boldsymbol{B}^+$ is a modal algebra and that $\mathbb{X}$ embeds into $\boldsymbol{B}^+$. Then conclude that $\mathrm{Th}_\forall(\mathsf{MA})$ is decidable and $\mathsf{MA} = \mathbb{U}(\mathsf{MA}^{<\omega})$. Use these facts to infer that the logic $\mathbf{K}_g$ is decidable. ⊠

3.3 Quasi-varieties

For a detailed presentation of the theory of quasi-varieties, we refer the reader to [68, 93].

Definition 3.22. A class of similar algebras closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$ is said to be a *quasi-variety*.

Examples of quasi-varieties include the classes of Boolean, Heyting and modal algebras, as well as the class of (bounded) distributive lattices and groups. Our aim will be to prove that quasi-varieties are precisely the classes of algebras axiomatized by the following kind of first order formulas.

Definition 3.23. A *quasi-equation* of type ρ is an expression Φ of the form

$$(\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n) \Longrightarrow \varepsilon \approx \delta,$$

where $\{\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n, \varepsilon \approx \delta\}$ is a set of equations of type ρ. Then Φ is *valid* in an algebra $\boldsymbol{A}$ of type ρ when so is its universal closure $\forall \vec{x}\,\Phi$, that is, for every $\vec{a} \in A$,

$$\text{if } \varphi_1^{\boldsymbol{A}}(\vec{a}) = \psi_1^{\boldsymbol{A}}(\vec{a}), \dots, \varphi_n^{\boldsymbol{A}}(\vec{a}) = \psi_n^{\boldsymbol{A}}(\vec{a}), \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).$$

In this case, we often say that $\boldsymbol{A}$ *satisfies* Φ and write $\boldsymbol{A} \vDash \Phi$. A quasi-equation is said to be an *equation* when its antecedent is empty.

Notably, for every class K of algebras and equations $\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n, \varepsilon \approx \delta$,

$$\mathsf{K} \vDash (\mathop{\&}_{i \leqslant n} \varphi_i \approx \psi_i) \Longrightarrow \varepsilon \approx \delta \text{ iff } \{\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n\} \vDash_{\mathsf{K}} \varepsilon \approx \delta.$$

Remark 3.24. The reader might have noticed that expressions of the form $\varepsilon \approx \delta$ and $\emptyset \Longrightarrow \varepsilon \approx \delta$ are both called *equations*. This is not a problem, because they are synonyms, in the sense that an algebra satisfies $\varepsilon \approx \delta$ if and only if it satisfies $\emptyset \Longrightarrow \varepsilon \approx \delta$. Because of this, we will continue to denote equations by $\varepsilon \approx \delta$, while keeping in mind that they can be viewed as quasi-equations whose antecedent is empty. ⊠

The aim of this section is to prove the following classical result.

Maltsev's Theorem 3.25. *A class of similar algebras is a quasi-variety if and only if it can be axiomatized by a set of quasi-equations.*

Proof. The "only if" part follows from the fact that the validity of quasi-equations is preserved by the class operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$. To prove the converse, consider a quasi-variety K and let Σ the set of quasi-equations valid in K. Let also K^+ be the class of algebras axiomatized by Σ. Our aim is to prove that $\mathsf{K} = \mathsf{K}^+$.

The inclusion $\mathsf{K} \subseteq \mathsf{K}^+$ is straightforward. To prove the other one, consider an algebra $\boldsymbol{A} \in \mathsf{K}^+$. In view of Theorem 3.8, in order to show that $\boldsymbol{A} \in \mathsf{K}$, it suffices to prove that every local subgraph of $\boldsymbol{A}$ embeds in some members of K. This is because, in this case, $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathsf{K}$. Since K is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$, this implies $\boldsymbol{A} \in \mathsf{K}$, as desired.

Then consider a local subgraph $\mathbb{X}$ of $\boldsymbol{A}$ with universe $\{a_1, \dots, a_n\}$. Observe that both $\mathcal{D}^+(\mathbb{X})$ and $\mathcal{D}^-(\mathbb{X})$ are finite sets. Then take an enumeration

$$\mathcal{D}^-(\mathbb{X}) = \{\varepsilon_1 \not\approx \delta_1, \dots, \varepsilon_t \not\approx \delta_t\}.$$

Moreover, for each $i \leqslant t$, consider the quasi-equation

$$\Phi_i \coloneqq \Big(\mathop{\&} \mathcal{D}^+(\mathbb{X}) \Big) \Longrightarrow \varepsilon_i \approx \delta_i.$$

As witnessed by the natural assignment

$$x_1 \longmapsto a_1, \dots, x_n \longmapsto a_n,$$

the quasi-equations $\Phi_1, \dots, \Phi_t$ fail in $\boldsymbol{A}$. Since $\boldsymbol{A}$ satisfies all the quasi-equations valid in K, this implies that each Φ_i fails in some $\boldsymbol{B}_i \in \mathsf{K}$ under an assignment

$$x_1 \longmapsto b^i_1, \dots, x_n \longmapsto b^i_n. \tag{3.5}$$

Now, consider the map $h\colon X \to (B_1 \times \dots \times B_t)$, defined by the rule

$$a_1 \longmapsto \langle b^1_1, \dots, b^t_1 \rangle, \dots, a_n \longmapsto \langle b^1_n, \dots, b^t_n \rangle.$$

We will prove that h is an embedding of $\mathbb{X}$ into $\boldsymbol{B}_1 \times \dots \times \boldsymbol{B}_t$. To prove that h is injective, consider two distinct elements $a_p, a_q \in X$. Then the formula $x_p \not\approx x_q$ belongs to $\mathcal{D}^-(\mathbb{X})$. Consequently, there exists $i \leqslant t$ such that

$$\Phi_i = \Big(\mathop{\&} \mathcal{D}^+(\mathbb{X}) \Big) \Longrightarrow x_p \approx x_q.$$

Since Φ_i fails in $\boldsymbol{B}_i$ under the assignment in (3.5), we obtain $b^i_p \neq b^i_q$. As a consequence,

$$h(a_p)(i) = b^i_p \neq b^i_q = h(a_q)(i)$$

and, therefore, $h(a_p) \neq h(a_q)$. Hence, h is injective. To prove that it preserves the partial operations, consider a basic m-ary operation f in the type of $\mathbb{X}$ and $a_{k_1}, \ldots, a_{k_m} \in X$ such that $f^{\boldsymbol{A}}(a_{k_1}, \ldots, a_{k_m}) \in X$. Then there exists some $p \leqslant n$ such that $a_p = f^{\boldsymbol{A}}(a_{k_1}, \ldots, a_{k_m})$. Moreover, the equation

$$f(x_{k_1}, \ldots, x_{k_m}) \approx x_p$$

belongs to $\mathcal{D}^+(\mathbb{X})$. As each quasi-equation Φ_i fails under the assignment in (3.5), the same assignment satisfies the antecedent of Φ_i, namely $\mathcal{D}^+(\mathbb{X})$. It follows that

$$f^{\boldsymbol{B}_i}(b^i_{k_1}, \ldots, b^i_{k_m}) = b^i_p, \text{ for each } i \leqslant t.$$

As a consequence, for every $i \leqslant t$,

$$\begin{aligned} h(f^{\boldsymbol{A}}(a_{k_1}, \ldots, a_{k_m}))(i) &= h(a_p)(i) \\ &= b^i_p \\ &= f^{\boldsymbol{B}_i}(b^i_{k_1}, \ldots, b^i_{k_m}) \\ &= f^{\boldsymbol{B}_i}(h(a_{k_1})(i), \ldots, h(a_{k_m})(i)) \\ &= f^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \ldots, h(a_{k_m}))(i). \end{aligned}$$

Thus, $h(f^{\boldsymbol{A}}(a_{k_1}, \ldots, a_{k_m})) = f^{\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t}(h(a_{k_1}), \ldots, h(a_{k_m}))$. We conclude that $h \colon \mathbb{X} \to (\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t)$ is an embedding. Since $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_t \in \mathsf{K}$ and K is closed under $\mathbb{P}$, the direct product $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_t$ belongs to K. Hence, $\mathbb{X}$ embeds into some member of K, as desired. ⊠

Exercise 3.26. In view of Łoś' Theorem quasi-equations persist in ultraproducts. If you are not familiar with the proof of Łoś' Theorem, offer a direct proof of this fact. ⊠

Given a class of similar algebras K, the least quasi-variety extending K exists. It will be denoted by $\mathbb{Q}(\mathsf{K})$ and called the quasi-variety *generated* by K.

Corollary 3.27. *Let K be a class of similar algebras. Then $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. If in addition K is a finite set of finite algebras, $\mathbb{Q}(\mathsf{K}) = \mathbb{ISP}(\mathsf{K})$.*

Proof. The inclusion $\mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$ is straightforward. To prove the other one, consider $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})$. By Maltsev's Theorem, $\mathbb{Q}(\mathsf{K})$ is the class of all algebras satisfying the quasi-equations valid in K. The proof of the hard part of Maltsev's Theorem show that $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}\mathbb{P}(\mathsf{K})$. Therefore, it only remains to show that $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$, which is left as an exercise. This shows that $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$.

To prove the second part of the statement, suppose that K is a finite set of finite algebras. This guarantees that $\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$ (see Exercise 3.4). As a consequence, $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) = \mathbb{ISP}(\mathsf{K})$, as desired. ⊠

Exercise 3.28. Prove that if K is a class of similar algebras, then $\mathbb{P}_{\mathrm{U}}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K})$. ⊠

Example 3.29 (Quasi-varieties)**.** In view of Examples 3.19 and 3.20, we know that

$$\mathsf{Latt} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{Latt}^{<\omega}) \ \text{ and } \ \mathsf{HA} = \mathbb{ISP}_{\mathrm{U}}(\mathsf{HA}^{<\omega}).$$

This implies that $\mathsf{Latt} \subseteq \mathbb{Q}(\mathsf{Latt}^{<\omega})$ and $\mathsf{HA} \subseteq \mathbb{Q}(\mathsf{HA}^{<\omega})$. Since both Latt and HA are closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$, this yields

$$\mathsf{Latt} = \mathbb{Q}(\mathsf{Latt}^{<\omega}) \ \text{ and } \ \mathsf{HA} = \mathbb{Q}(\mathsf{HA}^{<\omega}).$$

Let DL be the class of distributive lattices and $\boldsymbol{B}$ the two-element distributive lattice. In view of Example 1.23, we have $\mathsf{DL} = \mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B})$. Clearly, $\mathbb{IP}_{\mathrm{SD}}(\boldsymbol{B}) \subseteq \mathbb{ISP}(\boldsymbol{B}) \subseteq \mathbb{Q}(\boldsymbol{B})$. On the other hand, since DL is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$, we obtain $\mathbb{Q}(\boldsymbol{B}) \subseteq \mathsf{DL}$. Hence, $\mathsf{DL} = \mathbb{Q}(\boldsymbol{B})$. On the other hand, $\mathbb{U}(\boldsymbol{B}) = \mathbb{ISP}_{\mathrm{U}}(\boldsymbol{B}) = \mathbb{I}(\boldsymbol{B})$. Similarly, the class of Boolean algebras is the quasi-variety generated by the two-element Boolean algebra (see 1.24, if necessary). ⊠

Exercise 3.30. Prove that there is no finite Heyting algebra $\boldsymbol{A}$ such that $\mathsf{HA} = \mathbb{Q}(\boldsymbol{A})$, cf. with $\mathsf{HA} = \mathbb{Q}(\mathsf{HA}^{<\omega})$. Prove that, however, there exists an infinite Heyting algebra $\boldsymbol{A}$ such that $\mathsf{HA} = \mathbb{Q}(\boldsymbol{A})$ [83, 131], see also [50, 106]. ⊠

The next observation will be needed later on.

Theorem 3.31. *If* K *is a quasi-variety, the consequence relation* $\vDash_{\mathsf{K}}$ *is finitary.*

Proof. In view of Maltsev's Theorem, K can be axiomatized by a set Σ of quasi-equations. Formally speaking, this means that K is the class of models of the set of sentences

$$\Sigma_{\forall} := \{\forall \vec{x} \varphi : \varphi(\vec{x}) \in \Sigma\}.$$

Let then $\Theta \cup \{\varphi \approx \psi\} \subseteq E(Var)$ be such that $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$. Moreover, let $\{a_n : n \in \mathbb{N}\}$ be a set of new constants. Given a formula $\varphi(x_1, \dots, x_n)$, we write

$$\varphi(\vec{a}) \text{ as a shorthand for } \varphi(a_1, \dots, a_n).$$

Since $\Sigma_{\forall}$ axiomatizes K, the fact that $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$ is equivalent to the demand that

$$\Sigma_{\forall} \cup \{\varepsilon(\vec{a}) \approx \delta(\vec{a}) : \varepsilon \approx \delta \in \Theta\} \vdash \varphi(\vec{a}) \approx \psi(\vec{a}),$$

where $\vdash$ is the derivability symbol of classical first order logic. Consequently, by the Compactness Theorem, there exists a finite $\Delta \subseteq \Theta$ such that

$$\Sigma_{\forall} \cup \{\varepsilon(\vec{a}) \approx \delta(\vec{a}) : \varepsilon \approx \delta \in \Delta\} \vdash \varphi(\vec{a}) \approx \psi(\vec{a}).$$

Since $\Sigma_{\forall}$ axiomatizes K, this means that $\Delta \vDash_{\mathsf{K}} \varphi \approx \psi$, as desired. ⊠

Observe that, for every class K of similar algebras, $\mathbb{I}(\mathsf{K}) \cup \mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$. Because of this, the following classes of algebras [12, 30] are also quasi-varieties.

Definition 3.32. A class of similar algebras closed under $\mathbb{H}, \mathbb{S}$ and $\mathbb{P}$ is said to be a *variety*.

Notably, varieties coincide with equational classes of algebras [12, Thm. 4.41].

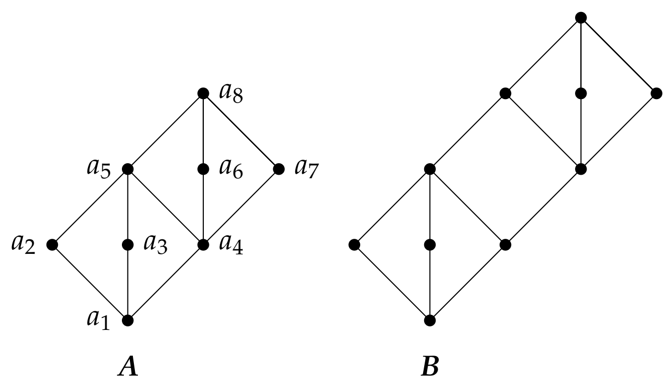
Birkhoff's Theorem 3.33. *A class of similar algebras is a variety if and only if it can be axiomatized by a set of equations.*

Given a class of similar algebras K, the least variety extending K exists. It will be denoted by $\mathbb{V}(\mathsf{K})$ and called the variety *generated* by K.

Corollary 3.34. *Let* K *be a class of similar algebras. Then* $\mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K})$.

As a consequence, examples of varieties include the classes of (distributive) lattices, Heyting, Boolean and modal algebras. While every variety is a quasi-variety, the converse is not true in general, as we proceed to explain.

Example 3.35 (Lattices)**.** Consider the lattices $\boldsymbol{A}$ and $\boldsymbol{B}$ depicted below. We will show that the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$ and, therefore, is not a variety.



To this end, let $\mathcal{D}^+(\boldsymbol{A})$ be the positive atomic diagram of $\boldsymbol{A}$ written with the variables $x_1, \dots, x_8$ corresponding to the elements $a_1, \dots, a_8$ and consider the quasi-equation

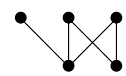
$$\Phi = \mathop{\&} \mathcal{D}^+(\boldsymbol{A}) \Longrightarrow x_1 \approx x_8.$$

Notice that $\boldsymbol{B}$ validates Φ. To prove this, consider an assignment $f\colon \{x_1, \dots, x_8\} \to B$ that validates $\mathcal{D}^+(\boldsymbol{A})$ in $\boldsymbol{B}$. Using the definition of $\mathcal{D}^+(\boldsymbol{A})$, it is easy to see that the map $h\colon A \to B$ that sends a_i to $f(a_i)$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Since $\mathsf{Con}(\boldsymbol{A}) = \{\mathrm{id}_A, A \times A\}$, the kernel $\mathsf{Ker}(h)$ is either id_A or $A \times A$. Notice that there is no embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$. Therefore, $\mathsf{Ker}(h)$ cannot be the identity relation. It follows that $\mathsf{Ker}(h) = A \times A$. In particular, $\langle a_1, a_8 \rangle \in \mathsf{Ker}(h)$ and, therefore, $f(x_1) = h(a_1) = f(a_8) = f(x_8)$. Hence, we conclude that $\boldsymbol{B} \vDash \Phi$, as desired. Moreover, Φ fails in $\boldsymbol{A}$, as witnessed by the assignment

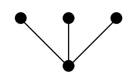
$$x_1 \longmapsto a_1, \dots, x_8 \longmapsto a_8.$$

In brief, Φ holds in $\boldsymbol{B}$ but fails in $\boldsymbol{A}$. By Maltsev's Theorem, we conclude that $\boldsymbol{A}$ does not belong to the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ generated by $\boldsymbol{B}$. On the other hand, $\boldsymbol{A}$ is a homomorphic image of $\boldsymbol{B}$ (obtained by glueing two pairs of elements of $\boldsymbol{B}$). Thus, the quasi-variety $\mathbb{Q}(\boldsymbol{B})$ is not closed under $\mathbb{H}$. ⊠

Example 3.36 (Heyting algebras)**.** We assume the reader is familiar with *Esakia duality* for Heyting algebras [53, 54]. Let $\boldsymbol{A}$ be the finite Heyting algebra whose dual Esakia space is the following poset $\boldsymbol{A}_*$ endowed with the discrete topology.



Or aim is to show that $\mathbb{Q}(\boldsymbol{A})$ is not closed under $\mathbb{H}$ and, therefore, fails to be a variety. To this end, consider the finite Heyting algebra $\boldsymbol{B}$ whose dual Esakia space is the the rooted poset $\boldsymbol{B}_*$ (endowed with the discrete topology) depicted below.

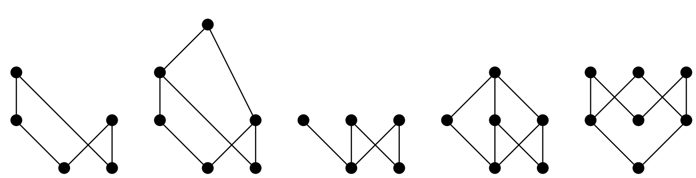


Notice that, as $\boldsymbol{B}_*$ is rooted and finite, the algebra $\boldsymbol{B}$ is subdirectly irreducible. Moreover, observe that, as $\boldsymbol{B}_*$ is an upset of $\boldsymbol{A}_*$, by Esakia duality we obtain $\boldsymbol{B} \in \mathbb{H}(\boldsymbol{A})$. Therefore, it suffices to show that $\boldsymbol{B} \notin \mathbb{Q}(\boldsymbol{A})$. Suppose the contrary, with a view to contradiction. By Corollary 3.27,

$$\boldsymbol{B} \in \mathbb{ISP}(\boldsymbol{A}) \subseteq \mathbb{IP}_{\text{SD}}\mathbb{S}(\boldsymbol{A}).$$

As $\boldsymbol{B}$ is subdirectly irreducible, we conclude that $\boldsymbol{B} \in \mathbb{IS}(\boldsymbol{A})$. By Esakia duality, this means that $\boldsymbol{B}_*$ is a p-morphic image of $\boldsymbol{A}_*$. But a quick inspection of the posets $\boldsymbol{A}_*$ and $\boldsymbol{B}_*$ shows that this is impossible, a contradiction. Hence, we conclude that $\mathbb{Q}(\boldsymbol{A})$ does not contain $\boldsymbol{B}$ and, therefore, is not closed under $\mathbb{H}$. ⊠

Exercise 3.37. Let $\boldsymbol{A}_1, \dots, \boldsymbol{A}_5$ be the Heyting algebras whose Esakia duals are the five posets depicted below.



A variety K is said to be *primitive* whenever every quasi-variety $\mathsf{M} \subseteq \mathsf{K}$ is a variety. Prove that if a variety K of Heyting algebras is primitive, then it omits $\boldsymbol{A}_1, \dots, \boldsymbol{A}_5$.

Notably, the converse is also true: a variety K of Heyting algebras is primitive if and only if it omits $\boldsymbol{A}_1, \dots, \boldsymbol{A}_5$ [36, 37], see also [13]. A similar description of primitive varieties of K4-algebras has been obtain in [122]. These results admit a logical interpretation in terms of structural completeness [11, 118, 123]. ⊠

If K is a quasi-variety and θ a congruence of some $\boldsymbol{A} \in \mathsf{K}$, the algebra $\boldsymbol{A}/\theta$ need not belong to K. This makes the following concept attractive.

Definition 3.38. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. A congruence $\theta \in \mathsf{Con}(\boldsymbol{A})$ is said to be a K-*congruence* of $\boldsymbol{A}$ if $\boldsymbol{A}/\theta \in \mathsf{K}$. We denote the poset of K-congruences of $\boldsymbol{A}$, ordered under the inclusion relation, by $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.

Proposition 3.39. *If* K *is a quasi-variety, then* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *is a complete lattice in which meets are intersections, for every algebra* $\boldsymbol{A}$ *of the type of* K.

Proof. Then let $\boldsymbol{A}$ be an algebra of the type of K. Since K is a quasi-variety, it is closed under $\mathbb{P}_{\text{SD}}$ and it contains a trivial algebra (the subdirect product of the empty family). Therefore, as K is closed under $\mathbb{I}$ by assumption, it contains all trivial algebras and, in particular, $\boldsymbol{A}/(A \times A)$. Thus, $A \times A \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Then consider a nonempty family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. By Proposition 1.25,

$$\boldsymbol{A}/\bigcap_{i \in I} \theta_i \in \mathbb{I}\mathbb{P}_{\text{SD}}(\{\boldsymbol{A}/\theta_i : i \in I\}).$$

Observe that $\{\boldsymbol{A}/\theta_i : i \in I\} \subseteq \mathsf{K}$, since the various θ_i are K-congruences of $\boldsymbol{A}$. Together with the above display and the assumption that K is closed under $\mathbb{I}$ and $\mathbb{P}_{\text{SD}}$, this yields $\boldsymbol{A}/\bigcap_{i \in I} \theta_i \in \mathsf{K}$, whence $\bigcap_{i \in I} \theta_i \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. It follows that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ has a minimum, namely $A \times A$ and infima of nonempty families. Therefore, arbitrary infima exist in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and, therefore, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a complete lattice. ⊠

Corollary 3.40. *If* K *is a quasi-variety and* $\boldsymbol{A}$ *an algebra of the same type, the map*

$$\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}} \colon \mathcal{P}(A \times A) \to \mathcal{P}(A \times A)$$

that associates the least K*-congruence of* $\boldsymbol{A}$ *containing a subset* X *with a subset* $X \subseteq A \times A$ *is a closure operator on* $A \times A$.

Remark 3.41. The proof of Proposition 3.39 depends only on the fact that K is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. Classes of similar algebras closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$ are called *prevarieties*. Notably, they coincide with the classes of algebras that can be axiomatized by *proper classes* of infinitary quasi-equations. The demand that proper classes can be replaced by sets in the axiomatization of prevarieties is equivalent to an independent set theoretical principle known as *Vopěnka Principle* [68, Prop. 2.3.18], see also [1]. ⊠

CHAPTER 4

Algebraizable logics

4.1 Algebraizability

We are now ready to introduce a robust theory of algebraization that will account for the relation between logic and algebra [21], see also [23, 42, 55, 56, 57, 73, 74]. To this end, for every set of formulas $\Delta(x, y)$ and set of equations $\Theta \cup \{\varepsilon \approx \delta\}$, we shall abbreviate

$$\{\varphi(\varepsilon, \delta) \colon \varphi(x, y) \in \Delta\} \text{ as } \Delta(\varepsilon, \delta), \text{ and } \bigcup_{\varphi \approx \psi \in \Theta} \Delta(\varphi, \psi) \text{ as } \Delta[\Theta].$$

Definition 4.1 (Blok & Pigozzi)**.** A finitary logic $\vdash$ is said to be *algebraizable* if there are a finite set of equations $\boldsymbol{\tau}(x)$, a finite set of formulas $\Delta(x, y)$, and a quasi-variety K such that

$$\Gamma \vdash \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi) \tag{Alg1}$$

$$\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff \Delta[\Theta] \vdash \Delta(\varepsilon, \delta) \tag{Alg2}$$

$$\varphi \dashv\vdash \Delta[\boldsymbol{\tau}(\varphi)] \tag{Alg3}$$

$$\varepsilon \approx \delta \mathrel{=\!\!|\!\!\vDash_{\mathsf{K}}} \boldsymbol{\tau}[\Delta(\varepsilon, \delta)] \tag{Alg4}$$

for every set of formulas $\Gamma \cup \{\varphi\}$ and every set of equations $\Theta \cup \{\varepsilon \approx \delta\}$. In this case, K is said to be an *equivalent algebraic semantics* for $\vdash$. In addition, we say that $\boldsymbol{\tau}, \Delta$ and K *witness* the algebraizability of the logic $\vdash$.

Condition (Alg1) expresses the demand that K is a $\boldsymbol{\tau}$-algebraic semantics for $\vdash$, namely, that $\vdash$ can be interpreted into $\vDash_{\mathsf{K}}$ by means of the set of equations $\boldsymbol{\tau}(x)$ that allows to translate a set of formulas Γ into a set of equations $\boldsymbol{\tau}[\Gamma]$. Condition (Alg2) states that this interpretation can be *reversed*, in the sense that $\vDash_{\mathsf{K}}$ can also be interpreted into $\vdash$ by means of the set of formulas $\Delta(x, y)$ that allows to translate sets of equations Θ into sets of formulas $\Delta[\Theta]$. Lastly, conditions (Alg3) and (Alg4) guarantee that these two interpretations are *inverses of each other* up to provability equivalence.

Remark 4.2. While the problem of determining whether logics presented by Hilbert calculi are algebraizable is undecidable [100], the same problem becomes decidable, although complete for EXPTIME [103], for logics presented by finite sets of finite logical matrices in the sense of [129]. ⊠

The definition of an algebraizable logic can be made more concise as follows.

Proposition 4.3. *The following conditions are equivalent for a finitary logic* $\vdash$*:*

(i) $\vdash$ *is algebraizable;*

(ii) *There are a set of equations* $\boldsymbol{\tau}(x)$*, a set of formulas* $\Delta(x, y)$ *and a quasi-variety* K *that satisfy* (Alg1) *and*
$$x \approx y \mathrel{=\mid\models_{\mathsf{K}}} \boldsymbol{\tau}[\Delta(x, y)]; \tag{Alg4*}$$

(iii) *There are a set of equations* $\boldsymbol{\tau}(x)$*, a set of formulas* $\Delta(x, y)$ *and a quasi-variety* K *that satisfy* (Alg2) *and*
$$x \dashv\vdash \Delta[\boldsymbol{\tau}(x)]. \tag{Alg3*}$$

In this case, $\boldsymbol{\tau}$*,* Δ*, and* K *witness the algebraizability of* $\vdash$*.*

Proof. The implication (i)⇒(ii) is straightforward. To prove (ii)⇒(iii), observe that (Alg4*) implies that, for every $\varphi \approx \psi \in \boldsymbol{\tau}$,
$$\varphi \approx \psi \mathrel{=\mid\models_{\mathsf{K}}} \boldsymbol{\tau}[\Delta(\varphi, \psi)].$$
Since $\boldsymbol{\tau}[\Delta[\boldsymbol{\tau}(x)]] = \bigcup\{\boldsymbol{\tau}[\Delta(\varphi, \psi)] : \varphi \approx \psi \in \boldsymbol{\tau}\}$, this yields
$$\boldsymbol{\tau}(x) \mathrel{=\mid\models_{\mathsf{K}}} \boldsymbol{\tau}[\Delta[\boldsymbol{\tau}(x)]].$$
By (Alg1), we conclude that $x \dashv\vdash \Delta[\boldsymbol{\tau}(x)]$. Then consider a set of equations $\Theta \cup \{\varepsilon \approx \delta\}$. We have
$$\Theta \models_{\mathsf{K}} \varepsilon \approx \delta \iff \boldsymbol{\tau}[\Delta[\Theta]] \models_{\mathsf{K}} \boldsymbol{\tau}[\Delta(\varepsilon, \delta)] \iff \Delta[\Theta] \vdash \Delta(\varepsilon, \delta),$$
where the first equivalence follows from (Alg4*) and the second from (Alg1).

(iii)⇒(i): Since $\vdash$ is finitary and $\Delta[\boldsymbol{\tau}(x)] \vdash x$, there exists a finite $\boldsymbol{\tau}' \subseteq \boldsymbol{\tau}$ such that $\Delta[\boldsymbol{\tau}(x)] \vdash x$. Together with the assumption that $x \vdash \Delta[\boldsymbol{\tau}(x)]$, this yields $x \dashv\vdash \Delta[\boldsymbol{\tau}'(x)]$. Therefore, the pair $\boldsymbol{\tau}'$ and Δ satisfy conditions (Alg2) and (Alg3*). Furthermore, a proof similar to the one of the implication (ii)⇒(iii) establishes (Alg1) and (Alg4*) for $\boldsymbol{\tau}'$ and Δ.

Now, by (Alg4*), we have $x \approx y \mathrel{=\mid\models_{\mathsf{K}}} \boldsymbol{\tau}'[\Delta(x, y)]$. Since, by Theorem 3.31, the relation $\models_{\mathsf{K}}$ is finitary, there exists a finite $\Delta' \subseteq \Delta$ such that $x \approx y \mathrel{=\mid\models_{\mathsf{K}}} \boldsymbol{\tau}'[\Delta'(x, y)]$. Therefore, the pair $\boldsymbol{\tau}'$ and Δ' satisfy conditions (Alg1) and (Alg4*). Furthermore, a proof similar to the one of the implication (ii)⇒(iii) establishes (Alg2) and (Alg3*) for $\boldsymbol{\tau}'$ and Δ'. As a consequence, the pair of finite sets $\boldsymbol{\tau}'$ and Δ' satisfy (Alg1), (Alg2), (Alg3*) and (Alg4*). Since conditions (Alg3*) and (Alg4*) imply (Alg3) and (Alg4), respectively, we conclude that $\vdash$ is algebraizable and that its algebraizability is witnessed by $\boldsymbol{\tau}$, Δ, and K. ⊠

Example 4.4 (Algebraizable logics)**.** Consider the following sets of equations and formulas, respectively,
$$\boldsymbol{\tau}(x) := \{x \approx 1\} \text{ and } \Delta(x, y) := \{x \to y, y \to x\}.$$

We shall prove that **IPC** is algebraizable and that its algebraizability is witnessed by $\boldsymbol{\tau}, \Delta$ and the variety HA of Heyting algebras. First, recall that HA is a $\boldsymbol{\tau}$-algebraic semantics for **IPC**. Moreover, observe that

$$x \approx y \mathrel{=\!\!|\!\!\models}_{\mathsf{HA}} \{x \to y \approx 1, y \to x \approx 1\}. \tag{4.1}$$

To prove this, consider a Heyting algebra $\boldsymbol{A}$ and $a, c \in A$. Using the residuation law and the fact that 1 is the maximum of $\boldsymbol{A}$, we obtain

$$\begin{aligned} a = c &\iff (a \leqslant c \text{ and } c \leqslant a) \\ &\iff (1 \wedge a \leqslant \text{ and } 1 \wedge c \leqslant a) \\ &\iff (1 \leqslant a \to c \text{ and } 1 \leqslant c \to a) \\ &\iff a \to c = 1 \text{ and } c \to a = 1. \end{aligned}$$

This establishes (4.1), which is precisely (Alg4*). Hence, **IPC** satisfies (Alg1) and (Alg4*) with respect to $\boldsymbol{\tau}, \Delta$ and HA. By Proposition 4.3, we conclude that **IPC** is algebraizable and that its algebraizability is witnessed by $\boldsymbol{\tau}, \Delta$ and HA.

A similar argument shows that $\mathbf{K}_g$ is algebraizable and that its algebraizability is witnessed by $\boldsymbol{\tau}, \Delta$ and the class of modal algebras MA. Similarly, **CPC** is algebraizable and its algebraizability is witnessed by $\boldsymbol{\tau}, \Delta$ and the class of Boolean algebras BA. ⊠

Example 4.5 (A nonalgebraizable logic)**.** On the other hand, not every logic is algebraizable. As an exemplification, we will prove that $\mathbf{K}_\ell$ is not algebraizable.

Suppose the contrary, with a view to contradition. Then there are a set of equations $\boldsymbol{\tau}(x)$, a set of formulas $\Delta(x, y)$ and a quasi-variety K witnessing the algebraizbaility of $\mathbf{K}_\ell$.

We shall see that K is a class of modal algebras. As the class MA of modal algebras is a variety, it suffices to show that every equation valid in MA is also valid in K. To this end, let $\varepsilon \approx \delta$ be an equation valid in MA. From (Alg1) and (Alg4) it follows

$$\emptyset \vdash_{\mathbf{K}_\ell} \Delta(x, x) \iff \emptyset \vDash_{\mathsf{K}} \boldsymbol{\tau}[\Delta(x, x)] \iff \emptyset \vDash_{\mathsf{K}} x \approx x.$$

Since $\emptyset \vDash_{\mathsf{K}} x \approx x$, we conclude that $\emptyset \vdash_{\mathbf{K}_\ell} \Delta(x, x)$. By substitution invariance, this yields

$$\emptyset \vdash_{\mathbf{K}_\ell} \Delta(\varepsilon, \varepsilon). \tag{4.2}$$

Since $\varepsilon \approx \delta$ is valid in MA, for every $\varphi(x, y) \in \Delta$, we have

$$\mathsf{MA} \vDash \varphi(\varepsilon, \varepsilon) \approx \varphi(\varepsilon, \delta).$$

Together with (2.4), this yields $\Delta(\varepsilon, \varepsilon) \dashv\vdash_{\mathbf{K}_\ell} \Delta(\varepsilon, \delta)$. Since, by (4.2), $\emptyset \vdash_{\mathbf{K}_\ell} \Delta(\varepsilon, \varepsilon)$, this implies $\emptyset \vdash_{\mathbf{K}_\ell} \Delta(\varepsilon, \delta)$. By (Alg1), we obtain $\emptyset \vDash_{\mathsf{K}} \boldsymbol{\tau}[\Delta(\varepsilon, \delta)]$. Using (Alg4), we conclude that K satisfies $\varepsilon \approx \delta$ and, therefore, that K is a class of modal algebras.

Since $\mathbf{K}_\ell$ is related to K by (Alg4), this implies that $\mathbf{K}_\ell$ has an algebraic semantics, namely K, that is a class of modal algebras. But this contradicts Theorem 2.22. ⊠

As we mentioned, the theory of algebraizable logics allows to associate with a logic a unique *distinguished* algebraic semantics.

Theorem 4.6. *If* $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ *and* $\boldsymbol{\tau}_2, \Delta_2, \mathsf{K}_2$ *witness the algebraizability of a logic* $\vdash$*, then*

$$\mathsf{K}_1 = \mathsf{K}_2 \qquad \boldsymbol{\tau}_1(x) =\mid\models_{\mathsf{K}_1} \boldsymbol{\tau}_2(x) \qquad \Delta_1(x,y) \dashv\vdash \Delta_2(x,y).$$

Proof. We begin by proving $\Delta_1(x,y) \dashv\vdash \Delta_2(x,y)$. By symmetry, it suffices to prove $\Delta_1(x,y) \vdash \Delta_2(x,y)$. To this end, consider $\varphi \in \Delta_2$. Since $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ witness the algebraizability of $\vdash$, from (Alg4) and (Alg1) it follows

$$\boldsymbol{\tau}_1(x), x \approx y \models_{\mathsf{K}_1} \boldsymbol{\tau}_1(y) \iff \boldsymbol{\tau}_1(x), \boldsymbol{\tau}_1[\Delta_1(x,y)] \models_{\mathsf{K}_1} \boldsymbol{\tau}_1(y) \iff x, \Delta_1(x,y) \vdash y$$

and

$$x \approx y \models_{\mathsf{K}_1} \varphi(x,x) \approx \varphi(x,y) \iff \Delta_1(x,y) \vdash \Delta_1(\varphi(x,x), \varphi(x,y)).$$

Since $\boldsymbol{\tau}_1(x), x \approx y \models_{\mathsf{K}_1} \boldsymbol{\tau}_1(y)$ and $x \approx y \models_{\mathsf{K}_1} \varphi(x,x) \approx \varphi(x,y)$, we obtain that

$$x, \Delta_1(x,y) \vdash y \;\text{ and }\; \Delta_1(x,y) \vdash \Delta_1(\varphi(x,x), \varphi(x,y)).$$

Moreover, by substitution invariance, we get $\varphi(x,x), \Delta_1(\varphi(x,x), \varphi(x,y)) \vdash \varphi(x,y)$. Together with the above display, we conclude that

$$\varphi(x,x), \Delta_1(x,y) \vdash \varphi(x,y).$$

Now, as explained in Example 4.5, the fact that $\boldsymbol{\tau}_2, \Delta_2, \mathsf{K}_2$ witness the algebraizability of $\vdash$ guarantees that $\emptyset \vdash \Delta_2(x,x)$. As a consequence, $\emptyset \vdash \varphi(x,x)$. Together with the above display, this yields $\Delta_1(x,y) \vdash \varphi(x,y)$ and, therefore, $\Delta_1(x,y) \vdash \Delta_2(x,y)$. Hence, we conclude that $\Delta_1(x,y) \dashv\vdash \Delta_2(x,y)$, as desired.

Then we turn to prove that $\mathsf{K}_1 = \mathsf{K}_2$. Since K_1 and K_2 are quasi-varieties, it suffices to show that they satisfy the same quasi-equations with variables in Var. To prove this, consider a finite set of equation $\Theta \cup \{\varepsilon \approx \delta\} \subseteq E(Var)$. We have

$$\begin{aligned} \mathsf{K}_1 \models \mathop{\&} \Theta \Longrightarrow \varepsilon \approx \delta &\iff \Theta \models_{\mathsf{K}_1} \varepsilon \approx \delta \\ &\iff \Delta_1[\Theta] \vdash \Delta_1(\varepsilon, \delta) \\ &\iff \Delta_2[\Theta] \vdash \Delta_2(\varepsilon, \delta) \\ &\iff \Theta \models_{\mathsf{K}_2} \varepsilon \approx \delta \\ &\iff \mathsf{K}_2 \models \mathop{\&} \Theta \Longrightarrow \varepsilon \approx \delta. \end{aligned}$$

The above equivalence can be justified as follows. The first and the last are straightforward, the second from (Alg2) and the assumption that $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ witness the algebraizability of $\vdash$, the third from $\Delta_1(x,y) \dashv\vdash \Delta_2(x,y)$ and the fourth from the assumption that $\boldsymbol{\tau}_2, \Delta_2, \mathsf{K}_2$ witness the algebraizability of $\vdash$. Hence, we conclude that $\mathsf{K}_1 = \mathsf{K}_2$.

It only remains to prove that $\boldsymbol{\tau}_1(x) =\mid\models_{\mathsf{K}_1} \boldsymbol{\tau}_2(x)$. To prove this, observe that

$$\begin{aligned} \boldsymbol{\tau}_1(x) =\mid\models_{\mathsf{K}_1} \boldsymbol{\tau}_2(x) &\iff \Delta_1[\boldsymbol{\tau}_1(x)] \dashv\vdash \Delta_1[\boldsymbol{\tau}_2(x)] \\ &\iff \Delta_1[\boldsymbol{\tau}_1(x)] \dashv\vdash \Delta_2[\boldsymbol{\tau}_2(x)] \\ &\iff x \dashv\vdash x. \end{aligned}$$

The above equivalence can be justified as follows. The first follows from (Alg2) and the assumption that $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ witness the algebraizability of $\vdash$, the third from $\Delta_1(x,y) \dashv\vdash \Delta_2(x,y)$ and the last from (Alg3) and the fact that both $\boldsymbol{\tau}_1, \Delta_1, \mathsf{K}_1$ and $\boldsymbol{\tau}_2, \Delta_2, \mathsf{K}_2$ witness the algebraizability of $\vdash$. Since $x \dashv\vdash x$, we conclude that $\boldsymbol{\tau}_1(x) =\mid\models_{\mathsf{K}_1} \boldsymbol{\tau}_2(x)$. ⊠

Corollary 4.7. *Every algebraizable logic has a unique equivalent algebraic semantics.*

Proof. Immediate from Theorem 4.6. ⊠

Example 4.8 (Equivalent algebraic semantics)**.** In view of Example 4.5,

(i) The unique equivalent algebraic semantics of **CPC** is BA;

(ii) The unique equivalent algebraic semantics of **IPC** is HA;

(iii) The unique equivalent algebraic semantics of $\mathbf{K}_g$ is MA.

In particular, while **CPC** has various algebraic semantics (for instance, BA and HA), the class of Boolean algebras is its unique equivalent algebraic semantics. ⊠

At this stage, it is natural to wonder whether every quasi-variety is the equivalent algebraic semantics of some algebraizable logic. While this is not the case in general [58] (as explained in the next exercise), still every quasi-variety is *categorically equivalent* to the equivalent algebraic semantics of an algebraizable logic [105], see also [27]. As a consequence, the categorical properties of every every quasi-variety can be studied though the eyes of an algebraizable logic. Notably, category equivalences and adjunctions between quasi-varieties have been described in detail in [49, 95, 101], see also [59].

Exercise 4.9. A class of algebras K that satisfies $f(x, \ldots, x) \approx x$ for each of its basic operations f is said to be *idempotent*. For instance, all classes of lattices are idempotent. Prove that a nontrivial idempotent quasi-variety cannot be the equivalent algebraic semantics of any algebraizable logic.

To this end, you might wish to use the following strategy: suppose, with a view to contradiction, that there exists a nontrivial idempotent quasi-variety K that, moreover, is the equivalent algebraic semantics of some algebraizable logic $\vdash$. First show that $\emptyset \vdash x$. By substitution invariance, derive $\emptyset \vdash \Delta(x, y)$. Use this fact to infer that K is trivial and, therefore, to obtain a contradiction. ⊠

Definition 4.10. Given two logics $\vdash$ and $\vdash^*$ of the same type. Then $\vdash^*$ is said to be

(i) an *extension* of $\vdash$ if, $\Gamma \vdash^* \varphi$, for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$ such that $\Gamma \vdash \varphi$; and

(ii) an *axiomatic extension* of $\vdash$ if there exists a set of formulas Σ such that $\sigma[\Sigma] \subseteq \Sigma$ for every substitution σ and, for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash^* \varphi \Longleftrightarrow \Gamma, \Sigma \vdash \varphi.$$

In this case, we say that Σ *axiomatizes* $\vdash^*$ *relative to* $\vdash$.

Axiomatic extensions of **IPC** have been called *superintuitionistic logics* in the literature.

Remark 4.11. Notice that when a logic $\vdash^*$ is an axiomatic extension of a logic $\vdash$, axiomatized relative to $\vdash$ by Σ, it can be axiomatized by extending any Hilbert axiomatization for $\vdash$ with the set of axioms $\{\emptyset \rhd \varphi : \varphi \in \Sigma\}$. ⊠

Definition 4.12. Let K and M be quasi-varieties of the same type. Then M is said to be

(i) a *subquasi-variety* of K if $\mathsf{M} \subseteq \mathsf{K}$; and

(ii) a *relative subvariety* of K if it is axiomatized relative to K by a set of equations.

Notice that the relative subvarieties of a variety are precisely its subvarieties.

Given a finitary logic $\vdash$, the posets $\mathsf{fEx}(\vdash)$ and $\mathsf{aEx}(\vdash)$ of finitary and axiomatic extensions of $\vdash$, respectively, ordered under the inclusion relation form two complete lattices. Similarly, given a quasi-variety K, the posets $\mathsf{sQ}(\mathsf{K})$ and $\mathsf{sV}(\mathsf{K})$ of subquasi-varieties and relative subvarieties of K, respectively, ordered under the inclusion relation form two complete lattices. In view of the next result, axiomatic extensions of an algebraizable logic can be studied through the lenses of the relative subvarieties of its equivalent algebraic semantics, which, in turn, are amenable to the methods of model theory, universal algebra and duality theory. This approach has proved very fruitful in the study of superintuitionistic and normal modal logics [33, 89].

Theorem 4.13. *Let $\vdash$ be an algebraizable logic and K its equivalent algebraic semantics. The lattices $\mathsf{fEx}(\vdash)$ and $\mathsf{sQ}(\mathsf{K})$ are dually isomorphic under the map that sends a finitary extension of $\vdash$ to its equivalent algebraic semantics. This dual isomorphism restricts to one between $\mathsf{aEx}(\vdash)$ and $\mathsf{sV}(\mathsf{K})$.*

Remark 4.14. Notice that the above result implicitly states that algebraizability persists under the formation of finitary extensions. Moreover, it generalizes the well-known fact that the lattices of superintuitionistic logics and of normal modal logics are dually isomorphic to those of varieties of Heyting and modal algebras, respectively. ⊠

Remark 4.15. The notion of an algebraizable logic can be extended to other formalisms, including Gentzen systems and hypersequent calculi [28, 66, 112, 117, 120] and has inspired a notion of equivalence between arbitrary deductive systems [15, 16, 61, 64]. ⊠

4.2 Deductive filters

Deductive closed sets of formulas have been called theories. This notion can be extended as follows.

Definition 4.16. A subset F of the universe of an algebra $\boldsymbol{A}$ is said to be a *deductive filter* of a logic $\vdash$ on $\boldsymbol{A}$ when, for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\text{if } \Gamma \vdash \varphi, \text{ then for every homomorphism } f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}, \\ \text{if } f[\Gamma] \subseteq F, \text{ then } f(\varphi) \in F.$$

We denote by $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ the poset of deductive filters of $\vdash$ on $\boldsymbol{A}$ ordered under inclusion.

The following observations is a direct consequence of the definition of a deductive filter.

Proposition 4.17. *If $\vdash$ is a logic and $\boldsymbol{A}$ an algebra, then $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ is a complete lattice in which meets are intersections.*

Recall that the set of theories of a logic $\vdash$ is denoted by $\mathcal{T}h(\vdash)$. When convenient, we regard $\mathcal{T}h(\vdash)$ as a lattice ordered under inclusion.

Proposition 4.18. *If $\vdash$ is a logic, the lattice $\mathcal{T}h(\vdash)$ of theories of $\vdash$ coincides with the lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{T}(Var))$ of filters of $\vdash$ on the formula algebra $\boldsymbol{T}(Var)$.*

Proof. Consider a set of formulas Γ. Suppose first that Γ is a theory of $\vdash$. Then consider a set of formulas $\Sigma \cup \{\varphi\}$ such that $\Sigma \vdash \varphi$ and a homomorphism $\sigma\colon \boldsymbol{T}(Var) \to \boldsymbol{T}(Var)$ such that $\sigma[\Sigma] \subseteq \Gamma$. Notice that σ is a substitution. Consequently, as $\vdash$ is substitution invariant, $\sigma[\Sigma] \vdash \sigma(\varphi)$. Now, since $\sigma[\Sigma] \subseteq \Gamma$, we have $\Gamma \vdash \sigma[\Sigma]$. Together with $\sigma[\Sigma] \vdash \sigma(\varphi)$ and the Cut principle, this implies $\Gamma \vdash \sigma(\varphi)$. Since Γ is a theory of $\vdash$, this yields $\sigma(\varphi) \in \Gamma$. Hence, we conclude that Γ is a filter of $\vdash$ on $\boldsymbol{T}(Var)$.

Conversely, suppose that Γ is a filter of $\vdash$ on $\boldsymbol{T}(Var)$. Then consider a formula φ such that $\Gamma \vdash \varphi$ and let $id\colon \boldsymbol{T}(Var) \to \boldsymbol{T}(Var)$ be the identity homomorphism. Since $id[\Gamma] = \Gamma$ and $\Gamma \vdash \varphi$, the assumption that Γ is a filter of $\vdash$ on $\boldsymbol{T}(Var)$ guarantees that $\varphi = id(\varphi) \in \Gamma$. Hence, we conclude that Γ is a theory of $\vdash$. ⊠

The next observation is instrumental to describe deductive filters in concrete cases.

Proposition 4.19. *Let $\vdash$ be the logic axiomatized by an Hilbert calculus* H*. A subset F of the universe of an algebra $\boldsymbol{A}$ is a deductive filter of $\vdash$ on $\boldsymbol{A}$ if and only if F is closed under the interpretation of the rules in* H*, that is, for every rule $\Gamma \rhd \varphi$ in* H *and every homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$,*

$$\textit{if } f[\Gamma] \subseteq F, \textit{ then } f(\varphi) \in F.$$

Proof. The "only if" part is straightforward. To prove the "if" one, suppose that F is closed under the interpretation of the rules in H. Then consider a set of formulas $\Gamma \cup \{\varphi\}$ such that $\Gamma \vdash \varphi$ and a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f[\Gamma] \subseteq F$. Since H axiomatizes $\vdash$, there exists a formal proof $\langle \psi_\alpha : \alpha \leqslant \gamma \rangle$ of φ from Γ in H. We will prove, by complete induction on α, that $f(\psi_\alpha) \in F$. Accordingly, assume that $\{f(\psi_\beta) : \beta < \alpha\} \subseteq F$. Then we have two cases: either $\psi_\alpha \in \Gamma$ or there exists a rule $\Sigma \rhd \delta$ in H and a substitution σ such that $\sigma[\Delta] \subseteq \{\psi_\beta : \beta < \alpha\}$ and $\sigma(\delta) = \psi_\alpha$. If $\psi_\alpha \in \Gamma$, then it is clear that $f(\psi_\alpha) \in f[\Gamma] \subseteq F$. Then we consider the second case. By the inductive hypothesis, we have

$$f(\sigma[\Delta]) \subseteq \{f(\psi_\beta) : \beta < \alpha\} \subseteq F.$$

Furthermore, consider the homomorphism $f \circ \sigma\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$. From the above display it follows $f \circ \sigma[\Delta] \subseteq F$. Since F is closed under the interpretation of the rules in H (and, in particular, under $\Delta \rhd \delta$), this yields

$$f(\psi_\alpha) = f(\sigma(\delta)) = f \circ \sigma(\delta) \in F.$$

This concludes the inductive proof. Since $\varphi = \psi_\gamma$, we obtain that $f(\varphi) \in F$ and, therefore, that F is a deductive filter of $\vdash$ on $\boldsymbol{A}$. ⊠

Example 4.20 (Deductive filters)**.** We will prove that

(i) The deductive filters of **IPC** on a Heyting algebra $\boldsymbol{A}$ are the lattice filters of $\boldsymbol{A}$;

(ii) The deductive filters of **CPC** on a Boolean algebra $\boldsymbol{A}$ are the lattice filters of $\boldsymbol{A}$;

(iii) The deductive filters of $\mathbf{K}_g$ on a modal algebra $\boldsymbol{A}$ are the open filters of $\boldsymbol{A}$.

(i): Consider a subset F of A. Suppose first that F is a deductive filter of **IPC**. As $\emptyset \vdash_{\mathbf{IPC}} 1$ and F is a deductive filter of **IPC** on $\boldsymbol{A}$, we obtain $1 \in F$. Then consider $a, c \in A$. To prove that F is an upset, suppose that $a \in F$ and $a \leqslant c$. From the residuation law it follows

$$a \leqslant c \Longrightarrow 1 \wedge a \leqslant c \Longrightarrow 1 \leqslant a \to c \Longrightarrow a \to c = 1.$$

Since $1 \in F$, this yields $a, a \to c \in F$. Together with $x, x \to y \vdash_{\mathbf{IPC}} y$ and the assumption that F is a deductive filter of **IPC** on $\boldsymbol{A}$, this yields $c \in F$, as desired. Lastly, to prove that F is closed under binary meets, suppose that $a, c \in F$. Since F is a deductive filter of **IPC** on $\boldsymbol{A}$, the fact that $x, y \vdash_{\mathbf{IPC}} x \wedge y$ and $a, c \in F$ implies $a \wedge c \in F$. Hence, we conclude that F is a lattice filter of $\boldsymbol{A}$.

Conversely, suppose that F is a lattice filter of $\boldsymbol{A}$. In view of Proposition 4.19, it suffices to show that F is closed under the interpretation of the rules of an Hilbert calculus axiomatizing **IPC**. Accordingly, let H be the Hilbert calculus whose set of axioms is

$$\{\emptyset \rhd \varphi : \varphi \in T(Var) \text{ and } \emptyset \vdash_{\mathbf{IPC}} \varphi\}$$

and whose sole rule is modus ponens $x, x \to y \rhd y$. As H axiomatizes **IPC**, it only remains to prove that F is closed under the interpretation of its rules. Let then $\emptyset \rhd \varphi(x_1, \dots, x_n)$ be an axiom of H. Clearly, $\emptyset \vdash_{\mathbf{IPC}} \varphi(x_1, \dots, x_n)$. Since the class of Heyting algebras is an $\{x \approx 1\}$-algebraic semantics for **IPC**, it follows that, for all $a_1, \dots, a_n \in A$,

$$\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = 1 \in F.$$

The only rule in H is modus ponens $x, x \to y \rhd y$. To prove that F is closed under its interpretation, consider $a, c \in A$ such that $a, a \to c \in F$. Since F is closed under binary meets, $a \wedge (a \to c) \in F$. Moreover, by the residuation law,

$$a \to c \leqslant a \to c \Longleftrightarrow a \wedge (a \to c) \leqslant c.$$

Since $a \to c \leqslant a \to c$ always holds, we get $a \wedge (a \to c) \leqslant c$. As F is an upset that contains $a \wedge (a \to c)$, we obtain that $c \in F$. This shows that F is a deductive filter of **IPC** on $\boldsymbol{A}$.

(ii): Analogous to the proof of (i).

(iii): Consider a subset F of A. Suppose first that F is a deductive filter of $\mathbf{K}_g$ on $\boldsymbol{A}$. The proof that F is a lattice filter of $\boldsymbol{A}$ is analogous to the one detailed in the case of (i). In order to prove that F is also closed under $\Box$, consider $a \in F$. Since $x \vdash_{\mathbf{K}_g} \Box x$, the fact that F is a deductive filter of $\mathbf{K}_g$ on $\boldsymbol{A}$ and $a \in F$ implies that $\Box a \in F$. Hence, we conclude that F is an open filter of $\boldsymbol{A}$, as desired.

To prove the converse, suppose that F is an open filter of $\boldsymbol{A}$. In view of Proposition 4.19, it suffices to show that F is closed under the interpretation of the rules in the Hilbert calculus axiomatizing $\mathbf{K}_g$. Accordingly, let H be the Hilbert calculus whose set of axioms is

$$\{\emptyset \rhd \varphi : \varphi \in T(Var) \text{ and } \emptyset \vdash_{\mathbf{K}_g} \varphi\}$$

and whose rules are modus ponens $x, x \to y \rhd y$ and necessitation $x \rhd \Box x$. As H axiomatizes $\mathbf{K}_g$, it only remains to show that F is closed under the interpretation of its rules. The proof that F is closed under the interpretation of the axioms in H and of modus ponens is analogous to the one detailed for the case of (i). Therefore, it only remains to prove that F is closed under the interpretation of the necessitation rule $x \rhd \Box x$. But this is an immediate consequence of the fact that the filter F is open. ⊠

Exercise 4.21. Prove that the deductive filters of $\mathbf{K}_\ell$ on a modal algebra $\boldsymbol{A}$ are precisely the lattice filters of $\boldsymbol{A}$. Use the fact that $\mathbf{K}_\ell$ can be axiomatized by the Hilbert calculus whose set of axioms is

$$\{\emptyset \rhd \varphi : \varphi \in T(Var) \text{ and } \emptyset \vdash_{\mathbf{K}_\ell} \varphi\}$$

and whose sole rule is modus ponens $x, x \to y \rhd y$. Hint: you may use the fact that $\mathbf{K}_\ell$ and $\mathbf{K}_g$ have the same theorems. ⊠

The notion of a deductive filter can be extended to relative equational consequences as follows.

Definition 4.22. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras. A set $\theta \subseteq A \times A$ is said to be a *deductive filter* of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$ when, for every $\Theta \cup \{\varepsilon \approx \delta\} \subseteq E(Var)$,

$$\begin{aligned} &\text{if } \Theta \vDash_{\mathsf{K}} \varphi \approx \psi, \text{ then for every homomorphism } f\colon \boldsymbol{T}(Var) \to \boldsymbol{A},\\ &\qquad \text{if } \langle f(\varphi), f(\psi)\rangle \in \theta \text{ for all } \varphi \approx \psi \in \Theta, \text{ then } \langle f(\varepsilon), f(\delta)\rangle \in \theta. \end{aligned}$$

Proposition 4.23. *Let K be a quasi-variety and $\boldsymbol{A}$ an algebra of the same type. The deductive filters of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$ are precisely the K-congruences of $\boldsymbol{A}$.*

Proof. Consider a subset θ of $A \times A$. First suppose that θ is a deductive filters of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$. Notice that

$$\emptyset \vDash_{\mathsf{K}} x \approx x \qquad x \approx y \vDash_{\mathsf{K}} y \approx x \qquad x \approx y, y \approx z \vDash_{\mathsf{K}} x \approx z.$$

Furthermore, for every basic n-ary operation f, we have

$$x_1 \approx y_1, \dots, x_n \approx y_n \vDash_{\mathsf{K}} f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n).$$

Since θ is a deductive filter of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$, the above displays guarantee that θ is a congruence of $\boldsymbol{A}$. To prove that it is also a K-congruence, it remains to show that $\boldsymbol{A}/\theta \in \mathsf{K}$. Since K is a quasi-variety, it suffices to prove that $\boldsymbol{A}$ satisfies all the quasi-equations valid in K. Accordingly, consider a quasi-equation $\& \Theta \Longrightarrow \varepsilon \approx \delta$ valid in K and let $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}/\theta$ be a homomorphism such that $f(\varphi) = f(\psi)$, for all $\varphi \approx \psi \in \Theta$. Since the canonical homomorphism $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is surjective, we can apply Corollary 1.6, obtaining a homomorphism $g\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f = \pi \circ g$. For every $\varphi \approx \psi \in \Theta$, we have

$$g(\varphi)/\theta = \pi \circ g(\varphi) = f(\varphi) = f(\psi) = \pi \circ g(\psi) = g(\psi)/\theta$$

and, therefore, $\langle g(\varphi), g(\psi)\rangle \in \theta$. Since θ is a deductive filter of $\vDash_{\mathsf{K}}$ and $\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta$, this yields $\langle g(\varepsilon), g(\delta)\rangle \in \theta$. In turn, this implies

$$f(\varepsilon) = \pi \circ g(\varepsilon) = \pi \circ g(\delta) = f(\delta).$$

Hence, we conclude that $\boldsymbol{A}/\theta \in \mathsf{K}$, as desired.

To prove the converse, suppose that θ is a K-congruence of $\boldsymbol{A}$. Consider a set of equations $\Theta \cup \{\varepsilon \approx \delta\} \subseteq E(Var)$ such that $\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta$ and a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $\langle f(\varphi), f(\psi)\rangle \in \theta$, for all $\varphi \approx \psi \in \Theta$. Now, let $\pi \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ be the canonical projection. We have $\pi \circ f(\varphi) = \pi \circ f(\psi)$, for all $\varphi \approx \psi \in \Theta$. Since $\boldsymbol{A}/\theta \in \mathsf{K}$, this yields $\pi \circ f(\varepsilon) = \pi \circ f(\delta)$, which is $\langle f(\varepsilon), f(\delta)\rangle \in \theta$. ⊠

Definition 4.24. Let K be a quasi-variety. A set of equations $\Theta \subseteq E(Var)$ is said to be a *theory* of $\vDash_{\mathsf{K}}$ when, for every $\varepsilon \approx \delta \in E(Var)$,

$$\text{if } \Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta, \text{ then } \varepsilon \approx \delta \in \Theta.$$

When ordered under the inclusion relation, the theories of $\vDash_{\mathsf{K}}$ form a lattice that we denote by $\mathcal{T}h(\vDash_{\mathsf{K}})$.

Remark 4.25. Notice that $\vDash_{\mathsf{K}}$ is a consequence relation on the set of equations $E(Var)$. Therefore, the above definition is a special instance of Definition 2.4.

Recall that, formally speaking, equations are ordered pairs of formulas, e.g., the expression $\varepsilon \approx \delta$ is a suggestive notation for the ordered pair $\langle \varepsilon, \delta\rangle$. The following result builds on this observation.

Proposition 4.26. *If K is a quasi-variety, the lattice $\mathcal{T}h(\vDash_{\mathsf{K}})$ of theories of $\vDash_{\mathsf{K}}$ coincides with the lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var))$ of K-congruences of the formula algebra $\boldsymbol{T}(Var)$.*

Proof. An argument analogous to the one detailed in the proof of Proposition 4.18 shows that $\mathcal{T}h(\vDash_{\mathsf{K}})$ coincides with the lattice of deductive filters of $\vDash_{\mathsf{K}}$ on $\boldsymbol{T}(Var)$. But, in view of Proposition 4.23, the latter coincides with $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var))$. ⊠

Deductive filters are closed under inverse endomorphisms, as we proceed to explain. First, the set of *endomorphism* of an algebra $\boldsymbol{A}$ will be denoted by $\mathsf{End}(\boldsymbol{A})$. Then, given an endomorphism σ and a congruence θ of $\boldsymbol{A}$, we set

$$\sigma^{-1}[\theta] := \{\langle a, c\rangle \in A \times A \colon \langle \sigma(a), \sigma(c)\rangle \in \theta\}.$$

Lemma 4.27. *Let $\vdash$ be a logic, K a quasi-variety, $\boldsymbol{A}$ an algebra and $\sigma \in \mathsf{End}(\boldsymbol{A})$.*

(i) *If $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$, then $\sigma^{-1}[F] \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$.*

(ii) *If $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, then $\sigma^{-1}[\theta] \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.*

Proof. (i): Suppose that $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$. Then consider $\Gamma \cup \{\varphi\} \subseteq T(Var)$ such that $\Gamma \vdash \varphi$ and a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f[\Gamma] \subseteq \sigma^{-1}[F]$. Clearly, $\sigma \circ f[\Gamma] \subseteq F$. Since $\sigma \circ f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ is a homomorphism and $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$, this implies $\sigma \circ f(\varphi) \subseteq F$. Hence, we conclude $f(\varphi) \subseteq \sigma^{-1}[F]$, as desired.

(ii): Recall from Proposition 4.23 that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is the set of deductive filters of $\vDash_{\mathsf{K}}$ on $\boldsymbol{A}$. Because of this, we can mimic the proof detailed for condition (i) and obtain that $\sigma^{-1}[\theta] \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, for every $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. ⊠

Remark 4.28. Conditions (i) and (ii) in the above lemma can be generalized as follows. Let $\vdash$ be a logic, K a quasi-variety and $f\colon \boldsymbol{A} \to \boldsymbol{B}$ a homomorphism.

(i) If F is a deductive filter of $\vdash$ on $\boldsymbol{B}$, then $f^{-1}[F]$ is a deductive filter of $\vdash$ on $\boldsymbol{A}$; and

(ii) If θ is a K-congruence of $\boldsymbol{B}$, then $f^{-1}[\theta]$ is a K-congruence of $\boldsymbol{A}$. ⊠

In view of Lemma 4.27, given a logic $\vdash$ and an algebra $\boldsymbol{A}$, the lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ of deductive filters of $\vdash$ on $\boldsymbol{A}$ can be expanded with the unary operations $\{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}$. Similarly, given a quasi-variety K, the lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ of K-congruences of $\boldsymbol{A}$ can also be expanded with the unary operations $\{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}$. Accordingly, we set

$$\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} := \langle \mathsf{Fi}_{\vdash}(\boldsymbol{A}); \wedge, \vee, \{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}\rangle$$
$$\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+} := \langle \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}); \wedge, \vee, \{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}\rangle.$$

The above structures can be viewed as algebras whose type comprises two binary operations $\wedge$ and $\vee$ an a family of unary operations $\{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}$. From this perspective, an isomorphism from $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+}$ to $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ is a lattice isomorphism $\Phi\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ that commutes with inverse endomorphisms, in the sense that

$$\Phi(\sigma^{-1}[F]) = \sigma^{-1}[\Phi(F)], \text{ for every } \sigma \in \mathsf{End}(\boldsymbol{A}).$$

Recall from Propositions 4.18 and 4.26 that

$$\mathcal{T}h(\vdash) = \mathsf{Fi}_{\vdash}(\boldsymbol{T}(\mathit{Var})) \;\text{ and }\; \mathcal{T}h(\vDash_{\mathsf{K}}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(\mathit{Var})). \tag{4.3}$$

Because of this, when $\boldsymbol{A} = \boldsymbol{T}(\mathit{Var})$, we will denote $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+}$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ by

$$\mathcal{T}h(\vdash)^{+} \;\text{ and }\; \mathcal{T}h(\vDash_{\mathsf{K}})^{+}.$$

The importance of these structures will become apparent in the next section.

4.3 Isomorphism theorem

In many familiar examples the congruences of an algebra correspond to certain distinguished subsets of its universe. This happens, for instance, in the algebra of logic, where the congruence lattice $\mathsf{Con}(\boldsymbol{A})$ of a Heyting algebra $\boldsymbol{A}$ is isomorphic to the lattice $\mathsf{Fi}(\boldsymbol{A})$ of its filters. Similarly, the congruence lattice of a modal algebra $\boldsymbol{A}$ is isomorphic to the the lattice $\mathsf{Op}(\boldsymbol{A})$ of its open filters (see Examples 1.12 and 1.13, if necessary). Analogous correspondences can be found in classical algebra, where the congruences of groups and ideals are related to normal subgroups and two-sided ideals in a similar manner. As we shall see, all these correspondences can be viewed as consequences of the algebraization of some logic.

Isomorphism Theorem 4.29 (Blok & Pigozzi)**.** *The following conditions are equivalent for a finitary logic* $\vdash$ *and a quasi-variety* K*:*

(i) $\vdash$ *is algebraizable with equivalent algebraic semantics* K*;*

(ii) $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \cong \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$, *for every algebra* $\boldsymbol{A}$ *of the suitable type;*

(iii) $\mathcal{T}h(\vdash)^{+} \cong \mathcal{T}h(\vDash_{\mathsf{K}})^{+}$.

In view of the implication (i)⇒(ii) in the Isomorphism Theorem, every algebraizable logic induces an isomorphism between lattices of deductive filters and of K-congruences. Most of the known correspondences between congruences and distinguished sets are consequences of this fact, as we proceed to explain.

Corollary 4.30. *The following conditions hold:*

(i) *If* $\boldsymbol{A}$ *is a Heyting algebra,* $\mathsf{Con}(\boldsymbol{A})$ *is isomorphic to the lattice of filters of* $\boldsymbol{A}$*;*

(ii) *If* $\boldsymbol{A}$ *is a modal algebra,* $\mathsf{Con}(\boldsymbol{A})$ *is isomorphic to the lattice of open filters of* $\boldsymbol{A}$*;*

(iii) *If* $\boldsymbol{A}$ *is a group,* $\mathsf{Con}(\boldsymbol{A})$ *is isomorphic to the lattice of normal subgroups of* $\boldsymbol{A}$*;*

(iv) *If* $\boldsymbol{A}$ *is a ring,* $\mathsf{Con}(\boldsymbol{A})$ *is isomorphic to the lattice of two-sided ideals of* $\boldsymbol{A}$*.*

Proof. (i): Let $\boldsymbol{A}$ be a Heyting algebra. As detailed in Example 4.20, the deductive filters of **IPC** on $\boldsymbol{A}$ are precisely the lattice filters of $\boldsymbol{A}$, in symbols,

$$\mathsf{Fi}_{\mathbf{IPC}}(\boldsymbol{A}) = \mathsf{Fi}(\boldsymbol{A}).$$

Since **IPC** is algebraizable and its equivalent algebraic semantics is the class HA of Heyting algebras, the implication (i)⇒(ii) in the Isomorphism Theorem guarantees that the lattices $\mathsf{Fi}_{\mathbf{IPC}}(\boldsymbol{A})$ and $\mathsf{Con}_{\mathsf{HA}}(\boldsymbol{A})$ are isomorphic. Furthermore, since $\boldsymbol{A}$ is a Heyting algebra and HA is a variety, we have $\mathsf{Con}_{\mathsf{HA}}(\boldsymbol{A}) = \mathsf{Con}(\boldsymbol{A})$, whence

$$\mathsf{Fi}(\boldsymbol{A}) = \mathsf{Fi}_{\mathbf{IPC}}(\boldsymbol{A}) \cong \mathsf{Con}_{\mathsf{HA}}(\boldsymbol{A}) = \mathsf{Con}(\boldsymbol{A}).$$

Thus, we conclude that $\mathsf{Fi}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A})$, as desired.

The proof of condition (ii) is analogous to that of (i), because $\mathbf{K}_g$ is algebraizable with equivalent algebraic semantics the variety of modal algebras and the deductive filters of $\mathbf{K}_g$ on a modal algebra $\boldsymbol{A}$ are precisely the open filters of $\boldsymbol{A}$ (see Example 4.20).

It only remains to prove conditions (iii) and (iv). We will outline only the proof of (iii), as the one of (iv) is analogous. Let Gr be the variety of groups in the type $\langle \cdot, (-)^{-1}, 1 \rangle$. We will show that Gr is the equivalent algebraic semantics of an algebraizable logic. To this end, consider the sets

$$\boldsymbol{\tau}(x) := \{x \approx 1\} \text{ and } \Delta(x,y) := \{x \cdot y^{-1}\}$$

and observe that $x \approx y \dashv\vDash_{\mathsf{Gr}} x \cdot y^{-1} \approx 1$, that is,

$$x \approx y \dashv\vDash_{\mathsf{Gr}} \boldsymbol{\tau}[\Delta(x,y)]. \tag{4.4}$$

The *logic of groups* **G** is defined, for every set of formulas $\Gamma \cup \{\varphi\}$, as follows:

$$\Gamma \vdash_{\mathbf{G}} \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{Gr}} \boldsymbol{\tau}(\varphi).$$

This definition and condition (4.4) imply that $\mathbf{G}, \boldsymbol{\tau}, \Delta$ and G satisfy the conditions (Alg1) and (Alg4*). By Proposition 4.3, we conclude that $\mathbf{G}$ is algebraizable with equivalent algebraic semantics the variety of groups Gr.

Furthermore, the deductive filters of $\mathbf{G}$ on a group $\boldsymbol{A}$ are precisely the normal subgroups of $\boldsymbol{A}$. On the one hand, every deductive filter of $\mathbf{G}$ on $\boldsymbol{A}$ is a normal subgroup, because the following rules are valid in $\mathbf{G}$:

$$x, y \rhd x \cdot y \qquad x \rhd x^{-1} \qquad \emptyset \rhd 1 \qquad x \rhd y \cdot (x \cdot y^{-1}).$$

On the other hand, every deductive filter is a normal subgroup. For if F is a normal subgroup of $\boldsymbol{A}$, we have $F = \pi^{-1}[\{1\}]$, where $\pi \colon \boldsymbol{A} \to \boldsymbol{A}/\theta_F$ the canonical homomorphism and θ_F the congruence of $\boldsymbol{A}$ induced by F. Now, the definition of $\mathbf{G}$ guarantees that $\{1\}$ is a deductive filter of $\mathbf{G}$ on $\boldsymbol{A}/\theta_F$. In view of Remark 4.28, its inverse image $\pi^{-1}[\{1\}]$ is a deductive filter of $\mathbf{G}$ on $\boldsymbol{A}$. Since $F = \pi^{-1}[\{1\}]$, we are done.

Therefore, we can apply the implication (i)⇒(ii) in the Isomorphism Theorem, obtaining that the lattice of normal subgroups of $\boldsymbol{A}$ is isomorphic to $\mathsf{Con}_{\mathsf{Gr}}(\boldsymbol{A})$. But, as Gr is a variety, $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{Gr}}(\boldsymbol{A})$. ⊠

The implication (i)⇒(ii) in the Isomorphism Theorem is also instrumental to disprove that certain logics are algebraizable.

Example 4.31. Recall that the logic $\mathbf{K}_\ell$ is not algebraizable. We will present an alternative proof of this fact, based on the Isomorphism Theorem. Suppose, with a view to contradiction, that $\mathbf{K}_\ell$ is algebraizable and let K be its equivalent algebraic semantics. Then consider the modal algebra $\boldsymbol{A} = \langle \{0, a, c, 1\}; \wedge, \vee, \neg, \Box, 0, 1 \rangle$ such that

$$\langle \{0, a, c, 1\}; \wedge, \vee, \neg, 0, 1 \rangle$$

is the four-element Boolean algebra with minimum 0 and maximum 1 and

$$\Box 0 = \Box a = \Box c = 0 \;\text{ and }\; \Box 1 = 1.$$

As the deductive filters of $\mathbf{K}_\ell$ on a modal algebra are precisely the lattice filters (Exercise 4.21), we know that $\mathsf{Fi}_{\mathbf{K}_\ell}(\boldsymbol{A})$ is a four-element set. On the other hand, $\mathsf{Con}(\boldsymbol{A}) = \{\mathrm{id}_A, A \times A\}$. Consequently, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ has cardinality at most two. On cardinality grounds, it follows that the lattices $\mathsf{Fi}_{\mathbf{K}_\ell}(\boldsymbol{A})$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ cannot be isomorphic. As this contradicts the Isomorphism Theorem, we conclude that $\mathbf{K}_\ell$ is not algebraizable. ⊠

The role of the inverse endomorphisms in the structures $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^+$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^+$ becomes apparent in the following example.

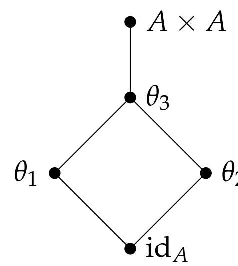
Example 4.32 (Implication free fragments)**.** We will prove that also the $\langle \wedge, \vee, \neg \rangle$-fragment of **IPC**, in symbols $\mathbf{IPC}_{\wedge\vee\neg}$, fails to be algebraizable. Suppose the contrary, with a view to contradiction. Then consider the algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg \rangle$, where $\langle A; \wedge, \vee \rangle$ is the lattice with order $0 < c < a < 1$ and $\neg$ is the unary operation defined as follows:

$$\neg 1 = \neg a = \neg c = 0 \;\text{ and }\; \neg 0 = 1.$$

By inspection, we see that $\boldsymbol{A}$ has precisely five congruences, namely, id_A, $A \times A$ and

$$\theta_1 := \text{ the congruence whose equivalence classes are } \{1, a\}, \{c\}, \{0\};$$
$$\theta_2 := \text{ the congruence whose equivalence classes are } \{1\}, \{a, c\}, \{0\};$$
$$\theta_3 := \text{ the congruence whose equivalence classes are } \{1, a, c\}, \{0\}.$$

Therefore, $\mathsf{Con}(\boldsymbol{A})$ is the lattice depicted below.



Then consider the endomorphism $\sigma_1 \colon \boldsymbol{A} \to \boldsymbol{A}$ and $\sigma_2 \colon \boldsymbol{A} \to \boldsymbol{A}$ such that

$$\sigma_1(1) = \sigma_1(a) = 1 \qquad \sigma_1(c) = c \qquad \sigma_1(0) = 0$$

and

$$\sigma_2(1) = 1 \qquad \sigma_2(a) = \sigma_2(c) = c \qquad \sigma_2(0) = 0$$

and observe that

$$\theta_1 = \sigma_1^{-1}[\theta_2] \text{ and } \theta_2 = \sigma_2^{-1}[\theta_1]. \tag{4.5}$$

We claim that if K is a quasi-variety and the lattice underlying $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^+$ contains a four-element chain, then $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ has cardinality five. To prove this, observe that, since $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a subposet of $\mathsf{Con}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ is the lattice depicted above, if $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^+$ has a four-element chain, then

$$\mathsf{Con}(\boldsymbol{A}) \smallsetminus \{\theta_1\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \text{ or } \mathsf{Con}(\boldsymbol{A}) \smallsetminus \{\theta_2\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}).$$

Since $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^+$ is closed under the unary operations σ_1^{-1} and σ_2^{-1}, from (4.5) it follows that $\theta_1, \theta_2 \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^+$. Together with the above display, this yields $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and, therefore, that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ has precisely five elements.

Lastly, as $\boldsymbol{A}$ is the $\langle \wedge, \vee, \neg \rangle$-reduct of a Heyting algebra, it is not hard to see that the deductive filters of $\mathbf{IPC}_{\wedge\vee\neg}$ on $\boldsymbol{A}$ are precisely the lattice filters of $\boldsymbol{A}$. Therefore, the lattice of deductive filters of $\mathbf{IPC}_{\wedge\vee\neg}$ on $\boldsymbol{A}$ is a four-element chain. By the Isomorphism Theorem, the lattice of K-congruences of $\boldsymbol{A}$ is also a four-element chain, where K is equivalent algebraic semantics of $\mathbf{IPC}_{\wedge\vee\neg}$. But, by the claim, this implies that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a five element set, a contradiction. ⊠

Exercise 4.33. An *ordered algebra* is a pair $\langle \boldsymbol{A}; \leqslant \rangle$ such that $\boldsymbol{A}$ is an algebra and $\leqslant$ a partial order on A. The logic *preserving degrees of truth* $\vdash_{\mathsf{K}}^{\leqslant}$ associated with a class of ordered algebras K is defined as follows: for every $\Gamma \cup \{\varphi\} \subseteq T(Var)$,

$$\Gamma \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff \text{for every } \langle \boldsymbol{A}; \leqslant \rangle \in \mathsf{K}, \text{ homomorphism } f \colon \boldsymbol{T}(Var) \to \boldsymbol{A} \text{ and } a \in A,$$
$$\text{if } a \leqslant f(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } a \leqslant f(\varphi).$$

A *bi-Heyting algebra* [121, 130] is a structure $\boldsymbol{A} = \langle A; \wedge, \vee, \to, \leftarrow, 0, 1\rangle$ such that both

$$\langle A; \wedge, \vee, \to, 0, 1\rangle \text{ and } \langle A; \vee, \wedge, \leftarrow, 1, 0\rangle$$

are Heyting algebras. Use the Isomorphism Theorem to show that the logic preserving degrees of truth associated with the class of bi-Heyting algebras (endowed with the lattice order) is not algebraizable. Hint: take inspiration from Example 4.31.

You might also wish to prove that **IPC** and $\mathbf{K}_\ell$ are the logics preserving degrees of truth associated, resepctively, with the classes of Heyting and modal algebras (endowed with the lattice order). Hint: try to adapt the proof of Theorem 2.15. ⊠

We shall now present a proof of the Isomorphism Theorem.

Proof. (i)⇒(ii): Let $\boldsymbol{\tau}$ and Δ be the sets of equations and formulas that, together with K, witness the algebraizability of $\vdash$. Moreover, let $\boldsymbol{A}$ be an algebra of the appropriate type. For every $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$ and $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, we define

$$\begin{aligned}\boldsymbol{\Omega}^{\boldsymbol{A}} F &:= \{\langle a, c\rangle \in A \times A : \Delta^{\boldsymbol{A}}(a, c) \subseteq F\}\\ \mathsf{S}^{\boldsymbol{A}}(\theta) &:= \{a \in A : \langle \varphi(a), \psi(a)\rangle \in \theta, \text{for all } \varphi \approx \psi \in \boldsymbol{\tau}\}.\end{aligned}$$

We will show that the maps

$$\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+} \text{ and } \mathsf{S}^{\boldsymbol{A}} \colon \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+} \to \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+}$$

are well-defined isomorphisms, one inverse to the other.

We begin by proving that the map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ is well defined. To this end, consider $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$. Observe that

$$\begin{aligned}\emptyset &\vDash_{\mathsf{K}} x \approx x\\ x \approx y &\vDash_{\mathsf{K}} y \approx x\\ x \approx y, y \approx z &\vDash_{\mathsf{K}} x \approx z\\ x_1 \approx y_1, \dots, x_n \approx y_n &\vDash_{\mathsf{K}} f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n),\end{aligned}$$

for every basic n-ary symbol. From condition (Alg2) in the definition of an algebraizable logic, it follows that

$$\begin{aligned}\emptyset &\vdash \Delta(x, x)\\ \Delta(x, y) &\vdash \Delta(y, x)\\ \Delta(x, y) \cup \Delta(y, z) &\vdash \Delta(x, z)\\ \Delta(x_1, y_1,) \cup \dots \cup \Delta(x_n, y_n) &\vdash \Delta(f(x_1, \dots, x_n), f(y_1, \dots, y_n)).\end{aligned}$$

Since F is a deductive filter of $\vdash$, these conditions guarantee that $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is a congruence of $\boldsymbol{A}$. For instance, in order to prove that $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is transitive, suppose that $\langle a, b\rangle, \langle b, c\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$. By the definition of $\boldsymbol{\Omega}^{\boldsymbol{A}} F$, we have $\Delta^{\boldsymbol{A}}(a, b) \cup \Delta^{\boldsymbol{A}}(b, c) \subseteq F$. As F is a deductive filter of $\vdash$ on $\boldsymbol{A}$, by the third condition in the above display we obtain $\Delta^{\boldsymbol{A}}(a, c) \subseteq F$. This in turn yields $\langle a, c\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$, establishing transitivity.

As $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is a congruence of $\boldsymbol{A}$, it only remains to prove that $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F \in \mathsf{K}$. Since K is a quasi-variety, Maltsev's Theorem guarantees that it can be axiomatized by a set of quasi-equations. Therefore, it suffices to show that every quasi-equation

$$(\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n) \Longrightarrow \varepsilon \approx \delta \tag{4.6}$$

valid in K is also valid in $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$. To this end, consider a homomorphism $f\colon \boldsymbol{T}(Var) \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$ such that $f(\varphi_i) = f(\psi_i)$ for every $i \leqslant n$. By Corollary 1.6, there exists a homomorphism $g\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f = \pi \circ g$, where $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the canonical projection. For every $i \leqslant n$, we have

$$\pi \circ g(\varphi_i) = f(\varphi_i) = f(\psi_i) = \pi \circ g(\psi_i)$$

and, therefore, $\langle g(\varphi_i), g(\psi_i)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$. By the definition of $\boldsymbol{\Omega}^{\boldsymbol{A}}F$, this amounts to

$$\Delta^{\boldsymbol{A}}(g(\varphi_1), g(\psi_1)) \cup \dots \cup \Delta^{\boldsymbol{A}}(g(\varphi_n), g(\psi_n)) \subseteq F.$$

Since g is a homomorphism, this can be rewritten as

$$g[\Delta(\varphi_1, \psi_1) \cup \dots \cup \Delta(\varphi_n, \psi_n)] \subseteq F. \tag{4.7}$$

Now, since the quasi-equation in (4.6) is valid in K, we have

$$\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n \vDash_{\mathsf{K}} \varepsilon \approx \delta.$$

By condition (Alg2) in the definition of an algebraizable logic, this implies

$$\Delta(\varphi_1, \psi_1) \cup \dots \cup \Delta(\varphi_n, \psi_n) \vdash \Delta(\varepsilon, \delta).$$

Together with the assumption that F is a deductive filter of $\vdash$ on $\boldsymbol{A}$ and (4.7), this yields

$$\Delta^{\boldsymbol{A}}(g(\varepsilon), g(\delta)) = g[\Delta(\varepsilon, \delta)] \subseteq F.$$

By definition of $\boldsymbol{\Omega}^{\boldsymbol{A}}F$, this yields $\langle g(\varepsilon), g(\delta)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$. Hence, we conclude that

$$f(\varepsilon) = \pi \circ g(\varepsilon) = g(\varepsilon)/\boldsymbol{\Omega}^{\boldsymbol{A}}F = g(\delta)/\boldsymbol{\Omega}^{\boldsymbol{A}}F = \pi \circ g(\delta) = f(\delta).$$

This shows that $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$ satisfies the quasi-equation in (4.6) and, therefore, that $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F \in \mathsf{K}$. We conclude that the map $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^+ \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^+$ is well defined.

A similar argument shows that $\mathsf{S}^{\boldsymbol{A}}\colon \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^+ \to \mathsf{Fi}_{\vdash}(\boldsymbol{A})^+$ is also well defined. To conclude the proof, it only remains to show that

$$\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^+ \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^+ \text{ and } \mathsf{S}^{\boldsymbol{A}}\colon \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^+ \to \mathsf{Fi}_{\vdash}(\boldsymbol{A})^+ \tag{4.8}$$

are isomorphisms, one inverse to the other. To this end, consider $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$. We have

$$\begin{aligned}\mathsf{S}^{\boldsymbol{A}}(\boldsymbol{\Omega}^{\boldsymbol{A}}F) &= \{a \in A : \langle \varphi(a), \psi(a)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F, \text{for all } \varphi \approx \psi \in \boldsymbol{\tau}\}\\ &= \{a \in A : \Delta^{\boldsymbol{A}}(\varphi(a), \psi(a)) \subseteq F, \text{for all } \varphi \approx \psi \in \boldsymbol{\tau}\}\\ &= \{a \in A : \Delta^{\boldsymbol{A}}[\boldsymbol{\tau}^{\boldsymbol{A}}(a)] \subseteq F\}\\ &= F.\end{aligned}$$

The first and the third equalities above are straightforward, the second follows from the definition of $\boldsymbol{\Omega}^{\boldsymbol{A}}F$, and the fourth follows from condition (Alg3) in the definition of an algebraizable logic and the fact that F is a deductive filter of $\vdash$. A similar argument shows that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}(\mathsf{S}^{\boldsymbol{A}}(\theta))$, for every $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Hence, we conclude that that the maps in (4.8) are bijections, one inverse to the other.

It is straightforward to see that they are order preserving. Then consider $F, G \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$. Since $\mathsf{S}^{\boldsymbol{A}}$ is order preserving, we obtain $\mathsf{S}^{\boldsymbol{A}}(\boldsymbol{\Omega}^{\boldsymbol{A}}F) \subseteq \mathsf{S}^{\boldsymbol{A}}(\boldsymbol{\Omega}^{\boldsymbol{A}}G)$. As $\boldsymbol{\Omega}^{\boldsymbol{A}}$ and $\mathsf{S}^{\boldsymbol{A}}$ are one inverse to the other, we conclude that

$$F = \mathsf{S}^{\boldsymbol{A}}(\boldsymbol{\Omega}^{\boldsymbol{A}}F) \subseteq \mathsf{S}^{\boldsymbol{A}}(\boldsymbol{\Omega}^{\boldsymbol{A}}G) = G.$$

It follows that $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ is also order reflecting and, therefore, an order embedding. As it is surjective, we conclude that it is a lattice isomorphism. To prove that it commutes with inverse endomorphism, consider $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$, a pair $a, c \in A$ and $\sigma \in \mathsf{End}(\boldsymbol{A})$. We have

$$\begin{aligned} \langle a, c\rangle \in \sigma^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}}F] &\iff \langle \sigma(a), \sigma(c)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F \\ &\iff \Delta^{\boldsymbol{A}}(\sigma(a), \sigma(c)) \subseteq F \\ &\iff \sigma[\Delta^{\boldsymbol{A}}(a, c)] \subseteq F \\ &\iff \Delta^{\boldsymbol{A}}(a, c) \subseteq \sigma^{-1}[F] \\ &\iff \langle a, c\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\sigma^{-1}[F]. \end{aligned}$$

The first and the fourth equivalences above are straightforward, the second holds by definition of $\boldsymbol{\Omega}^{\boldsymbol{A}}F$, and the third because σ is an endomorphism. The last equivalence follows from the definition of $\boldsymbol{\Omega}^{\boldsymbol{A}}$ too, because $\sigma^{-1}[F]$ is a deductive filter, by Lemma 4.27. Hence, we conclude that $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ is an isomorphism and, therefore, that $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \cong \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$.

(ii)⇒(iii): In view of condition (4.3), when $\boldsymbol{A} = \boldsymbol{T}(Var)$, we obtain $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} = \mathcal{T}h(\vdash)^{+}$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+} = \mathcal{T}h(\vDash_{\mathsf{K}})^{+}$. Therefore, condition (iii) is a special instance of (ii).

(iii)⇒(i): Consider an isomorphism $\Phi \colon \mathcal{T}h(\vdash)^{+} \to \mathcal{T}h(\vDash_{\mathsf{K}})^{+}$. Let also

$$\mathrm{Cn}_{\vdash} \colon \mathcal{P}(T(Var)) \to \mathcal{P}(T(Var)) \text{ and } \mathrm{Cn}_{\mathsf{K}} \colon \mathcal{P}(E(Var)) \to \mathcal{P}(E(Var))$$

be the closure operators associated with the consequence relations $\vdash$ and $\vDash_{\mathsf{K}}$ (equiv. with the closure systems $\mathcal{T}h(\vdash)$ and $\mathcal{T}h(\vDash_{\mathsf{K}})$), respectively. The proof proceeds through a series of technical claims.

Claim 4.34. *For every set of formula Γ, set of equations Θ and substitution σ,*

$$\begin{aligned} \Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma])) &= \mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}\Gamma)]) \\ \Phi^{-1}[\mathrm{Cn}_{\mathsf{K}}(\sigma[\Theta])] &= \mathrm{Cn}_{\vdash}(\sigma[\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\Theta))]). \end{aligned}$$

Proof of the Claim. We detail the proof of the first equality only, as that of the second is analogous. First, since Cn_{K} is a closure operator, we have

$$\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))] \subseteq \mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))])$$

and, therefore,

$$\Phi(\mathrm{Cn}_{\vdash}(\Gamma)) \subseteq \sigma^{-1}[\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))])]. \tag{4.9}$$

We will prove that the two sets in the above display are theories of $\vDash_{\mathsf{K}}$. For $\Phi(\mathrm{Cn}_{\vdash}(\Gamma))$ this is a consequence of the fact that $\mathrm{Cn}_{\vdash}(\Gamma) \in \mathcal{T}h(\vdash)$ and Φ sends theories of $\vdash$ to theories of $\vDash_{\mathsf{K}}$. To prove that $\sigma^{-1}[\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))])]$ is also a theory of $\vDash_{\mathsf{K}}$, observe that $\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))]) \in \mathcal{T}h(\vDash_{\mathsf{K}})$. Since $\mathcal{T}h(\vDash_{\mathsf{K}})$ is closed under inverse substitutions (as the structure $\mathcal{T}h(\vDash_{\mathsf{K}})^{+}$ is well defined), this yields the desired result.

Since the two sets in(4.9) are theories of $\vDash_{\mathsf{K}}$ and $\Phi^{-1} \colon \mathcal{T}h(\vDash_{\mathsf{K}})^{+} \to \mathcal{T}h(\vdash)^{+}$ is also an isomorphism, from (4.9) it follows

$$\begin{aligned}\mathrm{Cn}_{\vdash}(\Gamma) &= \Phi^{-1}\Phi(\mathrm{Cn}_{\vdash}(\Gamma))\\ &\subseteq \Phi^{-1}(\sigma^{-1}[\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))])])\\ &= \sigma^{-1}[\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))]))].\end{aligned}$$

Consequently,

$$\sigma[\mathrm{Cn}_{\vdash}(\Gamma)] \subseteq \Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))])).$$

Notice that the right hand side of the above display belongs to $\mathcal{T}h(\vdash)$, because Φ^{-1} sends theories of $\vDash_{\mathsf{K}}$ to theories of $\vdash$. Therefore, we obtain

$$\mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\Gamma)]) \subseteq \Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))])). \tag{4.10}$$

Lastly,

$$\begin{aligned}\Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma])) &\subseteq \Phi(\mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\Gamma)]))\\ &\subseteq \Phi\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))]))\\ &= \mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))]).\end{aligned}$$

The inclusions above are justified as follows. To prove the first, notice that $\sigma[\Gamma] \subseteq \sigma[\mathrm{Cn}_{\vdash}(\Gamma)]$. Since both $\mathrm{Cn}_{\vdash}$ and Φ are order preserving, this yields $\Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma])) \subseteq \Phi(\mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\Gamma)]))$, as desired. The second inclusion follows from (4.10) and the fact that Φ is order preserving, while the last equality follows from the fact that Φ is a bijection.

This establishes the left to right inclusion of the first equality in the statement of the Claim. To prove the other inclusion, observe that $\sigma[\Gamma] \subseteq \mathrm{Cn}_{\vdash}(\sigma[\Gamma])$ and, therefore, $\Gamma \subseteq \sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])]$. This, in turn, yields $\mathrm{Cn}_{\vdash}(\Gamma) \subseteq \mathrm{Cn}_{\vdash}(\sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])])$. As $\mathcal{T}h(\vdash)$ is closed under inverse substitutions (because $\mathcal{T}h(\vdash)^{+}$ is well defined), we obtain that $\sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])]$ is also a theory of $\vdash$. Consequently,

$$\mathrm{Cn}_{\vdash}(\Gamma) \subseteq \mathrm{Cn}_{\vdash}(\sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])]) = \sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])].$$

As at the left and right hand sides of the above displays we have two theories of $\vdash$ and Φ is order preserving and commutes with inverse substitutions, we obtain

$$\Phi(\mathrm{Cn}_{\vdash}(\Gamma)) \subseteq \Phi(\sigma^{-1}[\mathrm{Cn}_{\vdash}(\sigma[\Gamma])]) = \sigma^{-1}[\Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma]))].$$

Consequently, $\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))] \subseteq \Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma]))$ and, therefore,

$$\mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\Gamma))]) \subseteq \mathrm{Cn}_{\mathsf{K}}(\Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma]))) = \Phi(\mathrm{Cn}_{\vdash}(\sigma[\Gamma])),$$

where the last equality follows from the assumption that Φ sends theories of $\vdash$ to theories of $\vDash_{\mathsf{K}}$. This establishes the right to left inclusion of the first equality in the statement of the Claim. $\boxtimes$

We will rely on the following formulation of substitution invariance.

Claim 4.35. *For every set of formulas Γ, set of equations Θ and substitution σ,*

$$\mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\Gamma)]) = \mathrm{Cn}_{\vdash}(\sigma[\Gamma]) \ \textit{and} \ \mathrm{Cn}_{\mathsf{K}}(\sigma[\mathrm{Cn}_{\mathsf{K}}(\Theta)]) = \mathrm{Cn}_{\mathsf{K}}(\sigma[\Theta]).$$

Proof of the Claim. As before, we detail the proof of the first equality only. As $\Gamma \subseteq \mathrm{Cn}_{\vdash}(\Gamma)$, we have $\sigma[\Gamma] \subseteq \sigma[\mathrm{Cn}_{\vdash}(\Gamma)]$ and, therefore, $\mathrm{Cn}_{\vdash}(\sigma[\Gamma]) \subseteq \mathrm{Cn}_{\vdash}(\sigma[\mathrm{Cn}_{\vdash}(\Gamma)])$. To prove the reverse inclusion, it suffices to show that $\sigma[\mathrm{Cn}_{\vdash}(\Gamma)] \subseteq \mathrm{Cn}_{\vdash}(\sigma[\Gamma])$. To this end, consider $\varphi \in \sigma[\mathrm{Cn}_{\vdash}(\Gamma)]$. Then there exists $\psi \in \mathrm{Cn}_{\vdash}(\Gamma)$ such that $\varphi = \sigma(\psi)$. Moreover, $\Gamma \vdash \psi$. As $\vdash$ is substitution invariant, this yields $\sigma[\Gamma] \vdash \sigma(\psi)$, that is, $\sigma[\Gamma] \vdash \varphi$. Hence, we conclude that $\varphi \in \mathrm{Cn}_{\vdash}(\sigma[\Gamma])$. $\boxtimes$

Now, let σ_x (resp. $\sigma_{x,y}$) be the substitution that sends all variables to x (resp. all variables other than y to x, and leaves y untouched). We define a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Delta(x,y)$ as follows:

$$\boldsymbol{\tau}(x) := \sigma_x[\Phi(\mathrm{Cn}_{\vdash}(\{x\}))] \text{ and } \Delta(x,y) := \sigma_{x,y}[\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\}))].$$

Our aim is to prove that $\boldsymbol{\tau}$, Δ and K witness the algebraizability of $\vdash$.

Claim 4.36. *For every formula φ,*

$$\Phi(\mathrm{Cn}_{\vdash}(\{\varphi\})) = \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\varphi)).$$

Proof of the Claim. From Claim 4.34 it follows

$$\Phi(\mathrm{Cn}_{\vdash}(\{x\})) = \Phi(\mathrm{Cn}_{\vdash}(\{\sigma_x(x)\})) = \mathrm{Cn}_{\mathsf{K}}(\sigma_x[\Phi(\mathrm{Cn}_{\vdash}(\{x\}))]) = \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(x)). \tag{4.11}$$

Now, consider a formula φ and let σ be any substitution such that $\sigma(x) = \varphi$. We have

$$\begin{aligned}
\Phi(\mathrm{Cn}_{\vdash}(\{\varphi\})) &= \Phi(\mathrm{Cn}_{\vdash}(\{\sigma(x)\})) \\
&= \mathrm{Cn}_{\mathsf{K}}(\sigma[\Phi(\mathrm{Cn}_{\vdash}(\{x\}))]) \\
&= \mathrm{Cn}_{\mathsf{K}}(\sigma[\mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(x))]) \\
&= \mathrm{Cn}_{\mathsf{K}}(\sigma[\boldsymbol{\tau}(x)]) \\
&= \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\varphi)).
\end{aligned}$$

The above equalities are justified as follows. The first holds by the definition of σ, the second follows from Claim 4.34, the third from (4.11), the fourth from Claim 4.35, and the last one from the definition of σ. $\boxtimes$

Claim 4.36 can be extended to sets of formulas as follows.

Claim 4.37. *For every set of formulas Γ,*

$$\Phi(\mathrm{Cn}_{\vdash}(\Gamma)) = \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}[\Gamma]).$$

Proof of the Claim. We have

$$\Phi(\mathrm{Cn}_{\vdash}(\Gamma)) = \Phi(\bigvee_{\gamma\in\Gamma}^{\mathcal{T}h(\vdash)} \mathrm{Cn}_{\vdash}(\{\gamma\})) = \bigvee_{\gamma\in\Gamma}^{\mathcal{T}h(\vDash_{\mathsf{K}})} \Phi(\mathrm{Cn}_{\vdash}(\{\gamma\})) = \bigvee_{\gamma\in\Gamma}^{\mathcal{T}h(\vDash_{\mathsf{K}})} \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\gamma)).$$

The first and the equality above follows from the basic properties of closure operators, the second from the assumption that $\Phi\colon \mathcal{T}h(\vdash)^{+} \to \mathcal{T}h(\vDash_{\mathsf{K}})^{+}$ is an isomorphism (and, therefore, preserves arbitrary joins), and the third from Claim 4.36.

Furthermore, we have

$$\bigvee_{\gamma\in\Gamma}^{\mathcal{T}h(\vDash_{\mathsf{K}})} \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\gamma)) = \mathrm{Cn}_{\mathsf{K}}(\bigcup_{\gamma\in\Gamma} \boldsymbol{\tau}(\gamma)) = \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}[\Gamma]).$$

The first equality above follows from the description of joins in closure systems and the second from the definition of $\boldsymbol{\tau}[\Gamma]$. ⊠

In view of Proposition 4.3, in order to prove that $\boldsymbol{\tau}, \Delta$ and K witness the algebraizability of $\vdash$, it suffices to check that conditions (Alg1) and (Alg4*) hold. To prove (Alg1), consider a set of formulas $\Gamma \cup \{\varphi\}$. Applying the fact that Φ is an order isomorphism and Claim 4.37, we obtain

$$\begin{aligned}\Gamma \vdash \varphi &\Longleftrightarrow \mathrm{Cn}_{\vdash}(\{\varphi\}) \subseteq \mathrm{Cn}_{\vdash}(\Gamma) \Longleftrightarrow \Phi(\mathrm{Cn}_{\vdash}(\{\varphi\})) \subseteq \Phi(\mathrm{Cn}_{\vdash}(\Gamma))\\ &\Longleftrightarrow \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}(\varphi)) \subseteq \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}[\Gamma]) \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).\end{aligned}$$

This establishes (Alg1). To prove condition (Alg4*), observe that

$$\begin{aligned}\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\}) &= \mathrm{Cn}_{\mathsf{K}}(\sigma_{x,y}[\{x \approx y\}])\\ &= \mathrm{Cn}_{\mathsf{K}}(\sigma_{x,y}[\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\})])\\ &= \Phi\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\sigma_{x,y}[\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\})]))\\ &= \Phi(\mathrm{Cn}_{\vdash}(\sigma_{x,y}[\Phi^{-1}(\mathrm{Cn}_{\mathsf{K}}(\{x \approx y\}))]))\\ &= \Phi(\mathrm{Cn}_{\vdash}(\Delta(x,y))\\ &= \mathrm{Cn}_{\mathsf{K}}(\boldsymbol{\tau}[\Delta(x,y)]).\end{aligned}$$

The equalities above are justified as follows. The second follows from Claim 4.35, the fourth from Claim 4.34, and the sixth from Claim 4.37. From the above display it follows (Alg4*), thus concluding the proof. ⊠

Corollary 4.38. *Let $\vdash$ be an algebraizable logic whose algebraizability is witnessed by $\boldsymbol{\tau}, \Delta$ and K. For every algebra $\boldsymbol{A}$ of the suitable type, the map $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$, defined by the rule*

$$\boldsymbol{\Omega}^{\boldsymbol{A}} F \coloneqq \{\langle a,c\rangle \in A \times A : \Delta^{\boldsymbol{A}}(a,c) \subseteq F\},$$

is an isomorphism.

In algebraic logic, the map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ is known as the *Leibniz operator* [42, 55, 57, 115]. Its behavior influences the definability of logical equivalence [20, 38, 43, 74] and of truth predicates [72, 102, 113] in matrix semantics and, more in general, in equality free model theory [23, 32, 48, 51, 52].

Remark 4.39. The implication (ii)⇒(i) in the Isomorphism Theorem suggests the non-mathematical thesis that every correspondence between congruences and distinguished sets is induced by the algebraization of some logic. Large classes of varieties for which such a correspondence exists have been identified in [126, 3, 4, 5, 127] and [88], see also [2, 71]. However, in some of these cases, the isomorphism between congruences and distinguished sets does not commute with inverse substitutions. The notion of an algebraizable logic has been weakened in [46] to accommodate for these situations too. ⊠

Remark 4.40. At this stage, it is worth mentioning that algebraizable logics admit the following purely syntactic description [55, Thm. 3.21].

Theorem 4.41. *A finitary logic $\vdash$ is algebraizable if and only if there are a finite set of equations $\boldsymbol{\tau}$ and a finite set of formulas Δ such that, for every n-ary connective f,*

$$\begin{aligned} \emptyset &\vdash \Delta(x, x) \\ x, \Delta(x, y) &\vdash y \\ \Delta(x_1, y_1) \cup \dots \cup \Delta(x_n, y_n) &\vdash \Delta(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \\ x &\dashv\vdash \Delta[\boldsymbol{\tau}(x)]. \end{aligned}$$

In view of the Isomorphism Theorem, the above result identifies syntactic conditions equivalent to the existence of an isomorphism between lattices of K-congruences and deductive filters, for a suitable quasi-variety K. Similar results play a central role in universal algebra, where *Maltsev conditions* provide a syntactic description of structural properties of congruence lattices [9, 65, 85, 107, 125]. The connection with Maltsev conditions has been explored in [80, 81, 82]. ⊠

CHAPTER 5

Deduction theorems

5.1 Bridge theorem

For general information on deduction theorems in algebraic logic, the reader might consult [22, 26, 40, 41, 42, 45, 114].

Definition 5.1. A logic $\vdash$ has a *deduction-detachment theorem* (DDT) if there exists a finite set of formulas $I(x, y)$ such that for every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash \psi \iff \Gamma \vdash I(\varphi, \psi).$$

In this case, we say that $I(x, y)$ *witnesses* the DDT for $\vdash$.

It is well known that, for every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathbf{IPC}} \psi \iff \Gamma \vdash_{\mathbf{IPC}} \varphi \to \psi.$$

Therefore, **IPC** has a DDT witnessed by $I(x, y) := \{x \to y\}$. The same holds for $\mathbf{K}_\ell$ (see Exercise 2.23). On the other hand, we will prove that $\mathbf{K}_g$ lacks any DDT (Example 5.52).

Exercise 5.2. Prove that the DDT persists in axiomatic extensions. Conclude that all superintuitionistic logics and all axiomatic extensions of $\mathbf{K}_\ell$ have a DDT. On the other hand, the DDT does not persist in arbitrary (i.e., not necessarily axiomatic) extensions. ⊠

Let K be a quasi-variety and $\boldsymbol{A}$ an algebra. Recall from Corollary 3.40 that the closure operator of K-congruence generation on $\boldsymbol{A}$ is denoted by

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}} \colon \mathcal{P}(A \times A) \to \mathcal{P}(A \times A).$$

Furthermore, given $a, c \in A$, we abbreviate $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\{\langle a, c\rangle\})$ as $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a, c)$. Accordingly, the K-congruences of $\boldsymbol{A}$ of the form $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a, c)$ will be called *principal*.

When K is a variety and $\boldsymbol{A} \in \mathsf{K}$, we drop the subscript K in $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}$, because $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(X)$ is the least congruence of $\boldsymbol{A}$ extending X.

Definition 5.3. A quasi-variety K is said to have *equationally definable principal relative congruences* (EDPRC) when there exists a finite set of equations $\Phi(x, y, z, v)$ such that, for every $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) \iff \boldsymbol{A} \vDash \Phi(c, d, a, b).$$

In this case, we say that Φ *witnesses* EDPRC for K. When K is a variety, it is common to use the expression *equationally definable principal congruences* (EDPC), as opposed to (EDPRC).

This notion originates in [8] and was studied in the series of papers [19, 18, 24, 25, 87].

Remark 5.4. EDPRC persists in relative subvarieties, but not necessarily in subquasi-varieties (cf. Exercise 5.2). ⊠

Exercise 5.5. Prove that, for every Heyting algebra $\boldsymbol{A}$ and $c, d \in A$,

$$\mathrm{Cg}^{\boldsymbol{A}}(c, d) = \{\langle a, b\rangle \in A \times A : c \leftrightarrow d \leqslant a \leftrightarrow b\}.$$

Use this fact to infer that the set of equations

$$\Phi(x, y, z, v) := \{x \leftrightarrow y \leqslant z \leftrightarrow v\}$$

witnesses EDPC for the variety of Heyting algebras. As EDPC persists in subvarieties, this shows that every variety of Heyting algebras has EDPC. ⊠

Exercise 5.6. The case of modal algebras is more complex. For every $n \in \mathbb{N}$, we define

$$\boxplus^n x := x \wedge \Box x \wedge \Box\Box x \wedge \cdots \wedge \Box^n x.$$

Prove that, for every modal algebra $\boldsymbol{A}$ and $c, d \in A$,

$$\mathrm{Cg}^{\boldsymbol{A}}(c, d) = \{\langle a, b\rangle \in A \times A : \text{there exists } n \in \mathbb{N} \text{ such that } \boxplus^n (c \leftrightarrow d) \leqslant a \leftrightarrow b\}.$$

A variety K of modal algebras is said to be *weakly transitive* if

$$\mathsf{K} \vDash \boxplus^n x \leqslant \Box^{n+1} x, \text{ for some } n \in \mathbb{N}.$$

Prove that every weakly transitive variety of modal algebras has EDPC. Hint: use sets of equations of the form

$$\Phi(x, y, z, v) := \{\boxplus^n (x \leftrightarrow y) \leqslant z \leftrightarrow v\}.$$

As shown in Theorem 5.55, the converse is also true, whence the varieties of modal algebras with EDPC are precisely the weakly transitive ones. ⊠

Our aim is to prove the following bridge theorem, connecting the DDT on the logic side with EDPRC on the algebra side [26].

Theorem 5.7 (Blok & Pigozzi)**.** *An algebraizable logic has a DDT if and only if its equivalent algebraic semantics has EDPRC.*

To this end, it is convenient to extend the definition of a DDT to relative equational consequences.

Definition 5.8. Given a quasi-variety K, we say that $\vDash_{\mathsf{K}}$ has a *deduction-detachment theorem* (DDT) if there exists a finite set of equations $\Phi(x, y, z, v)$ such that, for every $\Theta \cup \{\varphi \approx \psi, \varepsilon \approx \delta\} \subseteq E(Var)$,

$$\Theta, \varphi \approx \psi \vDash_{\mathsf{K}} \varepsilon \approx \delta \Longleftrightarrow \Theta \vDash_{\mathsf{K}} \Phi(\varphi, \psi, \varepsilon, \delta).$$

In this case, we say that Ψ *witnesses* the DDT for $\vDash_{\mathsf{K}}$.

Proposition 5.9. *An algebraizable logic $\vdash$ has a DDT if and only if the equational consequence $\vDash_{\mathsf{K}}$ relative to its equivalent algebraic semantics K has one.*

Proof. We shall detail only the proof of the "only if" part, as the other one is analogous. To this end, assume that $\vdash$ has a DDT witnessed by a finite set of formulas $I(x, y)$. For every $n \in \mathbb{N}$, we define a set $I_n(x_1, \dots, x_n, y)$ recursively by the following rule:

$$I_0(y) := \{y\}$$
$$I_{k+1}(x_1, \dots, x_{k+1}, y) := \bigcup\{I(x_1, \varphi) \colon \varphi \in I_k(x_2, \dots, x_{k+1}, y)\}.$$

A straightforward inductive argument shows that for every $n \in \mathbb{N}$ and every set of formulas $\Gamma \cup \{\psi\} \cup \{\varphi_i \colon i < n\}$,

$$\Gamma \cup \{\varphi_i \colon i < n\} \vdash \psi \Longleftrightarrow \Gamma \vdash I_n(\varphi_0, \dots, \varphi_{n-1}, \psi). \tag{5.1}$$

As $\vdash$ is algebraizable, there are a finite set of equations $\boldsymbol{\tau}(x)$ and a finite set of formulas $\Delta(x, y)$ such that the triple$\boldsymbol{\tau}, \Delta$ and K witnesses the algebraizability of $\vdash$. As Δ is finite, it has the form $\{\delta_i \colon i < n\}$ for some $n \in \mathbb{N}$. Then, consider the following set of equations

$$\Phi(x, y, z, v) := \bigcup_{i<n} \boldsymbol{\tau}[I_n(\delta_0(x, y), \dots, \delta_{n-1}(x, y), \delta_i(z, v))].$$

Observe that Φ is finite, as so are Δ and $\boldsymbol{\tau}$. We shall see that Φ witnesses a DDT for $\vDash_{\mathsf{K}}$. To this end, consider a set of equations $\Theta \cup \{\varphi \approx \psi, \alpha \approx \beta\} \subseteq E(Var)$. We have

$$\begin{aligned}
\Theta, \varphi \approx \psi \vDash_{\mathsf{K}} \alpha \approx \beta &\Longleftrightarrow \Delta[\Theta], \Delta(\varphi, \psi) \vdash \Delta(\alpha, \beta) \\
&\Longleftrightarrow \Delta[\Theta] \cup \{\delta_i(\varphi, \psi) \colon i < n\} \vdash \delta_j(\alpha, \beta), \text{ for all } j < n \\
&\Longleftrightarrow \Delta[\Theta] \vdash \bigcup_{i<n} I_n(\delta_0(\varphi, \psi), \dots, \delta_{n-1}(\varphi, \psi), \delta_i(\alpha, \beta)) \\
&\Longleftrightarrow \boldsymbol{\tau}[\Delta[\Theta]] \vDash_{\mathsf{K}} \bigcup_{i<n} \boldsymbol{\tau}[I_n(\delta_0(\varphi, \psi), \dots, \delta_{n-1}(\varphi, \psi), \delta_i(\alpha, \beta))] \\
&\Longleftrightarrow \Theta \vDash_{\mathsf{K}} \bigcup_{i<n} \boldsymbol{\tau}[I_n(\delta_0(\varphi, \psi), \dots, \delta_{n-1}(\varphi, \psi), \delta_i(\alpha, \beta))] \\
&\Longleftrightarrow \Theta \vDash_{\mathsf{K}} \Phi(\varphi, \psi, \alpha, \beta).
\end{aligned}$$

The above equivalences justified as follows: the first follows from (Alg2), the second is obvious, the third holds because I witnesses a DDT for $\vdash$, the fourth follows from (Alg1), the fifth from (Alg4) and the last one from the definition of Φ. Hence, we conclude that Φ witnesses a DDT for $\vDash_{\mathsf{K}}$, as desired. ⊠

Accordingly, in order to establish Theorem 5.7, it suffices to prove the following.

Proposition 5.10. *A quasi-variety* K *has EDPRC if and only if* $\vDash_{\mathsf{K}}$ *has a DDT.*

To this end, we rely on two technical lemmas.

Lemma 5.11. *Let* K *be a quasi-variety,* $\boldsymbol{A}$ *an algebra and* $a, c \in A$*. Then*

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,c) = \bigcup\{\mathrm{Cg}^{\boldsymbol{B}}_{\mathsf{K}}(a,c) \colon \boldsymbol{B} \in \mathbb{S}(\boldsymbol{A}) \text{ is finitely generated and } a,c \in B\}.$$

Proof. The inclusion from right to left follows from the fact that if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and θ a K-congruence of $\boldsymbol{A}$, then $\theta \cap (B \times B)$ is a K-congruence of $\boldsymbol{B}$. This observation will be used without further notice in the proof.

In order to prove the inclusion from left to right, consider the relation

$$\theta := \bigcup\{\mathrm{Cg}^{\boldsymbol{B}}_{\mathsf{K}}(a,c) \colon \boldsymbol{B} \in \mathbb{S}(\boldsymbol{A}) \text{ is finitely generated and } a,c \in B\}.$$

It is easy to see that θ is a congruence of $\boldsymbol{A}$ containing the pair $\langle a, c\rangle$. In order to check that θ is also a K-congruence of $\boldsymbol{A}$, consider a quasi-equation

$$\mathop{\&}_{i \leqslant n} \varphi_i \approx \psi_i \Longrightarrow \varepsilon \approx \delta \tag{5.2}$$

valid in K and a homomorphism $f \colon \boldsymbol{T}(Var) \to \boldsymbol{A}/\theta$ such that $f(\varphi_i) = f(\psi_i)$, for all $i \leqslant n$. Moreover, let $\pi \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ be the canonical surjection. By Corollary 1.6, there exists a homomorphism $g \colon \boldsymbol{T}(Var) \to \boldsymbol{A}$ such that $f = \pi \circ g$.

For every $i \leqslant n$, we have

$$g(\varphi_i)/\theta = \pi(g(\varphi_i)) = f(\varphi_i) = f(\psi_i) = \pi(g(\psi_i)) = g(\psi_i)/\theta$$

and, therefore, $\langle g(\varphi_i), g(\psi_i)\rangle \in \theta$. By definition of θ, for every $i \leqslant n$ there exists a finite subset X_i of A such that $\langle g(\varphi_i), g(\psi_i)\rangle \in \mathrm{Cg}^{\boldsymbol{B}_i}_{\mathsf{K}}(a,c)$, where $\boldsymbol{B}_i$ is the subalgebra of $\boldsymbol{A}$ generated by X_i. As the definition of θ requires $\boldsymbol{B}_i$ to contain a and c, we may assume that $a, c \in X_i$.

Let then $\{x_1, \dots, x_m\}$ be the set of variables occurring in the quasi-equation in (5.2) and $\boldsymbol{C}$ the subalgebra of $\boldsymbol{A}$ generated by

$$X_1 \cup \dots \cup X_n \cup \{g(x_1), \dots, g(x_m)\}.$$

Clearly, $\boldsymbol{C}$ is finitely generated, it contains a, c, and

$$\boldsymbol{B}_1, \dots, \boldsymbol{B}_n \in \mathbb{S}(\boldsymbol{C}).$$

Since $\mathrm{Cg}^{\boldsymbol{B}_i}_{\mathsf{K}}(a,c) \subseteq \mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c)$ for every $i \leqslant n$, we have

$$\langle g(\varphi_i), g(\psi_i)\rangle \in \mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c), \text{ for all } i \leqslant n.$$

Consider a homomorphism $h \colon \boldsymbol{T}(Var) \to \boldsymbol{C}$ such that

$$h(x_1) = g(x_1), \dots, h(x_m) = g(x_m).$$

This is possible because $g(x_1), \dots, g(x_m) \in C$, by definition of $\boldsymbol{C}$. Let also $p \colon \boldsymbol{C} \to \boldsymbol{C}/\mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c)$ be the canonical surjection. For every $i \leqslant n$, we have

$$p(h(\varphi_i)) = p(g(\varphi_i)) = g(\varphi_i)/\mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c) = g(\psi_i)/\mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c) = p(g(\psi_i)) = p(h(\psi_i)).$$

Since $\mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c)$ is a K-congruence of $\boldsymbol{C}$, the above display implies $p(h(\varepsilon)) = p(h(\delta))$. In turn, this guarantees that

$$g(\varepsilon)/\mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c) = p(g(\varepsilon)) = p(h(\varepsilon)) = p(h(\delta)) = p(h(\delta)) = g(\delta)/\mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c).$$

As a consequence, $\langle g(\varepsilon), g(\delta)\rangle \in \mathrm{Cg}^{\boldsymbol{C}}_{\mathsf{K}}(a,c) \subseteq \theta$. Hence, we conclude that

$$f(\varepsilon) = \pi(g(\varepsilon)) = g(\varepsilon)/\theta = g(\delta)/\theta = \pi(g(\delta)) = f(\delta).$$

It follows that θ is a K-congruence of $\boldsymbol{A}$ containing $\langle a,c\rangle$, whence $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,c) \subseteq \theta$. ⊠

Lemma 5.12. *Let* K *be a quasi-variety and* $\boldsymbol{A}$ *and algebra. If* $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *and* $a,b \in A$*, then*

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle a,b\rangle\}) = \{\langle c,d\rangle \in A \times A : \langle c/\theta, d/\theta\rangle \in \mathrm{Cg}^{\boldsymbol{A}/\theta}_{\mathsf{K}}(a/\theta, b/\theta)\}.$$

Proof. We define

$$\phi := \{\langle c,d\rangle \in A \times A : \langle c/\theta, d/\theta\rangle \in \mathrm{Cg}^{\boldsymbol{A}/\theta}_{\mathsf{K}}(a/\theta, b/\theta)\}.$$

The proof that ϕ is a K-congruence of $\boldsymbol{A}$ is left to you as an exercise.

Now, the definition of ϕ guarantees that $\theta \cup \{\langle a,b\rangle\} \subseteq \phi$. Since ϕ is a K-congruence of $\boldsymbol{A}$, this implies $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle a,b\rangle\}) \subseteq \phi$. To prove the reverse inclusion, we reason by contraposition. Accordingly, consider a pair $c,d \in A$ such that $\langle c,d\rangle \notin \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle a,b\rangle\})$. Then let $f \colon \boldsymbol{A}/\theta \to \boldsymbol{A}/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle a,b\rangle\})$ be the map defined by the rule

$$f(e/\theta) := e/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle a,b\rangle\}).$$

Notice that f is well defined homomorphism, because $\theta \subseteq \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle a,b\rangle\})$. Together with $\boldsymbol{A}/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle a,b\rangle\}) \in \mathsf{K}$, this implies that $\mathsf{Ker}(f)$ is a K-congruence of $\boldsymbol{A}/\theta$. Furthermore,

$$\langle a/\theta, b/\theta\rangle \in \mathsf{Ker}(f) \;\text{ and }\; \langle c/\theta, d/\theta\rangle \notin \mathsf{Ker}(f),$$

because $\langle a,b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle a,b\rangle\})$ and $\langle c,d\rangle \notin \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle a,b\rangle\})$. In view of the above display, we obtain

$$\langle c/\theta, d/\theta\rangle \notin \mathrm{Cg}^{\boldsymbol{A}/\theta}_{\mathsf{K}}(a/\theta, b/\theta)$$

and, therefore, $\langle c,d\rangle \notin \phi$. Hence, we conclude that $\phi \subseteq \mathrm{Cg}^{\boldsymbol{A}/\theta}_{\mathsf{K}}(a/\theta, b/\theta)$, as desired. ⊠

Exercise 5.13. Complete the proof of the above lemma by showing that the relation ϕ is indeed a K-congruence of $\boldsymbol{A}$. ⊠

We are now ready to prove Proposition 5.10.

Proof. In order to prove the "if" part, suppose that $\vDash_{\mathsf{K}}$ has a DDT witnessed by a finite set of equations $\Phi(x, y, z, v)$. We shall see that Φ witnesses EDPRC for K. In view of Lemma 5.11, it suffices to show that for every *finitely generated* $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a, b\rangle \in \mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) \Longleftrightarrow \boldsymbol{A} \vDash \Phi(c, d, a, b). \tag{5.3}$$

Suppose first that $\langle a, b\rangle \in \mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$. Since $\boldsymbol{A}$ is finitely generated, we can choose some generators $e_1, \dots, e_n$ for it. Then there are φ_a, φ_b, φ_c and φ_d in variables $x_1, \dots, x_n$ such that

$$\begin{aligned}
\varphi_a^{\boldsymbol{A}}(e_1, \dots, e_n) &= a\\
\varphi_b^{\boldsymbol{A}}(e_1, \dots, e_n) &= b\\
\varphi_c^{\boldsymbol{A}}(e_1, \dots, e_n) &= c\\
\varphi_d^{\boldsymbol{A}}(e_1, \dots, e_n) &= d.
\end{aligned}$$

Furthermore, consider the set of equations

$$\Theta := \{\varepsilon(x_1, \dots, x_n) \approx \delta(x_1, \dots, x_n) \in E(Var) \colon \varepsilon^{\boldsymbol{A}}(e_1, \dots, e_n) = \delta^{\boldsymbol{A}}(e_1, \dots, e_n)\}.$$

We claim that

$$\Theta, \varphi_c \approx \varphi_d \vDash_{\mathsf{K}} \varphi_a \approx \varphi_b.$$

To prove this, consider an algebra $\boldsymbol{B} \in \mathsf{K}$ and elements $p_1, \dots, p_n \in B$ such that

$$\varepsilon^{\boldsymbol{B}}(p_1, \dots, p_n) = \delta^{\boldsymbol{B}}(p_1, \dots, p_n) \text{ and } \varphi_c^{\boldsymbol{B}}(p_1, \dots, p_n) = \varphi_d^{\boldsymbol{B}}(p_1, \dots, p_n), \tag{5.4}$$

for every $\varepsilon \approx \delta \in \Theta$. Moreover, let $f \colon A \to B$ be the map defined, for every formula $\gamma(x_1, \dots, x_n)$, by the rule

$$\gamma^{\boldsymbol{A}}(e_1, \dots, e_n) \longmapsto \gamma^{\boldsymbol{B}}(p_1, \dots, p_n).$$

From the left hand side of (5.4) and the fact that $\boldsymbol{A}$ is generated by $e_1, \dots, e_n$ it follows that f is a well-defined homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Therefore, $\mathsf{Ker}(f)$ is a congruence of $\boldsymbol{A}$. Furthermore, as $\boldsymbol{A}/\mathsf{Ker}(f) \in \mathbb{IS}(\boldsymbol{B})$ and $\boldsymbol{B} \in \mathsf{K}$, we obtain that $\mathsf{Ker}(f)$ is a K-congruence of $\boldsymbol{A}$. We have that

$$\begin{aligned}
f(a) &= f(\varphi_a^{\boldsymbol{A}}(e_1, \dots, e_n)) = \varphi_a^{\boldsymbol{B}}(p_1, \dots, p_n)\\
f(b) &= f(\varphi_b^{\boldsymbol{A}}(e_1, \dots, e_n)) = \varphi_b^{\boldsymbol{B}}(p_1, \dots, p_n)\\
f(c) &= f(\varphi_c^{\boldsymbol{A}}(e_1, \dots, e_n)) = \varphi_c^{\boldsymbol{B}}(p_1, \dots, p_n)\\
f(d) &= f(\varphi_d^{\boldsymbol{A}}(e_1, \dots, e_n)) = \varphi_d^{\boldsymbol{B}}(p_1, \dots, p_n).
\end{aligned}$$

Therefore, the right hand side of (5.4) guarantees that $\langle c, d\rangle \in \mathsf{Ker}(f)$. As a consequence, $\mathsf{Ker}(f)$ is a K-congruence of $\boldsymbol{A}$ containing the pair $\langle c, d\rangle$. It follows that $\mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) \subseteq \mathsf{Ker}(f)$. Together with the assumption that $\langle a, b\rangle \in \mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$, this implies

$$\varphi_a^{\boldsymbol{B}}(p_1, \dots, p_n) = f(a) = f(b) = \varphi_b^{\boldsymbol{B}}(p_1, \dots, p_n),$$

establishing the claim.

As Φ witnesses a DDT for $\vDash_{\mathsf{K}}$, the claim implies

$$\Theta \vDash_{\mathsf{K}} \Phi(\varphi_c, \varphi_d, \varphi_a, \varphi_b).$$

Now, the definition of Θ guarantees that $\varepsilon^{\boldsymbol{A}}(e_1, \dots, e_n) = \delta^{\boldsymbol{A}}(e_1, \dots, e_n)$, for every $\varepsilon \approx \delta \in \Theta$. Together with the above display, this yields

$$\boldsymbol{A} \vDash \Phi(\varphi_c(e_1, \dots, e_n), \varphi_d(e_1, \dots, e_n), \varphi_a(e_1, \dots, e_n), \varphi_b(e_1, \dots, e_n)),$$

i.e., $\boldsymbol{A} \vDash \Phi(c, d, a, b)$. This establishes the implication from left to right in (5.3).

To prove the other implication in (5.3), notice that from $\Phi(x, y, z, v) \vDash_{\mathsf{K}} \Phi(x, y, z, v)$ and the assumption that Φ witnesses a DDT for $\vDash_{\mathsf{K}}$, it follows

$$x \approx y, \Phi(x, y, z, v) \vDash_{\mathsf{K}} z \approx v. \tag{5.5}$$

Suppose that $\boldsymbol{A} \vDash \Phi(c, d, a, b)$. Then the set of premises of the above display is valid in $\boldsymbol{A}/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$ under the assignment

$$\begin{array}{ll} x \longmapsto c/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) & y \longmapsto d/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \\ z \longmapsto a/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) & v \longmapsto b/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d). \end{array}$$

Therefore, by (5.5), we obtain that $a/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) = b/\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$, i.e., $\langle a, b \rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$. As this establishes (5.3), we conclude that K has EDPRC.

To prove the "only if" part of the statement, suppose that K has EDPRC witnessed by set $\Phi(x, y, z, v)$. We shall see that Φ witnesses a DDT for $\vDash_{\mathsf{K}}$.

Recall from Proposition 4.26 that the closure systems $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var))$ and $\mathcal{T}h(\vDash_{\mathsf{K}})$ coincide. Accordingly, the closure operators

$$\mathrm{Cg}^{\boldsymbol{T}(Var)}_{\mathsf{K}} \colon \mathcal{P}(Eq) \to \mathcal{P}(Eq) \;\text{ and }\; \mathrm{Cn}_{\mathsf{K}} \colon \mathcal{P}(Eq) \to \mathcal{P}(Eq)$$

associated with them coincide too. Bearing this in mind, consider a set of equations $\Theta \cup \{\varphi \approx \psi, \varepsilon \approx \delta\}$ and define

$$\theta \coloneqq \mathrm{Cg}^{\boldsymbol{T}(Var)}_{\mathsf{K}}(\Theta).$$

Bearing in mind that equations are ordered pairs of formulas, we obtain

$$\begin{aligned}
\Theta, \varphi \approx \psi \vDash_{\mathsf{K}} \varepsilon \approx \delta &\iff \varepsilon \approx \delta \in \mathrm{Cn}_{\mathsf{K}}(\Theta \cup \{\varphi \approx \psi\}) \\
&\iff \langle \varepsilon, \delta \rangle \in \mathrm{Cg}^{\boldsymbol{T}(Var)}_{\mathsf{K}}(\Theta \cup \{\langle \varphi, \psi \rangle\}) \\
&\iff \langle \varepsilon, \delta \rangle \in \mathrm{Cg}^{\boldsymbol{T}(Var)}_{\mathsf{K}}(\theta \cup \{\langle \varphi, \psi \rangle\}) \\
&\iff \langle \varepsilon/\theta, \delta/\theta \rangle \in \mathrm{Cg}^{\boldsymbol{T}(Var)/\theta}_{\mathsf{K}}(\varphi/\theta, \psi/\theta) \\
&\iff \boldsymbol{T}(Var)/\theta \vDash \Phi(\varphi/\theta, \psi/\theta, \varepsilon/\theta, \delta/\theta) \\
&\iff \Phi(\varphi, \psi, \varepsilon, \delta) \subseteq \theta \\
&\iff \Phi(\varphi, \psi, \varepsilon, \delta) \subseteq \mathrm{Cg}^{\boldsymbol{T}(Var)}_{\mathsf{K}}(\Theta) \\
&\iff \Phi(\varphi, \psi, \varepsilon, \delta) \subseteq \mathrm{Cn}_{\mathsf{K}}(\Theta) \\
&\iff \Theta \vDash_{\mathsf{K}} \Phi(\varphi, \psi, \varepsilon, \delta).
\end{aligned}$$

The above equivalences are justified as follows. The first and the latter are obvious, the second and second to last follow from the the fact that the closure operators $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{T}(Var)}$ and Cn_{K} coincide, and the third and the third to last from the definition of θ and the fact that $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{T}(Var)}$ is a closure operator. Lastly, the fourth equality is a consequence of Lemma 5.12, while the fifth and the sixth are immediate applications of the definitions. ⊠

5.2 Quasi-varieties with EDPRC

The aim of this section is to prove that the demand that a quasi-variety K has EDPRC is equivalent to a purely order theoretic property of lattices of K-congruences.

Definition 5.14. Let $\boldsymbol{A}$ be a complete lattice.

(i) An element $a \in A$ is said to be *compact* if for every $X \subseteq A$,

$$\text{if } a \leqslant \bigvee X \text{, then there is a finite } Y \subseteq X \text{ such that } a \leqslant \bigvee Y.$$

(ii) $\boldsymbol{A}$ is said to be *algebraic* if every element is a join of compact elements.

We denote by $\mathsf{C}(\boldsymbol{A})$ the set of compact elements of $\boldsymbol{A}$.

In order to present canonical examples of algebraic lattices, recall that a *semilattice* is an algebra $\boldsymbol{A} = \langle A; * \rangle$ such that $*$ is a binary idempotent, commutative and associative operation. Every semilattice $\boldsymbol{A}$ can be associated with two partial orders on A, namely the *meet order* $\leqslant_m$ and the *join order* $\leqslant_j$, defined respectively by the following rules:

$$a \leqslant_m c \Longleftrightarrow a * c = a \quad \text{and} \quad a \leqslant_j c \Longleftrightarrow a * c = c.$$

Accordingly, we say that $\boldsymbol{A}$ is a *meet semilattice* (resp. *join semilattice*) when we give priority to the meet order (resp. join order), which, in this case, will be denoted simply by $\leqslant$.

Example 5.15 (Algebraic lattices)**.** Let $\boldsymbol{A}$ be a join semilattice with minimum element. A set $I \subseteq A$ is said to be an *ideal* of $\boldsymbol{A}$ if it is a nonempty downset such that if $a, c \in I$, then $a * c \in I$. The poset $\mathcal{I}(\boldsymbol{A})$ of ideals of $\boldsymbol{A}$ ordered under the inclusion relation is an algebraic lattice, whose compact elements are the principal downsets of $\boldsymbol{A}$. ⊠

We will prove that every algebraic lattice arises in this way (Theorem 5.17). To this end, we rely on the following immediate consequence of the definition of a compact element.

Proposition 5.16. *If $\boldsymbol{A}$ is a complete lattice and 0 its minimum element, then*

$$\textit{if } a, c \in \mathsf{C}(\boldsymbol{A}) \textit{, then } a \vee c \in \mathsf{C}(\boldsymbol{A}).$$

Consequently, $\mathsf{C}(\boldsymbol{A})$ can be viewed as a join semilattice with minimum $\langle \mathsf{C}(\boldsymbol{A}); \vee, 0 \rangle$, whose order coincides with the restriction of the order of $\boldsymbol{A}$ to $\mathsf{C}(\boldsymbol{A})$.

As a consequence, we obtain a representation theorem for algebraic lattices.

Theorem 5.17. *A poset is an algebraic lattice if and only if it is isomorphic to the lattice of ideals of a join semilattice with minimum.*

Proof. The "if" part is supplied in Example 5.15. To prove the "only if" part, consider an algebraic lattice $\boldsymbol{A}$. In view of Proposition 5.16, $\mathsf{C}(\boldsymbol{A})$ is a join semilattice with minimum. It is not hard to prove that the map $f \colon \boldsymbol{A} \to \mathcal{I}(\mathsf{C}(\boldsymbol{A}))$, defined by the rule

$$a \longmapsto \mathsf{C}(\boldsymbol{A}) \cap {\downarrow} a,$$

is an isomorphism. ⊠

Exercise 5.18. Complete the above proof by showing that f is indeed an isomorphism. ⊠

Algebraic lattices play a central role in algebra, partly because of the next observation.

Proposition 5.19. *Let* K *be a quasi-variety. If* $\boldsymbol{A}$ *is an algebra of the same type,* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *is an algebraic lattice, whose compact elements are precisely the finitely generated* K*-congruences of* $\boldsymbol{A}$*, i.e., the* K*-congruence of the form* $\mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(X)$ *for some finite* $X \subseteq A \times A$*.*

Remark 5.20. Notably, nothing more can be said about lattices of K-congruences in general, as every algebraic lattice is isomorphic to the congruence lattice of some algebra [69]. ⊠

Exercise 5.21. Prove Proposition 5.19. ⊠

The following semilattices are instrumental to characterize quasi-varieties with ED-PRC.

Definition 5.22. A *dually Brouwerian semilattice* is an algebra $\boldsymbol{A} = \langle A; \vee, \leftarrow \rangle$ such that $\langle A; \vee \rangle$ is a join semilattice and $\leftarrow$ a binary operation such that, for every $a, b, c \in A$,

$$c \leqslant a \vee b \Longleftrightarrow c \leftarrow b \leqslant a.$$

Typical examples of dually Brouwerian semilattices arise from Heyting algebras. More precisely, if $\boldsymbol{A}$ is a Heyting algebra with order $\leqslant$, the join order of the semilattice $\langle A; \wedge \rangle$ is the dual relation $\geqslant$. Because of this, from the residuation law it follows that $\langle A; \wedge, \to \rangle$ is a dually Brouwerian semilattice.

Our aim is to establish the following result [87].

Theorem 5.23 (Köhler & Pigozzi)**.** *A quasi-variety* K *has EDPRC if and only if, for every algebra* $\boldsymbol{A} \in \mathsf{K}$*, the join semilattice* $\mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}))$ *can be endowed with a binary operation* $\leftarrow$ *that turns it into a dually Brouwerian semilattice.*

In order to prove Theorem 5.23, we rely on the following observation.

Lemma 5.24. *Let* $\boldsymbol{A}$ *be a join semilattice generated by a set* $X \subseteq A$*. Suppose that for every* $b, c \in X$ *there exists an element* $c \dashleftarrow b \in A$ *such that for all* $a \in A$*,*

$$c \leqslant a * b \Longleftrightarrow c \dashleftarrow b \leqslant a.$$

Then there is a binary operation $\leftarrow$ *on* A *such that* $\langle A; *, \leftarrow \rangle$ *is a dually Brouwerian semilattice.*

Proof. As X generates $\boldsymbol{A}$, it suffices to show that, for every $b_1, \ldots, b_n \in X$ and $c \in A$, there exists an element $c \leftarrow (b_1 * \cdots * b_n) \in A$ such that, for all $a \in A$,

$$c \leqslant a * (b_1 * \cdots * b_n) \Longleftrightarrow c \leftarrow (b_1 * \cdots * b_n) \leqslant a.$$

We will prove this by induction on n.

For the base case, consider two elements $b \in X$ and $c \in A$. As X generates $\langle A; * \rangle$, there are $c_1, \ldots, c_n \in X$ such that $c = c_1 * \cdots * c_n$. We set

$$c \leftarrow b \coloneqq (c_1 \dashleftarrow b) * \cdots * (c_n \dashleftarrow b).$$

Observe that for all $a \in A$,

$$\begin{aligned} c \leqslant a * b &\Longleftrightarrow c_1 * \cdots * c_n \leqslant a * b \\ &\Longleftrightarrow c_i \leqslant a * b, \text{ for all } i \leqslant n \\ &\Longleftrightarrow c_i \dashleftarrow b \leqslant a, \text{ for all } i \leqslant n \\ &\Longleftrightarrow (c_1 \dashleftarrow b) * \cdots * (c_n \dashleftarrow b) \leqslant a \\ &\Longleftrightarrow c \leftarrow b \leqslant a. \end{aligned}$$

For the induction step, consider $b_1, \ldots, b_{k+1} \in X$ and $c \in A$. By the inductive hypothesis, we can define

$$c \leftarrow (b_1 * \cdots * b_{k+1}) \coloneqq (c \leftarrow (b_1 * \cdots * b_k)) \leftarrow b_{k+1}.$$

For every $a \in A$, we have

$$\begin{aligned} c \leqslant a * (b_1 * \cdots * b_{k+1}) &\Longleftrightarrow c \leqslant (a * b_{k+1}) * (b_1 * \cdots * b_k) \\ &\Longleftrightarrow c \leftarrow (b_1 * \cdots * b_k) \leqslant a * b_{k+1} \\ &\Longleftrightarrow (c \leftarrow (b_1 * \cdots * b_k)) \leftarrow b_{k+1} \leqslant a \\ &\Longleftrightarrow c \leftarrow (b_1 * \cdots * b_{k+1}) \leqslant a. \end{aligned}$$

The above equivalences can be justified as follows. The first holds because $*$ is idempotent and commutative, the second follows from the induction hypothesis for the case $n = k$, the third from the induction hypothesis for the case $n = 1$, and the last from the definition of $c \leftarrow (b_1 * \cdots * b_{k+1})$. ⊠

We are now ready to prove Theorem 5.23.

Proof. To prove the "only if" part, suppose that K has EDPRC and consider an algebra $\boldsymbol{A} \in \mathsf{K}$. Moreover, let $\vee$ be the join operation of the lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Recall from Proposition 5.16 that $\langle \mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})); \vee \rangle$ is a join semilattice, whose order is the inclusion relation. Furthermore, in view of Proposition 5.19, the elements of $\mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}))$ are precisely the finitely generated K-congruences of $\boldsymbol{A}$. Therefore, the semilattice $\langle \mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})); \vee \rangle$ is generated by the principal K-congruences of $\boldsymbol{A}$, because for every finite $X \subseteq A \times A$,

$$\mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(X) = \bigvee_{\langle a, c \rangle \in X} \mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a, c).$$

By Lemma 5.24, to conclude the proof it suffices to show that for every $a, b, c, d \in A$, there exists an element $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \leftarrow \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \in \mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}))$ such that for all $\theta \in \mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}))$,

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \subseteq \theta \vee \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \iff \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \leftarrow \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \subseteq \theta. \tag{5.6}$$

To this end, consider $a, b, c, d \in A$ and let $\Phi(x, y, z, v)$ be the finite set of equations witnessing EDPRC for K. We will work under the identification of $\Phi^{\boldsymbol{A}}(c, d, a, b)$ with the following subset of $A \times A$:

$$\{\langle \varphi^{\boldsymbol{A}}(c,d,a,b), \psi^{\boldsymbol{A}}(c,d,a,b)\rangle : \varphi \approx \psi \in \Phi\}.$$

Bearing this in mind, we define

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \leftarrow \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) := \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\Phi^{\boldsymbol{A}}(c,d,a,b)).$$

Since $\Phi^{\boldsymbol{A}}(c,d,a,b)$ is finite, $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \leftarrow \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d)$ is a finitely generated K-congruence of $\boldsymbol{A}$ and, therefore, a compact element of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, by Proposition 5.19. In brief,

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \leftarrow \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \in \mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})).$$

Now, for every $\theta \in \mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}))$, we have

$$\begin{aligned}
\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \subseteq \theta \vee \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) &\iff \langle a,b\rangle \in \theta \vee \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \\
&\iff \langle a,b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(\theta \cup \{\langle c,d\rangle\}) \\
&\iff \langle a/\theta, b/\theta\rangle \in \mathrm{Cg}^{\boldsymbol{A}/\theta}_{\mathsf{K}}(c/\theta, d/\theta) \\
&\iff \Phi^{\boldsymbol{A}}(c,d,a,b) \subseteq \theta \\
&\iff \mathrm{Cg}^{\boldsymbol{A}/\theta}_{\mathsf{K}}(\Phi^{\boldsymbol{A}}(c,d,a,b)) \subseteq \theta \\
&\iff \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \leftarrow \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \subseteq \theta.
\end{aligned}$$

The above equivalences are justified as follows. The first, second and fifth hold because $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}$ is a closure operator. The third is a consequence of Lemma 5.12, while the fourth holds because Φ witnesses EDPRC for K. Finally, the last one is a consequence of the definition of $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(a,b) \leftarrow \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d)$. This establishes (5.6), whence $\mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}))$ can be turned into a dually Brouwerian semilattice.

To prove the "if" part of the statement, let θ be the minimum of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var))$ and consider the algebra $\boldsymbol{F} := \boldsymbol{T}(Var)/\theta$. Clearly, $\boldsymbol{F} \in \mathsf{K}$. By assumption the join semilattice $\langle \mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{F})); \vee\rangle$ can be endowed with a binary operation $\leftarrow$ that turns it into a dually Brouwerian semilattice. As the K-congruences $\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x/\theta, y/\theta)$ and $\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z/\theta, v/\theta)$ are finitely generated, they belong to $\mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{F}))$, by Proposition 5.19. Thus,

$$\mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z/\theta, v/\theta) \leftarrow \mathrm{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x/\theta, y/\theta)$$

is a well-defined compact K-congruence of $\boldsymbol{F}$. By Proposition 5.19, it also finitely generated, i.e., there is a finite set of pairs of formulas

$$\Phi = \{\langle \varepsilon_i(x,y,z,v,w_1,\dots,w_n), \delta_i(x,y,z,v,w_1,\dots,w_n)\rangle : i < m\}$$

such that

$$\mathsf{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(z,v) \leftarrow \mathsf{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x,y) = \mathsf{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(\{\langle \varepsilon_i/\theta, \delta_i/\theta\rangle : i < m\}).$$

We will prove that the set of equations

$$\Psi := \{\varepsilon_i(x,y,z,v,x\dots x) \approx \delta_i(x,y,z,v,x,\dots,x) : i < m\}$$

witnesses EDPRC for K.

First, in view of Lemma 5.11, it suffices to prove that for every finitely generated $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a,b\rangle \in \mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \iff \boldsymbol{A} \vDash \Psi(c,d,a,b).$$

Accordingly, consider a finitely generated $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$. The proof will proceed through a series of claims.

Claim 5.25. *There exists a surjective homomorphism $g\colon \boldsymbol{F} \to \boldsymbol{A}$ such that*

$$g(x/\theta) = c \qquad g(y/\theta) = d \qquad g(z/\theta) = a \qquad g(v/\theta) = b$$

and $g(w_i/\theta) = c$, for every $i \leqslant n$.

Proof of the Claim. Let $\{e_1,\dots,e_k\}$ be a set of generators for $\boldsymbol{A}$. Then consider a function $f\colon Var \to A$ such that $\{e_1,\dots,e_n\} \subseteq f[Var]$,

$$f(x) = c \qquad f(y) = d \qquad f(z) = a \qquad f(v) = b.$$

and $f(w_i) = c$, for every $i \leqslant n$. By Proposition 1.4, f can be extended to a homomorphism $f^*\colon \boldsymbol{T}(Var) \to \boldsymbol{A}$. Since $\boldsymbol{T}(Var)/\mathsf{Ker}(f^*) \in \mathbb{IS}(\boldsymbol{A}) \subseteq \mathsf{K}$, we obtain $\mathsf{Ker}(f^*) \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var))$. As θ is the minimum of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{T}(Var))$, this implies $\theta \subseteq \mathsf{Ker}(f^*)$. Because of this, the map $g\colon \boldsymbol{F} \to \boldsymbol{A}$, defined by the rue

$$\varphi/\theta \longmapsto f^*(\varphi),$$

is a well-defined homomorphism such that

$$g(x/\theta) = c \qquad g(y/\theta) = d \qquad g(z/\theta) = a \qquad g(v/\theta) = b$$

and $g(w_i/\theta) = c$, for every $i \leqslant n$. Furthermore, g is surjective because $g[Var/\theta]$ contains the generators $e_1,\dots,e_k$ of $\boldsymbol{A}$. ⊠

Claim 5.26. *The following equivalence holds:*

$$\langle a,b\rangle \in \mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \iff \langle z/\theta, v/\theta\rangle \in \mathsf{Ker}(g) \vee^{\mathsf{Con}_{\mathsf{K}}(\boldsymbol{F})} \mathsf{Cg}^{\boldsymbol{F}}_{\mathsf{K}}(x/\theta, y/\theta).$$

Proof of the Claim. Since $g\colon \boldsymbol{F} \to \boldsymbol{A}$ is a sujective homomorphism, from Corollary 1.19 it follows that the map $h\colon \boldsymbol{F}/\mathsf{Ker}(g) \to \boldsymbol{A}$, defined by the rule

$$(\varphi/\theta)/\mathsf{Ker}(g) \longmapsto g(\varphi/\theta),$$

is an isomorphism. As a consequence, $\langle a, b\rangle \in \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(c, d)$ if and only if

$$\langle (z/\theta)/\mathsf{Ker}(g), (v/\theta)/\mathsf{Ker}(g)\rangle \in \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}/\mathsf{Ker}(g)}((x/\theta)/\mathsf{Ker}(g), (y/\theta)/\mathsf{Ker}(g)).$$

In view of Lemma 5.12, the above display is equivalent to

$$\langle z/\theta, v/\theta\rangle \in \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(\mathsf{Ker}(g) \cup \{\langle x/\theta, y/\theta\rangle\}).$$

Hence, we conclude that

$$\begin{aligned}\langle a, b\rangle \in \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(c, d) &\Longleftrightarrow \langle z/\theta, v/\theta\rangle \in \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(\mathsf{Ker}(g) \cup \{\langle x/\theta, y/\theta\rangle\}) \\ &\Longleftrightarrow \langle z/\theta, v/\theta\rangle \in \mathsf{Ker}(g) \vee^{\mathsf{Con}_{\mathsf{K}}(\boldsymbol{F})} \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta),\end{aligned}$$

as desired. ⊠

Claim 5.27. *The following equivalence holds:*

$$\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta) \leftarrow \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta) \subseteq \mathsf{Ker}(g) \Longleftrightarrow \boldsymbol{A} \vDash \Psi(c, d, a, b).$$

Proof of the Claim. We have

$$\begin{aligned}&\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta) \leftarrow \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta) \subseteq \mathsf{Ker}(g) \\ \Longleftrightarrow\ &\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(\{\langle \varepsilon_i/\theta, \delta_i/\theta\rangle : i < m\}) \subseteq \mathsf{Ker}(g) \\ \Longleftrightarrow\ &\{\langle \varepsilon_i/\theta, \delta_i/\theta\rangle : i < m\} \subseteq \mathsf{Ker}(g) \\ \Longleftrightarrow\ &g(\varepsilon_i(x, y, z, v, w_1, \dots, w_n)/\theta) = g(\delta_i(x, y, z, v, w_1, \dots, w_n)/\theta), \text{ for all } i < m \\ \Longleftrightarrow\ &\delta_i^{\boldsymbol{A}}(c, d, a, b, c, \dots, c) = \varepsilon_i^{\boldsymbol{A}}(c, d, a, b, c, \dots, c), \text{ for all } i < m \\ \Longleftrightarrow\ &\boldsymbol{A} \vDash \Psi(c, d, a, b).\end{aligned}$$

The above equivalences are justified as follows. The first holds by the definition of $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta) \leftarrow \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta)$, the second because $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}$ is a closure operator, the third follows from the definition of g, the fourth from Claim 5.25, and the last from the definition of Ψ. ⊠

In view of Claims 5.26 and 5.27, to conclude the proof, it suffices to show that

$$\langle z/\theta, v/\theta\rangle \in \mathsf{Ker}(g) \vee \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta) \Longleftrightarrow \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta) \leftarrow \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta) \subseteq \mathsf{Ker}(g).$$

This follows from the following series of equivalences:

$$\begin{aligned}&\langle z/\theta, v/\theta\rangle \in \mathsf{Ker}(g) \vee \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta) \\ \Longleftrightarrow\ &\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta) \subseteq \mathsf{Ker}(g) \vee \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta) \\ \Longleftrightarrow\ &\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta) \subseteq \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(\Sigma) \vee \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta), \text{ for some finite } \Sigma \subseteq \mathsf{Ker}(g) \\ \Longleftrightarrow\ &\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta) \leftarrow \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta) \subseteq \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(\Sigma), \text{ for some finite } \Sigma \subseteq \mathsf{Ker}(g) \\ \Longleftrightarrow\ &\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta) \leftarrow \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta) \subseteq \mathsf{Ker}(g).\end{aligned}$$

The first equivalence follows from the fact that $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}$ is a closure operator, the second and the last from the compactness of $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta)$ and $\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(z/\theta, v/\theta) \leftarrow \mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{F}}(x/\theta, y/\theta)$. Lastly, the third follows from the fact that $\langle \mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{F})); \vee, \leftarrow\rangle$ is a dually Brouwerian semilattice. ⊠

Remark 5.28. The proof of Claim 5.25 can be slightly simplified by observing that $\boldsymbol{F}$ is the free algebra of K with a denumerable set of free generators. However, since we did not introduce the notion of a free algebra, we opted for spelling the details in full. ⊠

Remark 5.29. In view of Theorem 5.23, EDPRC can be viewed as a property of lattices of K-congruences. Since their structure is preserved by category equivalences between quasi-varieties, this implies that EDPRC is preserved by category equivalences too. As a consequence, if two algebraizable logics $\vdash$ and $\vdash'$ have categorically equivalent algebraic semantics, then $\vdash$ has a DDT if and only if so does $\vdash'$. This kind of observations have been exploited, for instance, in [62, 63]. ⊠

5.3 Sketches of structure theory

We shall now review some basic properties of quasi-varieties with EDPRC.

Definition 5.30. A quasi-variety K is said to be *relatively congruence distributive* when $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a distributive lattice, for every $\boldsymbol{A} \in \mathsf{K}$. When K is a variety, we drop the adverb "relatively" and say that K is *congruence distributive*.

The following result was established in [87].

Theorem 5.31 (Köhler & Pigozzi)**.** *Quasi-varieties with EDPRC are relatively congruence distributive.*

Proof. Let K be a quasi-variety with EDPRC and consider an algebra $\boldsymbol{A} \in \mathsf{K}$. By Proposition 5.19, the lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is algebraic. Since the proof of Proposition 5.17 shows that every algebraic lattice $\boldsymbol{B}$ is isomorphic to $\mathcal{I}(\mathsf{C}(\boldsymbol{B}))$, we obtain

$$\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \cong \mathcal{I}(\mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}))).$$

Therefore, it will be enough to show that the lattice $\mathcal{I}(\mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})))$ is distributive.

We will prove a more general result, namely, that if $\boldsymbol{B}$ is a dually Brouwerian semilattice, then $\mathcal{I}(\boldsymbol{B})$ is a distributive lattice. Since $\mathsf{C}(\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}))$ is a dually Brouwerian semilattice by Theorem 5.23, this will conclude the proof.

Accordingly, let $\boldsymbol{B}$ be a dually Brouwerian semilattice. As $\boldsymbol{B}$ has a minimum (namely, $a \leftarrow a$, for any $a \in B$), the poset of ideals $\mathcal{I}(\boldsymbol{B})$ is a well-defined lattice. To prove that it is distributive, it suffices to show that, for every $I, J, L \in \mathcal{I}(\boldsymbol{B})$,

$$(I \vee J) \cap (I \vee L) \subseteq I \vee (J \cap L),$$

where the join $\vee$ is computed in $\mathcal{I}(\boldsymbol{B})$. To this end, consider an element $a \in (I \vee J) \cap (I \vee L)$. Then there are $b \in I, c \in J$ and $d \in L$ such that

$$a \leqslant b * c \text{ and } a \leqslant b * d.$$

As $\boldsymbol{B}$ is dually Brouwerian, this implies

$$a \leftarrow b \leqslant c \text{ and } a \leftarrow b \leqslant d,$$

whence $a \leftarrow b \in J \cap L$. Since $b \in I$, this yields

$$b * (a \leftarrow b) \in I \vee (J \cap L).$$

As $\boldsymbol{B}$ is dually Brouwerian, from $a \leftarrow b \leqslant a \leftarrow b$ it follows $a \leqslant b * (a \leftarrow b)$. Together with the above display, this implies $a \in I \vee (J \cap L)$, as desired. ⊠

Relative congruence distributivity has a number interesting consequences related to axiomatization problems.

Definition 5.32. A quasi-variety is said to be *finitely based* when it can be axiomatized by a finite set of quasi-equations.

Remark 5.33. In view of the Compactness Theorem of first order logic, if a variety is finitely based as a quasi-variety, then it can be axiomatized by a finite set of *equations* too. ⊠

Given a finite set K of finite algebras of finite type, there is no guarantee that $\mathbb{V}(\mathsf{K})$ or $\mathbb{Q}(\mathsf{K})$ are finitely based. For varieties, this is known since [92]. In fact a transparent characterization of the finite algebras $\boldsymbol{A}$ of finite type for which $\mathbb{V}(\boldsymbol{A})$ is finitely based seems out of reach, in part because the problem of determining whether $\mathbb{V}(\boldsymbol{A})$ is finitely based is undecidable [96]. This makes the following result appealing [110] .

Theorem 5.34 (Pigozzi)**.** *Let* K *be a finite set of finite algebras of finite type. If* $\mathbb{Q}(\mathsf{K})$ *is relatively congruence distributive, then it is finitely based.*

Remark 5.35. The above result subsumes Baker's Finite Basis Theorem [6] for varieties. Both results were generalized in [109] and [7]. ⊠

In view of Theorem 5.31, we obtain the following.

Corollary 5.36. *Let* K *be a finite set of finite algebras of finite type. If* $\mathbb{Q}(\mathsf{K})$ *has EDPRC, then it is finitely based.*

This result admits a logical reading. A logic $\vdash$ is said to be *finitely axiomatizable* if it can be axiomatized by a finite Hilbert calculus.

Corollary 5.37. *Let* $\vdash$ *be an algebraizable logic of finite type and* K *its equivalent algebraic semantics. If* $\vdash$ *has a DDT and* $\mathsf{K} = \mathbb{Q}(\mathsf{M})$ *for a finite set of finite algebras* M*, then* $\vdash$ *is finitely axiomatizable.*

Proof. From Theorem 5.7 it follows that K has EDPRC. Therefore, we can apply Corollary 5.36, obtaining that K is axiomatized by a finite set Σ of quasi-equations. Let $\boldsymbol{\tau}$ and Δ be the sets of equations and formulas that, together with K, witness the algebraizability of $\vdash$. It is easy to show that $\vdash$ can be axiomatized by the Hilbert calculus consisting of the rules

$$\begin{aligned} x \rhd \Delta[\boldsymbol{\tau}(x)] \\ \Delta[\boldsymbol{\tau}(x)] \rhd \\ \Delta(\varphi_1, \psi_1) \cup \cdots \cup \Delta(\varphi_n, \psi_n) \rhd \Delta(\varepsilon, \delta), \end{aligned}$$

for every quasi-equation $(\varphi_1 \approx \psi_1 \,\&\, \ldots \,\&\, \varphi_n \approx \psi_n) \Longrightarrow \varepsilon \approx \delta$ in Σ. ⊠

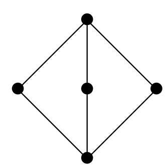
Exercise 5.38. Prove that the Hilbert calculus in the above proof axiomatizes $\vdash$. ⊠

Example 5.39 (Tabular superintuitionistic logics)**.** A superintuitionistic logic is said to be *tabular* if its equivalent algebraic semantics is a variety of the form $\mathbb{V}(\boldsymbol{A})$, for a finite Heyting algebra $\boldsymbol{A}$. We will prove that every tabular intermediate logic is finitely axiomatizable. Accordingly, let $\vdash$ be tabular and $\mathbb{V}(\boldsymbol{A})$ its equivalent algebraic semantics (where $\boldsymbol{A}$ is a finite Heyting algebra). In view of Jónsson's Lemma [84], the subdiretcly irreducible members of $\mathbb{V}(\boldsymbol{A})$ form a finite set K of finite algebras. Therefore, $\mathbb{V}(\boldsymbol{A}) = \mathbb{P}_{\text{SD}}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$. As the inclusion $\mathbb{Q}(\mathsf{K}) \subseteq \mathbb{V}(\boldsymbol{A})$ is obvious, we conclude that $\mathbb{V}(\boldsymbol{A}) = \mathbb{Q}(\mathsf{K})$. Since $\vdash$ has a DDT (Exercise 5.2), we can apply Corollary 5.37, obtaining that $\vdash$ is finitely axiomatizable. ⊠

Exercise 5.40. Prove that if K is a finite set of finite algebras, there exists a finite algebra $\boldsymbol{A}$ such that $\mathbb{V}(\mathsf{K}) = \mathbb{V}(\boldsymbol{A})$. On the other hand, exhibit a finite set K of finite algebras for which there is no algebra $\boldsymbol{A}$ (finite or infinite) such that $\mathbb{Q}(\mathsf{K}) = \mathbb{Q}(\boldsymbol{A})$. ⊠

Theorem 5.31 can be used to disprove that certain quasi-varieties have EDPRC.

Example 5.41 (Groups)**.** As an exemplification, we will prove that the variety Gr of groups lacks EDPC. Let $\boldsymbol{A}$ be the *Klein four-group*, i.e., the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the groups of integers modulo two. The congruence lattice of $\boldsymbol{A}$ is isomorphic to the following nondistributive diamond.



Hence, the variety Gr is not congruence distributive. By 5.31, this implies that Gr lacks EDPC, as desired. In view of the bridge theorem between the DDT and EDPRC, the algebraizable logic of groups $\mathbf{G}$, defined in the proof of Corollary 4.30, lacks any DDT. ⊠

Principal K-congruences (and, therefore, EDPRC) are also related to the building blocks of quasi-varieties, as we proceed to explain.

Definition 5.42. Let K be a quasi-variety. A member $\boldsymbol{A}$ of K is said to be *subdirectly irreducible relative to* K when for every subdirect emebedding $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{B}_i$ with $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{K}$, there exists some $i \in I$ such that the composition $p_i \circ f \colon \boldsymbol{A} \to \boldsymbol{B}_i$ is an isomorphism. The class of all subdirectly irreducible algebras relative to K will be denoted by K_{RSI} and its elements are called the RSI members of K.

An algebra $\boldsymbol{A}$ is said to be *subdirectly irreducible* (in the absolute sense) when it is subdirectly irreducible relative to the quasi-variety of all algebras of its type.

The following classical result was discovered by Birkhoff in the setting of varieties and generalized to quasi-varieties by Maltsev [68, Thm. 3.1.1].

Subdirect Decomposition Theorem 5.43. *If* K *is a quasi-variety, then* $\mathsf{K} = \mathbb{IP}_{\text{SD}}(\mathsf{K}_{\text{RSI}})$.

As we mentioned, the RSI members of a quasi-variety K admit a description in terms of principal K-congruences.

Proposition 5.44. *Let K be a quasi-variety and $\boldsymbol{A} \in \mathsf{K}$. The following conditions are equivalent:*

(i) *$\boldsymbol{A}$ is an RSI member of K;*

(ii) id_A *is completely meet irreducible in* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$*;*

(iii) *There are distinct $a, b \in A$ such that $\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$, for every pair of distinct $c, d \in A$.*

Exercise 5.45. Prove the above result. Hint: you might wish to use Proposition 1.25. ⊠

A special class of RSI algebras is the following.

Definition 5.46. Let K be a quasi-variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is *simple relative to* K if it has exactly two K-congruences. The class of all simple algebras relative to K will be denoted by K_{RS} and its elements are called the RS members of K.

Corollary 5.47. *If K is a quasi-variety, then $\mathsf{K}_{\mathrm{RS}} \subseteq \mathsf{K}_{\mathrm{RSI}}$.*

Proof. Let $\boldsymbol{A}$ be an RS member of K. Then $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is the two-element chain with minimum id_A and maximum $A \times A$. Therefore, id_A is completely meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. By Proposition 5.44, we conclude that $\boldsymbol{A}$ is also RSI. ⊠

As for the case of subdirect irreducibility, the notion of a simple algebra admits an absolute variant. More precisely, an algebra $\boldsymbol{A}$ is *simple* (in the absolute sense) if $\mathsf{Con}(\boldsymbol{A})$ has precisely two elements.

As we will see, the definability of principal K-congruences influences the model theoretic properties of the class of RSI members of a quasi-variety K. To make this precise, it is convenient to introduce the following concept.

Definition 5.48. A quasi-variety K is said to have *definable principal relative congruences* (DPRC) if there exists a first order formula $\phi(x, y, z, v)$ such that, for every $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) \iff \boldsymbol{A} \vDash \phi(c, d, a, b).$$

When K is a variety, it is common to use the expression *definable principal congruences* (DPC), as opposed to (DPRC).

Clearly, every quasi-variety with EDPRC has DPRC, while the converse need not be true in general. For instance, an argument analogous to the one detailed in Example 5.41 shows that the variety CR of commutative rings with unit lacks EDPC. On the other hand, the formula

$$\phi(x, y, z, v) \coloneqq \exists w(w(x - y) \approx z - v)$$

witnesses DPC for CR.

A class of structures is said to be *elementary* when it can be axiomatized by a set of sentences.

Theorem 5.49. *If K is a quasi-variety with DPRC, then $\mathsf{K}_{\mathrm{RSI}}$ and K_{RS} are elementary classes.*

Proof. We detail the proof for the case of $\mathsf{K}_{\mathrm{RSI}}$ and leave that of K_{RS} as an exercise. Let $\phi(x, y, z, v)$ be the formula witnessing DPRC for K. Then consider the sentence

$$\varphi := \exists x, y(x \not\approx y \,\&\, \forall z, v(z \not\approx v \to \phi(z, v, x, y))). \tag{5.7}$$

Let also Σ be the set of universal the closures of the quasi-equations valid in K. We will prove that $\Sigma \cup \{\varphi\}$ axiomatizes $\mathsf{K}_{\mathrm{RSI}}$.

In view of Maltsev's Theorem, Σ axiomatizes K. Therefore, as φ witnesses DPRC, $\Sigma \cup \{\varphi\}$ axiomatizes the members $\boldsymbol{A}$ of K that contain two distinct a, b such that $\langle a, b\rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d)$, for every pair of distinct $c, d \in A$. In view of Proposition 5.44, these are precisely the RSI members of K. ⊠

As a consequence of Löwenheim-Skolem Theorem, we obtain the following.

Corollary 5.50. *Let K be a quasi-variety with DPRC whose language has cardinality κ. If K has an infinite RSI member, then it also has an RSI member of cardinality λ, for every $\lambda \geqslant \kappa + \aleph_0$.*

Exercise 5.51. Complete the proof of Theorem 5.49, by showing that if a quasi-variety K has DPRC, then K_{RS} is elementary. Prove also that if K has EDPRC, when extended with all trivial algebras of the suitable type, K_{RS} becomes a universal class. ⊠

Theorem 5.31 can be used to disprove that certain quasi-varieties have EDPRC.

Example 5.52 (Modal algebras)**.** As an exemplification, we will prove that the variety MA of modal algebras lacks DPC and, therefore, EDPC. Notably, in view of Theorem 5.7, this implies that the logic $\mathbf{K}_g$ lacks any DDT. By Theorem 5.49, it suffices to prove that the class of simple modal algebras is not elementary. We will do this, by showing that it is not closed under ultraproducts (see Corollary 3.6, if necessary). To this end, recall that a modal algebra $\boldsymbol{A}$ is simple when it is nontrivial and for every element $a \in A \smallsetminus \{1\}$ there is $n \in \mathbb{N}$ such that $\boxplus^n a = 0$.

For every $n \in \mathbb{N}$, consider the Kripke frame $\langle W_n, R_n\rangle$ with universe $W_n = \{x_1, \dots, x_n\}$ and accessibility relation R_n defined as follows:

$$\langle y, z\rangle \in R_n \iff \text{ either } y = z \text{ or } (y = x_i \text{ and } z = x_{i+1}) \text{ or } (y = x_n \text{ and } z = x_1).$$

Furthermore, let $\boldsymbol{A}_n$ be the complex algebra of $\langle W_n, R_n\rangle$. It is easy to prove that $\boldsymbol{A}_n$ is simple. Furthermore, for every $n \in \mathbb{N}$, there exists an element $a_n \in A_{n+2} \smallsetminus \{1\}$ such that

$$\boxplus^{n\boldsymbol{A}_{n+2}} a_n \neq 0^{\boldsymbol{A}_{n+2}}.$$

Suppose, with a view to contradiction, that the class simple modal algebras is closed under ultraproducts. Then consider a free ultrafilter U on $\mathbb{N}$ and take the ultrapower

$$\boldsymbol{B} := \prod_{n \in \mathbb{N}} \boldsymbol{A}_n / \theta_U.$$

By assumption, $\boldsymbol{B}$ is a simple modal algebra. Then consider an element $\vec{b} \in \prod_{n \in \mathbb{N}} A_n$ such that $\vec{b}(n+2) = a_n$, for every $n \in \mathbb{N}$. Since $\boldsymbol{B}$ is simple, there is $m \in \mathbb{N}$ such that

$$\boxplus^{m\boldsymbol{B}} \vec{b}/\theta_U = 0^{\boldsymbol{B}}.$$

By Łoś' Theorem, this implies that U contains a finite subset of $\mathbb{N}$. But this contradicts the assumption that U is free. ⊠

At this stage, it is natural to wonder how restrictive is EDPRC with respect to DPRC. To answer this question, it is convenient to introduce the following concept.

Definition 5.53. A quasi-variety K is said to have the *relative congruence extension property* (RCEP) if for every $\boldsymbol{B} \leqslant \boldsymbol{A} \in \mathsf{K}$ and $\theta \in \mathrm{Con}_{\mathsf{K}}(\boldsymbol{B})$ there exists $\phi \in \mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta = \phi \cap (B \times B)$. When K is a variety, $\mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$ and $\mathrm{Con}_{\mathsf{K}}(\boldsymbol{B})$ can be replaced, respectively, by $\mathrm{Con}(\boldsymbol{A})$ and $\mathrm{Con}(\boldsymbol{B})$, and K is said to have the *congruence extension property* (CEP).

The following result was established in [60].

Theorem 5.54 (Fried, Grätzer & Quackenbush)**.** *A quasi-variety has EDPRC if and only if it is relatively congruence distributive and it has DPRC and the RCEP.*

As the majority of quasi-varieties in the algebra of logic have the RCEP and are relatively congruence distributive, in this setting it is often that case that EDPRC is equivalent to DPRC. For instance, as all varieties of modal algebras have the CEP and are congruence distributive, EDPC and DPC coincide for them.

We conclude our journey with a description of varieties of modal algebras with EDPC.

Theorem 5.55 (Blok & Pigozzi)**.** *A variety of modal algebras has EDPC if and only if it is weakly transitive.*

Proof. Recall from Exercise 5.6 that weakly transitive varieties have EDPC. Therefore, it only remains to prove the converse. Accordingly consider variety K of modal algebras with EDPC. By Theorem 4.13, it is the equivalent algebraic semantics of an axiomatic extension $\vdash$ of $\mathbf{K}_g$. As any other axiomatic extension of $\mathbf{K}_g$, the logic $\vdash$ has the following property: for every $\Gamma \cup \{\varphi, \psi\} \subseteq T(Var)$,

$$\Gamma, \varphi \vdash \psi \iff \text{there is } n \in \mathbb{N} \text{ such that } \Gamma \vdash \boxplus^n \varphi \to \psi. \tag{5.8}$$

Furthermore, $\vdash$ has a DDT, by Theorem 5.7. Let then $I(x, y)$ be the set witnessing the DDT for $\vdash$. Since I is finite, we can assume that it has cardinality at most one, for if $I = \{\varphi_1, \dots, \varphi_n\}$, we can replace I with $I^* = \{\varphi_1 \wedge \dots \wedge \varphi_n\}$. Furthermore, we can also assume that I contains a tautology and, therefore, that it is nonempty. Accordingly, we will assume that $I = \{x \multimap y\}$, for some formula $x \multimap y$ in two variables x and y. Therefore, the DDT amounts to the statement that, for every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash \psi \iff \Gamma \vdash \varphi \multimap \psi.$$

Now, observe that $x \multimap y \vdash x \multimap y$. In view of the DDT, this yields $x, x \multimap y \vdash y$. By (5.8) there exists $n \in \mathbb{N}$ such that

$$\emptyset \vdash \boxplus^n (x \wedge (x \multimap y)) \to y.$$

Observe that in the class of modal algebras MA the equation $\boxplus^n (x \wedge y) \approx \boxplus^n x \wedge \boxplus^n y$ holds. Therefore,

$$\mathsf{MA} \vDash \Big(\boxplus^n \, (x \wedge (x \multimap y)) \to y \Big) \to \Big((\boxplus^n x \wedge \boxplus^n (x \multimap y)) \to y \Big) \approx 1.$$

By Theorem 2.15, this yields

$$\emptyset \vdash_{\mathbf{K}_g} \Big(\boxplus^n (x \wedge (x \multimap y)) \to y \Big) \to \Big((\boxplus^n x \wedge \boxplus^n (x \multimap y)) \to y \Big). \tag{5.9}$$

Since $\vdash$ is an extension of $\mathbf{K}_g$, by applying modus ponens to the above display and (5.9), we obtain

$$\emptyset \vdash (\boxplus^n x \wedge \boxplus^n (x \multimap y)) \to y.$$

Recall that in **CPC** every formula of the form $((\alpha \wedge \beta) \to \gamma) \to (\beta \to (\alpha \to \gamma))$ is a tautology. Therefore, the same is true in $\vdash$. As the formula above have the form $(\alpha \wedge \beta) \to \gamma$, by modus ponens we obtain

$$\boxplus^n (x \multimap y) \vdash \boxplus^n x \to y.$$

As $\vdash$ in substitution invariant, we can replace y by $\Box^{n+1} x$ in the above deduction and obtain

$$\boxplus^n (x \multimap \Box^{n+1} x) \vdash \boxplus^n x \to \Box^{n+1} x. \tag{5.10}$$

Lastly, since $\vdash$ is an extension of $\mathbf{K}_g$, we have $x \vdash \Box^{n+1} x$. By the DDT, this yields $\emptyset \vdash x \multimap \Box^{n+1} x$. Together with $x \vdash \Box x$ and substitution invariance, this yields

$$\emptyset \vdash \boxplus^n (x \multimap \Box^{n+1} x).$$

By (5.10), we conclude that

$$\emptyset \vdash \boxplus^n x \to \Box^{n+1} x.$$

Since the algebraizability of $\vdash$ is witnessed by $\boldsymbol{\tau} = \{x \approx 1\}$, $\Delta = \{x \leftrightarrow y\}$ and K, from the above display it follows $\mathsf{K} \vDash \boxplus^n x \to \Box^{n+1} x \approx 1$, that is, $\mathsf{K} \vDash \boxplus^n x \leqslant \Box^{n+1} x$. Hence, we conclude that $\mathbf{K}_g$ is weakly transitive. ⊠

Corollary 5.56. *An axiomatic extension of $\mathbf{K}_g$ has the DDT if and only if its equivalent algebraic semantics is a weakly transitive variety of modal algebras.*

Remark 5.57. EDPRC can also be used to formulate the theory of *Jankov's formulas* [77, 78, 79] in universal algebraic terms [19]. This is made possible by the lattice theoretic notion of a splitting [128] and its application to subvariety lattices [94]. ⊠

Remark 5.58. In this course, we focused on the bridge theorem connecting the DDT with EDPRC. Similar theorems holds for a variety of other metalogical properties, including weaker variants of the DDT [22, 40, 41, 45, 114] and of the proof by cases [39, 44, 108], interpolation [47], Beth definability [14, 75, 76], inconsistency lemmas [31, 111, 116] and relevance principles [50, 91, 106]. For information of the algebraic counterparts of these properties, the reader might consult [86], see also [99]. ⊠

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