Abstract Algebraic Logic

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Introduction

One of the main achievements of algebraic logic was the development of the *Leibniz hierarchy*, a classification of (propositional) logics that parallels the hierarchy of Maltsev conditions in universal algebra [47, 56, 67, 88].*

The core of the Leibniz hierarchy is the class of *protoalgebraic* logics [10, 26, 27, 28], i.e., logics possessing a set of formulas that globally expresses logical equivalence. Their role in algebraic logic consists in providing a framework suitable for the formulation of *bridge theorems* (see, e.g., [39, Sec. 7]) that connect logic and algebra by associating a purely algebraic interpretation to metalogical properties such as the deduction theorem [12, 14, 26, 27, 80], the inconsistency lemma [19, 82], the proof by cases [22, 25, 29, 28], or the interpolation property [33, 41]. In addition, the theory of protoalgebraic logic allows us to extend some of the main finite basis theorems of universal algebra [1, 2, 69, 74] to the realm of propositional logics [70, 71] and to derive a signature-independent version [44] of Sahlqvist theory [87].

In order to describe the other pillar of the Leibniz hierarchy, recall that a logical matrix is a pair $\langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A}$ is an algebra and F a subset of the universe of $\boldsymbol{A}$, known as the set of *designated elements* of the matrix. It is well known that every logic has a canonical *matrix semantics*, which is obtained through the Lindenbaum-Tarski process. When the sets of designated elements of the matrices in this semantics are equationally definable, the logic under consideration is said to be *truth equational* [79]. For these logics, we can dispense with matrices and obtain instead a purely *algebraic semantics* [11, 16, 65]. Protoalgebraic and truth equational logics are related by a definability theorem unique to algebraic logic, stating that the sets of designated elements of the matrices in the canonical matrix semantics of a protoalgebraic logic are implicitly definable iff they are equationally definable, that is, if the logic is truth equational [32, 79].

In this lecture notes, we will present a brief introduction to the Leibniz hierarchy centered on the classes of protoalgebraic and truth equational logics. For some more advanced material we refer to the excellent monographs [11, 23, 28, 38, 40] and surveys [42, 43, 81]. We chose to formulate the main results for logics, but similar observations apply to Gentzen systems and other more complex formalisms [13, 48, 77, 78]. These lectures notes are based on the book chapter [66].

*For the relation between the Leibniz and Maltsev hierarchies, see [52, 53, 54].

CHAPTER 1

Logics and matrices

In this section, we shall review the basics of matrix semantics for propositional logics [23, 28, 38, 40, 89].

1.1 Propositional logics

Given an algebraic language, we denote the set of its formulas built up from a denumerable set of variables by Fm and the corresponding algebra of formulas by $\boldsymbol{Fm}$. Moreover, the endomorphisms of $\boldsymbol{Fm}$ will be called *substitutions*.

A *propositional logic* $\vdash$ (from now on, simply a *logic*) is a consequence relation on the set Fm of formulas of some algebraic language that, moreover, is *substitution invariant* in the sense that, for every substitution σ and every $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Given $\Gamma \cup \Sigma \subseteq Fm$, we write $\Gamma \vdash \Sigma$ when $\Gamma \vdash \varphi$, for all $\varphi \in \Sigma$.

Every logic $\vdash$ can be associated with a closure operator $\mathrm{Cn}_{\vdash} \colon \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ defined, for every $\Gamma \subseteq Fm$, as follows:

$$\mathrm{Cn}_{\vdash}(\Gamma) := \{\varphi \in Fm \colon \Gamma \vdash \varphi\}.$$

Moreover, a set of formulas Γ is said to be a *theory* of $\vdash$ if $\Gamma = \mathrm{Cn}_{\vdash}(\Gamma)$. When ordered under the inclusion relation, the set of theories of $\vdash$ forms a lattice that we denote by $\mathcal{T}h(\vdash)$.

Let $\mathscr{L}$ be the language in which a logic $\vdash$ is formulated. Given a sublanguage $\mathcal{L}^- \subseteq \mathscr{L}$, the $\mathcal{L}^-$*-fragment* of $\vdash$ is the restriction of $\vdash$ to the formulas of $\mathscr{L}^-$. On the other hand, a logic $\vdash^+$ is said to be an *extension* of $\vdash$ if it is formulated in the same language as $\vdash$ and $\vdash \subseteq \vdash^+$. Lastly, an extension $\vdash^+$ of $\vdash$ is said to be *axiomatic* if there exists a set Σ of formulas closed under substitutions such that, for every $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash^+ \varphi \Longleftrightarrow \Gamma, \Sigma \vdash \varphi.$$

By a *matrix* we understand a pair $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an algebra and $F \subseteq A$. The logic *induced* by a class M of similar matrices is the consequence relation $\vdash_{\mathsf{M}}$ on Fm

defined, for every $\Gamma \cup \{\varphi\} \subseteq Fm$, as

$$\Gamma \vdash_{\mathsf{M}} \varphi \iff \text{for every } \langle \boldsymbol{A}, F\rangle \in \mathsf{M} \text{ and homomorphism } f\colon \boldsymbol{Fm} \to \boldsymbol{A},\\ \text{if } f[\Gamma] \subseteq F, \text{ then } f(\varphi) \in F.$$

When a logic $\vdash$ coincides with the logic induced by a class M of matrices, we say that $\vdash$ is *complete* with respect to M and that M is a *matrix semantics* for $\vdash$.

Theorem 1.1. *Every logic has a matrix semantics.*

Proof. We will prove that every logic $\vdash$ is complete with respect to the class of matrices

$$\mathsf{M} := \{\langle \boldsymbol{Fm}, \Gamma\rangle : \Gamma \in \mathcal{T}h(\vdash)\}.$$

To this end, consider $\Gamma \cup \{\varphi\} \subseteq Fm$. We need to show that

$$\Gamma \vdash \varphi \iff \Gamma \vdash_{\mathsf{M}} \varphi.$$

Suppose first that $\Gamma \vdash \varphi$. Then consider a matrix $\langle \boldsymbol{Fm}, \Sigma\rangle \in \mathsf{M}$ and a homomorphism $\sigma\colon \boldsymbol{Fm} \to \boldsymbol{Fm}$ such that $\sigma[\Gamma] \subseteq \Sigma$. Since σ is a substitution and $\vdash$ is substitution invariant, from $\Gamma \vdash \varphi$ it follows $\sigma[\Gamma] \vdash \sigma(\varphi)$. Together with $\sigma[\Gamma] \subseteq \Sigma$ and the fact that $\vdash$ is a consequence relation, this yields $\Sigma \vdash \sigma(\varphi)$. As $\Sigma \in \mathcal{T}h(\vdash)$, we obtain $\sigma(\varphi) \in \Sigma$. Hence, we conclude that $\Gamma \vdash_{\mathsf{M}} \varphi$.

To prove the converse, we reason by contraposition. Suppose that $\Gamma \nvdash \varphi$, that is, $\varphi \notin \mathrm{Cn}_{\vdash}(\Gamma)$. Then consider the identity homomorphism $id\colon \boldsymbol{Fm} \to \boldsymbol{Fm}$. From $\langle \boldsymbol{Fm}, \mathrm{Cn}_{\vdash}(\Gamma)\rangle \in \mathsf{M}$ and

$$id[\Gamma] = \Gamma \subseteq \mathrm{Cn}_{\vdash}(\Gamma) \text{ and } id(\varphi) = \varphi \notin \mathrm{Cn}_{\vdash}(\Gamma)$$

it follows $\Gamma \nvdash_{\mathsf{M}} \varphi$, as desired. ⊠

Example 1.2. Let K be a class of similar algebras whose language contains a constant 1. The *assertional logic* $\vdash_{\mathsf{K}}$ of K is defined, for every $\Gamma \cup \{\varphi\} \subseteq Fm$, as follows (see, e.g., [15, 75, 79]):

$$\Gamma \vdash_{\mathsf{K}} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and homomorphism } f\colon \boldsymbol{Fm} \to \boldsymbol{A},\\ \text{if } f[\Gamma] \subseteq \{1\}, \text{ then } f(\varphi) = 1.$$

Equivalently, $\vdash_{\mathsf{K}}$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \{1\}\rangle : \boldsymbol{A} \in \mathsf{K}\}$. ⊠

Example 1.3. An *ordered algebra* is a pair $\langle \boldsymbol{A}; \leqslant\rangle$, where $\boldsymbol{A}$ is an algebra and $\leqslant$ a partial order on A. The *logic preserving degrees of truth* $\vdash_{\mathsf{K}}^{\leqslant}$ of a class K of similar ordered algebras is defined, for every $\Gamma \cup \{\varphi\} \subseteq Fm$, as follows (see, e.g., [17, 68]):

$$\Gamma \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff \text{for every } \langle \boldsymbol{A}; \leqslant\rangle \in \mathsf{K}, \text{ homomorphism } f\colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ and } a \in A,\\ \text{if } a \leqslant f(\gamma) \text{ for every } \gamma \in \Gamma, \text{ then } a \leqslant f(\varphi).$$

Equivalently, $\vdash_{\mathsf{K}}^{\leqslant}$ is the logic induced by the class of matrices

$$\{\langle \boldsymbol{A}, {\uparrow}a\rangle : \langle \boldsymbol{A}; \leqslant\rangle \in \mathsf{K} \text{ and } a \in A\},$$

where ${\uparrow}a$ is a shorthand for $\{b \in A : a \leqslant b\}$. ⊠

Remark 1.4. The definition of $\vdash_{\mathsf{K}}^{\leqslant}$ implies that $\Gamma \vdash_{\mathsf{K}}^{\leqslant} \varphi$ iff K validates the infinitary formula

$$\forall \vec{x} y \Big(\mathop{\&}_{\gamma \in \Gamma} y \leqslant \gamma(\vec{x}) \Longrightarrow y \leqslant \varphi(\vec{x}) \Big).$$

When K is an elementary class, the Compactness Theorem of first-order logic allows us to replace the infinitary formula above by a sentence, thus yielding

$$\begin{aligned} \Gamma \vdash_{\mathsf{K}}^{\leqslant} \varphi \Longleftrightarrow {} & \text{there exists a finite } \Sigma \subseteq \Gamma \text{ s.t.} \\ & \mathsf{K} \vDash \forall \vec{x} y \Big(\mathop{\&}_{\gamma \in \Sigma} y \leqslant \gamma(\vec{x}) \Longrightarrow y \leqslant \varphi(\vec{x}) \Big). \end{aligned}$$

Observe that, once equipped with the lattice order, every algebra $\boldsymbol{A}$ with a lattice reduct can be viewed as an ordered algebra. Consequently, for every elementary class K of ordered algebras with a lattice reduct, the above display admits the following improvement:

$$\begin{aligned} \Gamma \vdash_{\mathsf{K}}^{\leqslant} \varphi \Longleftrightarrow {} & \text{either } \varphi \text{ defines the maximum in every member of } \mathsf{K} \\ & \text{or there exists } \{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma \text{ s.t. } \mathsf{K} \vDash \gamma_1 \wedge \cdots \wedge \gamma_n \leqslant \varphi. \end{aligned} \tag{1.1}$$

This observation will be used repeatedly in what follows. ⊠

Examples of concrete assertional logics and of logics preserving degrees of truth abound in the literature:

Example 1.5. A *modal algebra* [63] is a structure $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, \Box, 0, 1 \rangle$ that comprises a Boolean algebra $\langle A; \wedge, \vee, \neg, 0, 1 \rangle$ and a unary operation $\Box$ such that, for every $a, b \in A$,

$$\Box(a \wedge b) = \Box a \wedge \Box b \;\; \text{and} \;\; \Box 1 = 1.$$

We denote the class of modal algebras by MA.

The *global consequence* K_g of the basic modal system K is the assertional logic of MA, that is, the logic defined as

$$\begin{aligned} \Gamma \vdash_{\mathsf{K}_g} \varphi \Longleftrightarrow {} & \text{for every } \boldsymbol{A} \in \mathsf{MA} \text{ and homomorphism } f \colon \boldsymbol{Fm} \to \boldsymbol{A}, \\ & \text{if } f[\Gamma] \subseteq \{1\}, \text{ then } f(\varphi) = 1. \end{aligned}$$

On the other hand, the *local consequence* K_ℓ of K is the logic preserving degrees of truth of MA. In view of Remark 1.4, it can be described as follows:

$$\begin{aligned} \Gamma \vdash_{\mathsf{K}_\ell} \varphi \Longleftrightarrow {} & \text{either } \mathsf{MA} \vDash \varphi \approx 1 \text{ or } \mathsf{MA} \vDash \gamma_1 \wedge \cdots \wedge \gamma_n \leqslant \varphi, \\ & \text{for some } \{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma. \end{aligned}$$

The axiomatic extensions of K_g (resp. K_ℓ), known as the *global* (resp. *local*) *consequences of normal modal logics*, have been widely studied (see, e.g., [8, 21, 57]). ⊠

Example 1.6. A *Heyting algebra* [3, 84] is a structure $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1 \rangle$ that comprises a bounded lattice $\langle A; \wedge, \vee, 0, 1 \rangle$ and a binary connective $\to$ that satisfies the *residuation law*: for every $a, b, c \in A$,

$$a \wedge b \leqslant c \Longleftrightarrow a \leqslant b \to c.$$

The assertional logic and the logic preserving degrees of truth of the class HA of Heyting algebras coincide and are known as the *intuitionistic propositional calculus*, in symbols IPC. Their axiomatic extensions have been called *superintuitionistic logics* (see, e.g., [21, 36]).* ⊠

1.2 Deductive filters

A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm$. A rule $\Gamma \rhd \varphi$ is said to be *valid* in a logic $\vdash$ when $\Gamma \vdash \varphi$. Given an algebra $\boldsymbol{A}$ in the same language as $\vdash$, a set $F \subseteq A$ is said to be a *deductive filter* of $\vdash$ on $\boldsymbol{A}$ when it is closed under the interpretation of the rules valid in $\vdash$, that is, when, for every $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash \varphi$ and every homomorphism $f : \boldsymbol{Fm} \to \boldsymbol{A}$,

$$\text{if } f[\Gamma] \subseteq F, \text{ then } f(\varphi) \in F.$$

When ordered under the inclusion relation, the set of deductive filters of $\vdash$ on $\boldsymbol{A}$ forms a complete lattice, which we denote by $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$. Furthermore, we denote the closure operator of deductive filter generation on $\boldsymbol{A}$ by $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash} : \mathcal{P}(A) \to \mathcal{P}(A)$. Then the join operation $+$ of the lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ can be described, for every $F, G \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$, as

$$F + G = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup G).$$

Proposition 1.7. *The deductive filters of a logic $\vdash$ on $\boldsymbol{Fm}$ coincide with the theories of $\vdash$. Consequently, $\mathcal{T}h(\vdash) = \mathsf{Fi}_{\vdash}(\boldsymbol{Fm})$ and $\mathrm{Cn}_{\vdash}(-) = \mathrm{Fg}^{\boldsymbol{Fm}}_{\vdash}(-)$.*

Proof. Let Γ be a theory of $\vdash$. Then consider $\Sigma \cup \{\varphi\} \subseteq Fm$ such that $\Sigma \vdash \varphi$ and a homomorphism $\sigma : \boldsymbol{Fm} \to \boldsymbol{Fm}$ such that $\sigma[\Sigma] \subseteq \Gamma$. Since σ is a substitution and $\vdash$ is substitution invariant, we have $\sigma[\Sigma] \vdash \sigma(\varphi)$. Consequently, the fact that $\vdash$ is a consequence relation implies $\Gamma \vdash \sigma(\varphi)$. Lastly, as Γ is a theory, we conclude that $\sigma(\varphi) \in \Gamma$. Hence, Γ is a deductive filter of $\vdash$ on $\boldsymbol{Fm}$.

Conversely, suppose that Γ is a deductive filter of $\vdash$ on $\boldsymbol{Fm}$. Moreover, let $\varphi \in Fm$ be such that $\Gamma \vdash \varphi$. Then consider the identity homomorphism $id : \boldsymbol{Fm} \to \boldsymbol{Fm}$. Clearly, we have $id[\Gamma] = \Gamma$. Since Γ is a deductive filter of $\vdash$ on $\boldsymbol{Fm}$ and $\Gamma \vdash \varphi$, we obtain $\varphi = id(\varphi) \in \Gamma$, whence Γ is a theory of $\vdash$. ⊠

The deductive filters of logics preserving degrees of truth can be described as follows. Recall that an *upset* of a lattice $\boldsymbol{A}$ is a subset $U \subseteq A$ such that, for every, $a, b \in A$,

$$\text{if } a \in U \text{ and } a \leqslant b, \text{ then } b \in U.$$

If, in addition, U is nonempty and closed under binary meets, it is said to be a *lattice filter*.

Then, given a class K of algebras with a lattice reduct and $\boldsymbol{A} \in \mathsf{K}$, let $\mathsf{Lat}_{\mathsf{K}}(\boldsymbol{A})$ be the set of lattice filters of $\boldsymbol{A}$, with the addition of the emptyset in case there is no formula that defines a maximum in every member of K.

Proposition 1.8. *Let K be an elementary class of algebras with a lattice reduct and $\boldsymbol{A} \in \mathsf{K}$. Then $\mathsf{Lat}_{\mathsf{K}}(\boldsymbol{A})$ is the set of deductive filters of $\vdash^{\leqslant}_{\mathsf{K}}$ on $\boldsymbol{A}$.*

*Consistent axiomatic extensions of IPC are also known as *intermediate logics*.

Proof. To simplify the notation, we will denote the set of deductive filters of $\vdash^{\leqslant}_{\mathsf{K}}$ on $\boldsymbol{A}$ by $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$. We begin by proving that the emptyset belongs to $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ iff it belongs to $\mathsf{Lat}_{\mathsf{K}}(\boldsymbol{A})$. We have that

$$\begin{aligned} \emptyset \in \mathsf{Fi}_{\vdash}(\boldsymbol{A}) &\iff \text{there is no } \varphi \in Fm \text{ s.t. } \emptyset \vdash^{\leqslant}_{\mathsf{K}} \varphi \\ &\iff \text{there is no formula that defines a maximum in every member of } \mathsf{K} \\ &\iff \emptyset \in \mathsf{Lat}_{\mathsf{K}}(\boldsymbol{A}). \end{aligned}$$

The equivalence above are justified as follows: the first holds by the definition of a deductive filter, the second by Condition (1.1), and the third by the definition of $\mathsf{Lat}_{\mathsf{K}}(\boldsymbol{A})$. Notice that the validity of Condition (1.1) depends on the assumption that K is elementary.

Therefore, it only remains to prove that the nonempty members of $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ are precisely the lattice filters of $\boldsymbol{A}$. On the one hand, the definition of $\vdash^{\leqslant}_{\mathsf{K}}$ guarantees the validity of the rules

$$x \wedge y \rhd y \quad \text{and} \quad x, y \rhd x \wedge y.$$

Consequently, the deductive filters of $\vdash^{\leqslant}_{\mathsf{K}}$ on $\boldsymbol{A}$ are upsets closed under binary meets. Thus, every nonempty member of $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ is a lattice filter of $\boldsymbol{A}$.

On the other hand, let F be a lattice filter of $\boldsymbol{A}$. Then suppose that $\Gamma \vdash^{\leqslant}_{\mathsf{K}} \varphi$ and consider a homomorphism $f\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $f[\Gamma] \subseteq F$. In view of Condition (1.1), either φ defines the maximum of $\boldsymbol{A}$ or there exist some $\gamma_1, \dots, \gamma_n \in \Gamma$ such that

$$\mathsf{K} \vDash \gamma_1 \wedge \cdots \wedge \gamma_n \leqslant \varphi.$$

In the first case, the assumption that F is a nonempty upset ensures that $f(\varphi) \in F$. While, in the latter case, from $f[\Gamma] \subseteq F$ it follows that $f(\gamma_1), \dots, f(\gamma_n) \in F$. Since F is an upset closed under binary meets and $\boldsymbol{A} \in \mathsf{K}$, the above display yields $f(\varphi) \in F$. Hence, we conclude that F is a deductive filter of $\vdash^{\leqslant}_{\mathsf{K}}$, as desired. ⊠

Example 1.9. Since the class of modal algebras is elementary and its members possess a term-definable maximum, Proposition 1.8 implies that the deductive filters of K_{ℓ} on a modal algebra $\boldsymbol{A}$ are precisely the lattice filters of $\boldsymbol{A}$. By the same token, the deductive filters of IPC on a Heyting algebra $\boldsymbol{A}$ are the lattice filters of $\boldsymbol{A}$. ⊠

The following observation will be instrumental in describing the deductive filters of K_g.

Proposition 1.10. *Deductive filters are closed under inverse images of homomorphisms, in the sense that if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, $\vdash$ a logic, and $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{B})$, then $f^{-1}[F] \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$.*

Proof. Suppose that $\Gamma \vdash \varphi$ and consider a homomorphism $g\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $g[\Gamma] \subseteq f^{-1}[F]$. Then $f[g[\Gamma]] \subseteq F$. Since $f \circ g\colon \boldsymbol{Fm} \to \boldsymbol{B}$ is a homomorphism and F a deductive filter of $\vdash$ on $\boldsymbol{B}$, we obtain $f(g(\varphi)) \in F$, whence $g(\varphi) \in f^{-1}[F]$. Thus, we conclude that $f^{-1}[F]$ a deductive filter of $\vdash$ on $\boldsymbol{A}$. ⊠

From Propositions 1.7 and 1.10 we deduce:

Corollary 1.11. *If Γ is a theory of a logic $\vdash$ and σ a substitution, then $\sigma^{-1}[\Gamma]$ is also a theory of $\vdash$.*

Example 1.12. A lattice filter F of a modal algebra $\boldsymbol{A}$ is said to be *open* when, for every $a \in A$,

$$\text{if } a \in F, \text{ then } \Box a \in F.$$

We will show that deductive filters of K_g on a modal algebra $\boldsymbol{A}$ are precisely the open lattice filters of $\boldsymbol{A}$.

On the one hand, the definition of K_g guarantees the validity of the rules

$$\emptyset \rhd 1 \qquad x \wedge y \rhd y \qquad x, y \rhd x \wedge y \qquad x \rhd \Box x.$$

The first three rules ensure that the deductive filters of K_g on $\boldsymbol{A}$ are nonempty upsets closed under binary meets (i.e., lattice filters), while the fourth gives closure under the $\Box$ operation. Hence, the deductive filters of K_g on $\boldsymbol{A}$ are open lattice filters.

On the other hand, consider an open lattice filter F of $\boldsymbol{A}$. The relation

$$\theta_F := \{\langle a, b\rangle \in A \times A : a \to b, b \to a \in F\}$$

is easily seen to be a congruence of $\boldsymbol{A}$ such that $F = 1/\theta$. Then consider the canonical homomorphism $p_\theta \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ defined as $p(a) = a/\theta$, for every $a \in A$. Since $F = 1/\theta$, we have that

$$p_\theta^{-1}[\{1/\theta\}] = \{a \in A : p_\theta(a) = 1/\theta\} = \{a \in A : \langle a, 1\rangle \in \theta\} = 1/\theta = F.$$

Now, the definition of K_g guarantees that the singleton containing the maximum element of a modal algebra $\boldsymbol{B}$ is always a deductive filter of K_g on $\boldsymbol{B}$. In particular, $\{1/\theta\}$ is a deductive filter of K_g on $\boldsymbol{A}/\theta$. Therefore, from Proposition 1.10 and the above display it follows that F is a deductive filter of K_g on $\boldsymbol{A}$, as desired. ⊠

1.3 The Leibniz congruence

A congruence θ of an algebra $\boldsymbol{A}$ is *compatible* with a set $F \subseteq A$ when, for every $a, b \in A$,

$$\text{if } \langle a, b\rangle \in \theta \text{ and } a \in F, \text{ then } b \in F.$$

The *Leibniz congruence* of F on $\boldsymbol{A}$, in symbols $\boldsymbol{\Omega}^{\boldsymbol{A}}F$, is the largest congruence of $\boldsymbol{A}$ compatible with F. When $\boldsymbol{A}$ is $\boldsymbol{Fm}$, we will drop the superscript and write simply $\boldsymbol{\Omega}F$. The Leibniz congruence always exists, as we proceed to explain.

Given a formula φ, we write $\varphi(\vec{x})$ when the variables occurring in φ appear in the sequence $\vec{x}$. Given an algebra $\boldsymbol{A}$, a map $p \colon A^n \to A$ is said to be a *polynomial function* of $\boldsymbol{A}$ if there exist a formula $\varphi(x_1, \ldots, x_n, \vec{y})$ and a sequence $\vec{c} \in A$ such that

$$p(a_1, \ldots, a_n) = \varphi^{\boldsymbol{A}}(a_1, \ldots, a_n, \vec{c}),$$

for every $a_1, \ldots, a_n \in A$. The following description of $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ serves also as a proof of existence.

Proposition 1.13. *Let $\boldsymbol{A}$ be an algebra, $F \subseteq A$, and $a, b \in A$. Then*

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F \iff \textit{for every unary polynomial function } p \textit{ of } \boldsymbol{A}, \\ p(a) \in F \textit{ if and only if } p(b) \in F.$$

Proof. Consider the relation

$$\theta := \{\langle a, b\rangle \in A \times A : \text{for every unary polynomial function } p \text{ of } \boldsymbol{A},\\ p(a) \in F \text{ iff } p(b) \in F\}.$$

We will prove that θ is the largest congruence of $\boldsymbol{A}$ compatible with F.

First, let ϕ be a congruence of $\boldsymbol{A}$ compatible with F. We will show that $\phi \subseteq \theta$. Let $\langle a, b\rangle \in \phi$ and consider a unary polynomial function p of $\boldsymbol{A}$. Then there exist a formula $\varphi(x, y_1, \dots, y_n)$ and elements $e_1, \dots, e_n \in A$ such that

$$p(x) = \varphi^{\boldsymbol{A}}(x, e_1, \dots, e_n).$$

As congruences preserve formulas and $\langle a, b\rangle, \langle e_1, e_1\rangle, \dots, \langle e_n, e_n\rangle \in \phi$, we obtain

$$\langle p(a), p(b)\rangle = \langle \varphi^{\boldsymbol{A}}(a, e_1, \dots, e_n), \varphi^{\boldsymbol{A}}(b, e_1, \dots, e_n)\rangle \in \phi.$$

Since ϕ is compatible with F, this ensures that $p(a) \in F$ iff $p(b) \in F$, whence $\langle a, b\rangle \in \theta$. This establishes the inclusion $\phi \subseteq \theta$.

To conclude the proof, it suffices to show that θ is a congruence of $\boldsymbol{A}$ compatible with F. We begin by proving that θ is a congruence. Observe that θ is an equivalence relation, by definition. Then consider a basic n-ary operation f of $\boldsymbol{A}$ and $\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle \in \theta$. We need to show that $\langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(b_1, \dots, b_n)\rangle \in \theta$, that is, for every unary polynomial function p of $\boldsymbol{A}$,

$$p(f^{\boldsymbol{A}}(a_1, \dots, a_n)) \in F \Longleftrightarrow p(f^{\boldsymbol{A}}(b_1, \dots, b_n)) \in F.$$

Accordingly, consider a unary polynomial function p of $\boldsymbol{A}$. By symmetry, it will be enough to prove the implication from left to right in the above display. Then suppose that $p(f^{\boldsymbol{A}}(a_1, \dots, a_n)) \in F$. There exist a formula $\varphi(x, y_1, \dots, y_m)$ and elements $e_1, \dots, e_m \in A$ such that

$$p(x) = \varphi^{\boldsymbol{A}}(x, e_1, \dots, e_m).$$

For every positive integer $k \leqslant n$, we define a new unary polynomial function q_k of $\boldsymbol{A}$ as follows:

$$q_k(x) := \varphi^{\boldsymbol{A}}(f^{\boldsymbol{A}}(b_1, \dots, b_{k-1}, x, a_{k+1}, \dots, a_n), e_1, \dots, e_m).$$

We will prove, by induction on k, that $q_k(b_k) \in F$. For the base case, observe that

$$q_1(a_1) = \varphi^{\boldsymbol{A}}(f^{\boldsymbol{A}}(a_1, \dots, a_n), e_1, \dots, e_n) = p(f^{\boldsymbol{A}}(a_1, \dots, a_n)) \in F.$$

Together with $\langle a_1, b_1\rangle \in \theta$ and the definition of θ, this yields $q_1(b_1) \in F$, as desired.

For the step case, suppose that $q_k(b_k) \in F$. Since

$$q_k(b_k) = \varphi^{\boldsymbol{A}}(f^{\boldsymbol{A}}(b_1, \dots, b_k, a_{k+1}, \dots, a_n), e_1, \dots, e_m) = q_{k+1}(a_{k+1}),$$

this yields $q_{k+1}(a_{k+1}) \in F$. Together with $\langle a_k, b_k\rangle \in \theta$ and the definition of θ, this guarantees that $q_{k+1}(b_{k+1}) \in F$, thus concluding the inductive proof.

As a consequence, we obtain

$$p(f^{\boldsymbol{A}}(b_1, \dots, b_n)) = \varphi^{\boldsymbol{A}}(f^{\boldsymbol{A}}(b_1, \dots, b_n), e_1, \dots, e_m) = q_n(b_n) \in F.$$

Hence, we conclude that θ is a congruence of $\boldsymbol{A}$.

Then we turn to prove that θ is compatible with F. Suppose that $a \in F$ and $\langle a, b\rangle \in \theta$. Notice that the identity function $id\colon A \to A$ is a unary polynomial function of $\boldsymbol{A}$. Since $id(a) = a \in F$ and $\langle a, b\rangle \in \theta$, we obtain that $b = id(b) \in F$, as desired. ⊠

We denote the identity congruence of an algebra $\boldsymbol{A}$ by id_A. Given a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ and a congruence θ of $\boldsymbol{B}$, the relation

$$f^{-1}[\theta] = \{\langle a,b\rangle \in A \times A : \langle f(a), f(b)\rangle \in \theta\}$$

is always a congruence of $\boldsymbol{A}$. We will write $\mathsf{Ker}(f)$ as a shorthand for $f^{-1}[\mathrm{id}_B]$, i.e.,

$$\mathsf{Ker}(f) \coloneqq \{\langle a,b\rangle \in A \times A : f(a) = f(b)\}.$$

A *strict homomorphism* from a matrix $\langle \boldsymbol{A}, F\rangle$ to a matrix $\langle \boldsymbol{B}, G\rangle$ is a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ such that, for every $a \in A$,

$$a \in F \iff f(a) \in G.$$

Lemma 1.14. *If $f\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is a strict homomorphism, then $f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} G] \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F$. If, moreover, f is surjective, then $f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} G] = \boldsymbol{\Omega}^{\boldsymbol{A}} F$.*

Proof. As we mentioned, the relation $f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} G]$ is a congruence of $\boldsymbol{A}$. We will prove that it is compatible with F. Let $a, b \in A$ be such that $a \in F$ and $\langle a,b\rangle \in f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} G]$. Since f is strict, from $a \in F$ it follows $f(a) \in G$. Moreover, $\langle a,b\rangle \in f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} G]$ amounts to $\langle f(a), f(b)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} G$. As $\boldsymbol{\Omega}^{\boldsymbol{B}} G$ is compatible with G, this yields $f(b) \in G$. By the strictness of f, we obtain $b \in F$. Hence, $f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} G]$ is a congruence of $\boldsymbol{A}$ compatible with F. As $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is the largest such congruence, we conclude that $f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} G] \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F$.

Then we consider the case where f is surjective. We need to show that $\boldsymbol{\Omega}^{\boldsymbol{A}} F \subseteq f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} G]$, i.e., that if $\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$, then $\langle f(a), f(b)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}} G$. To this end, consider $\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$. In view of Proposition 1.13, in order to prove that $\langle f(a), f(b)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}} G$, it suffices to show that

$$p(f(a)) \in G \iff p(f(b)) \in G,$$

for every unary polynomial function p of $\boldsymbol{B}$. Accordingly, consider a unary polynomial function p of $\boldsymbol{B}$. Then there exist a formula $\varphi(x, \vec{z})$ and a sequence $\vec{c} \in B$ such that $p(x) = \varphi^{\boldsymbol{B}}(x, \vec{c})$. Since f is surjective, there exists $\vec{e} \in A$ such that $f(\vec{e}) = \vec{c}$. Then consider the unary polynomial function q of $\boldsymbol{A}$ defined as $q(x) \coloneqq \varphi^{\boldsymbol{A}}(x, \vec{e})$. We will prove that

$$q(a) \in F \iff f(q(a)) \in G \iff f(\varphi^{\boldsymbol{A}}(a, \vec{e})) \in G \iff \varphi^{\boldsymbol{B}}(f(a), \vec{c}) \in G \iff p(f(a)) \in G.$$

The first equivalence above holds because f is strict, the second by the definition of q, the third because f is a homomorphism such that $f(\vec{e}) = \vec{c}$, and the fourth by the definition of p. An analogous argument yields $q(b) \in F$ iff $p(f(b)) \in G$, whence

$$q(a) \in F \iff p(f(a)) \in G \;\text{ and }\; q(b) \in F \iff p(f(b)) \in G.$$

Since $\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$, Proposition 1.13 guarantees that $q(a) \in F$ iff $q(b) \in F$. Together with the above display, this implies $p(f(a)) \in G$ iff $p(f(b)) \in G$, as desired. ⊠

Corollary 1.15. *If $f\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is a strict surjective homomorphism and $\boldsymbol{\Omega}^{\boldsymbol{B}} G = \mathrm{id}_B$, then $\boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathsf{Ker}(f)$.*

Proof. Since $f^{-1}[\mathrm{id}_B] = \mathsf{Ker}(f)$, Lemma 1.14 yields the desired conclusion. ⊠

The *reduction* of a matrix $\langle \boldsymbol{A}, F\rangle$ is the matrix

$$\langle \boldsymbol{A}, F\rangle^* := \langle \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F, F/\boldsymbol{\Omega}^{\boldsymbol{A}}F\rangle,$$

where $F/\boldsymbol{\Omega}^{\boldsymbol{A}}F := \{a/\boldsymbol{\Omega}^{\boldsymbol{A}}F : a \in F\}$. Furthermore, a matrix $\langle \boldsymbol{A}, F\rangle$ is said to be *reduced* when $\boldsymbol{\Omega}^{\boldsymbol{A}}F = \mathrm{id}_A$.

Proposition 1.16. *For every matrix $\langle \boldsymbol{A}, F\rangle$, the canonical map $p_F \colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{A}, F\rangle^*$, defined as $p_F(a) = a/\boldsymbol{\Omega}^{\boldsymbol{A}}F$, is a strict surjective homomorphism. Moreover, the matrix $\langle \boldsymbol{A}, F\rangle^*$ is reduced.*

Proof. Clearly, $p_F \colon \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is a surjective homomorphism. We will prove that, for every $a \in A$,

$$a \in F \Longleftrightarrow p_F(a) \in F/\boldsymbol{\Omega}^{\boldsymbol{A}}F.$$

The implication from left to right is immediate. To prove the other implication, suppose that $p_F(a) \in F/\boldsymbol{\Omega}^{\boldsymbol{A}}F$. Then there exists $b \in F$ such that $a/\boldsymbol{\Omega}^{\boldsymbol{A}}F = p_F(a) = b/\boldsymbol{\Omega}^{\boldsymbol{A}}F$, that is, $\langle b, a\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$. As $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is compatible with F, we conclude that $a \in F$. Thus, p_F is a strict surjective homomorphism.

It only remains to prove that $\boldsymbol{\Omega}^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F}(F/\boldsymbol{\Omega}^{\boldsymbol{A}}F)$ is the identity relation on $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$. Then consider $a, b \in A$ such that $\langle a/\boldsymbol{\Omega}^{\boldsymbol{A}}F, b/\boldsymbol{\Omega}^{\boldsymbol{A}}F\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F}(F/\boldsymbol{\Omega}^{\boldsymbol{A}}F)$. Since p_F is a strict surjective homomorphism, we can apply Lemma 1.14, obtaining

$$\langle a, b\rangle \in p_F^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F}(F/\boldsymbol{\Omega}^{\boldsymbol{A}}F)] = \boldsymbol{\Omega}^{\boldsymbol{A}}F.$$

Hence, we conclude that $a/\boldsymbol{\Omega}^{\boldsymbol{A}}F = b/\boldsymbol{\Omega}^{\boldsymbol{A}}F$. ⊠

1.4 Reduced models

In what follows, we will use repeatedly the next technical observation:

Lemma 1.17. *Let $f \colon \boldsymbol{A} \to \boldsymbol{B}$ and $g \colon \boldsymbol{Fm} \to \boldsymbol{B}$ be homomorphisms. If f is surjective, there exists a homomorphism $h \colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $g = f \circ h$.*

Proof. Since $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is surjective, for every variable x, we can choose an element $a_x \in A$ such that $f(a_x) = g(x)$. By the universal property of the algebra of formulas $\boldsymbol{Fm}$, there exists a unique homomorphism $h \colon \boldsymbol{Fm} \to \boldsymbol{A}$ that sends each variable x to the corresponding element a_x. As the maps g and $f \circ h$ coincide on the set of variables and $\boldsymbol{Fm}$ is generated by this set, we conclude that $g = f \circ h$. ⊠

From a logical standpoint, strict homomorphisms are motivated as follows:

Proposition 1.18. *If $f \colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is a strict surjective homomorphism, the logics induced by the matrices $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, G\rangle$ coincide.*

Proof. Let $\vdash_{\boldsymbol{A}}$ and $\vdash_{\boldsymbol{B}}$ be the logics induced by $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, G\rangle$, respectively. To prove that $\vdash_{\boldsymbol{A}} = \vdash_{\boldsymbol{B}}$, it suffices to show that F is a deductive filter of $\vdash_{\boldsymbol{B}}$ on $\boldsymbol{A}$ and that G is a deductive filter of $\vdash_{\boldsymbol{A}}$ on $\boldsymbol{B}$. Since the strictness of f ensures that $F = f^{-1}[G]$, the first is an immediate consequence of Proposition 1.10. To prove that the latter holds too, suppose that $\Gamma \vdash_{\boldsymbol{A}} \varphi$ and consider a homomorphism $g \colon \boldsymbol{Fm} \to \boldsymbol{B}$ such that $g[\Gamma] \subseteq G$. Since f is surjective, we can apply Lemma 1.17, obtaining a homomorphism $h \colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $g = f \circ h$. Therefore, from $f[h[\Gamma]] = g[\Gamma] \subseteq G$ it follows

$h[\Gamma] \subseteq f^{-1}[G] = F$ (where the latter equality is a consequence of the strictness of f). As $\Gamma \vdash_{\boldsymbol{A}} \varphi$ and the definition of $\vdash_{\boldsymbol{A}}$ guarantees that F is a deductive filter of $\vdash_{\boldsymbol{A}}$ on $\boldsymbol{A}$, this yields $h(\varphi) \in F$. Together with the assumption that f is strict, this implies $g(\varphi) = f(h(\varphi)) \in f[F] \subseteq G$, as desired. ⊠

Corollary 1.19. *The logic induced by a matrix $\langle \boldsymbol{A}, F\rangle$ coincides with the one induced by its reduction $\langle \boldsymbol{A}, F\rangle^*$.*

Proof. Immediate from Propositions 1.16 and 1.18. ⊠

A matrix $\langle \boldsymbol{A}, F\rangle$ is said to be a *model* of a logic $\vdash$ when F is a deductive filter of $\vdash$ on $\boldsymbol{A}$ or, equivalently, when the logic induced by $\langle \boldsymbol{A}, F\rangle$ is an extension of $\vdash$. The class of models of $\vdash$ will be denoted by $\mathsf{Mod}(\vdash)$.

Proposition 1.20. *Every logic $\vdash$ is complete with respect to* $\mathsf{Mod}(\vdash)$.

Proof. The definition of $\mathsf{Mod}(\vdash)$ guarantees that the logic induced by $\mathsf{Mod}(\vdash)$ is an extension of $\vdash$. Therefore, it only remains to prove that $\vdash$ is an extension of the logic induced by $\mathsf{Mod}(\vdash)$. To this end, recall from Theorem 1.1 that $\vdash$ has a matrix semantics M. Since $\vdash$ is the logic induced by M, we have $\mathsf{M} \subseteq \mathsf{Mod}(\vdash)$. Consequently, the logic induced by M (namely, $\vdash$) is an extension of the logic induced by $\mathsf{Mod}(\vdash)$. ⊠

The class of the *reduced models* of a logic $\vdash$ is

$$\mathsf{Mod}^*(\vdash) := \{\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash) : \boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{id}_A\}.$$

In the next result, $\mathbb{I}$ is the class operator of closure under isomorphic copies.

Theorem 1.21. *For every logic $\vdash$, we have*

$$\mathsf{Mod}^*(\vdash) = \mathbb{I}\{\langle \boldsymbol{A}, F\rangle^* : \langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)\}.$$

Furthermore, $\vdash$ is complete with respect to $\mathsf{Mod}^*(\vdash)$.

Proof. The inclusion from left to right in the statement is an immediate consequence of the definitions. Since $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{I}$, to prove the reverse inclusion, it suffices to consider a matrix of the form $\langle \boldsymbol{A}, F\rangle^*$ with $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$. From Corollary 1.19 and the assumption that $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ it follows $\langle \boldsymbol{A}, F\rangle^* \in \mathsf{Mod}(\vdash)$. Furthermore, by Proposition 1.16, the matrix $\langle \boldsymbol{A}, F\rangle^*$ is reduced, whence $\langle \boldsymbol{A}, F\rangle^* \in \mathsf{Mod}^*(\vdash)$. This establishes the equality in the statement.

Therefore, it only remains to prove that $\vdash$ is complete with respect to $\mathsf{Mod}^*(\vdash)$. In view of the equality in the statement and Corollary 1.19, the logics induced by $\mathsf{Mod}(\vdash)$ and $\mathsf{Mod}^*(\vdash)$ coincide. Together with Proposition 1.20, this yields that $\vdash$ is complete with respect to $\mathsf{Mod}^*(\vdash)$. ⊠

Remark 1.22. The class $\mathsf{Mod}^*(\vdash)$ is often understood as the intended matrix semantics of the logic $\vdash$. Even if we shall not pursue this here, this claim is supported by the evidence that $\mathsf{Mod}^*(\vdash)$ is the class of models canonically associated with an equality-free theory that formalizes $\vdash$ (see, e.g., [13, 20, 34, 35]). ⊠

We close this section with observations that will be instrumental in describing the class of the reduced models of $\mathsf{K}_g, \mathsf{K}_\ell$, and IPC in Example 2.7.

Proposition 1.23. *Let $\vdash$ be the logic induced by a class of matrices whose algebraic reducts belong to a class* K *of algebras. If an equation holds in* K*, then it also holds in the algebraic reducts of the matrices in* $\mathsf{Mod}^*(\vdash)$*.*

Proof. Suppose that $\mathsf{K} \vDash \varphi(\vec{x}) \approx \psi(\vec{x})$. Since $\vdash$ is induced by a class of matrices whose algebraic reducts belong to K, this implies

$$\delta(\varphi(\vec{x}), \vec{z}) \vdash \delta(\psi(\vec{x}), \vec{z}) \quad \text{and} \quad \delta(\psi(\vec{x}), \vec{z}) \vdash \delta(\varphi(\vec{x}), \vec{z}), \tag{1.2}$$

for every formula $\delta(y, \vec{z})$.

Then consider a matrix $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*(\vdash)$ and a sequence $\vec{a} \in A$. We need to prove that $\varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a})$. Since $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*(\vdash)$, we have $\boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{id}_A$. Therefore, it suffices to show that $\langle \varphi^{\boldsymbol{A}}(\vec{a}), \psi^{\boldsymbol{A}}(\vec{a}) \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$. In view of Proposition 1.13, this amounts to the demand that

$$p(\varphi^{\boldsymbol{A}}(\vec{a})) \in F \Longleftrightarrow p(\psi^{\boldsymbol{A}}(\vec{a})) \in F,$$

for every unary polynomial function p of $\boldsymbol{A}$. Accordingly, consider a unary polynomial function p of $\boldsymbol{A}$. Then there exist a formula $\delta(y, \vec{z})$ and a sequence $\vec{c} \in A$ such that $p(x) = \delta^{\boldsymbol{A}}(x, \vec{c})$. By symmetry, it will be enough to prove the implication from left to right in the above display. Then suppose that $\delta^{\boldsymbol{A}}(\varphi^{\boldsymbol{A}}(\vec{a}), \vec{c}) = p(\varphi^{\boldsymbol{A}}(\vec{a})) \in F$. Since $\langle \boldsymbol{A}, F \rangle$ is a model of $\vdash$ and, by Condition (1.2), we have $\delta(\varphi(\vec{x}), \vec{z}) \vdash \delta(\psi(\vec{x}), \vec{z})$, this implies $p(\psi^{\boldsymbol{A}}(\vec{a})) = \delta^{\boldsymbol{A}}(\psi^{\boldsymbol{A}}(\vec{a}), \vec{c}) \in F$. ⊠

Proposition 1.24. *Let $\vdash$ be the logic induced by a class* M *of matrices such that $|F| \leqslant 1$, for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{M}$. Then $|F| \leqslant 1$, for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*(\vdash)$.*

Proof. We will prove that, for every formula $\varphi(x, \vec{z})$,

$$x, y, \varphi(x, \vec{z}) \vdash \varphi(y, \vec{z}). \tag{1.3}$$

Consider a matrix $\langle \boldsymbol{A}, F \rangle \in \mathsf{M}$ and a homomorphism $f \colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that

$$f(x), f(y), f(\varphi(x, \vec{z})) \in F.$$

Then $F \neq \emptyset$. By assumption, this implies that F is a singleton, whence $f(x) = f(y)$. As a consequence, we obtain

$$f(\varphi(y, \vec{z})) = \varphi^{\boldsymbol{A}}(f(y), f(\vec{z})) = \varphi^{\boldsymbol{A}}(f(x), f(\vec{z})) = f(\varphi(x, \vec{z})) \in F,$$

where the second equality follows from $f(x) = f(y)$ and the last step $f(\varphi(x, \vec{z})) \in F$ holds by assumption. Since $\vdash$ is the logic induced by M, we conclude that $x, y, \varphi(x, \vec{z}) \vdash \varphi(y, \vec{z})$.

Now, consider a matrix $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*(\vdash)$. In order to prove that $|F| \leqslant 1$, it suffices to show that, for every $a, b \in A$, if $a, b \in F$, then $a = b$. Accordingly, consider $a, b \in F$. From Condition (1.3) and the assumption that $\langle \boldsymbol{A}, F \rangle$ is a model of $\vdash$ it follows that, for every formula $\varphi(x, \vec{z})$ and $\vec{c} \in A$,

$$\varphi^{\boldsymbol{A}}(a, \vec{c}) \in F \Longleftrightarrow \varphi^{\boldsymbol{A}}(b, \vec{c}) \in F.$$

In other words, $p(a) \in F$ iff $p(b) \in F$, for every unary polynomial function p of $\boldsymbol{A}$. By Proposition 1.13, this means that $\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$. Since, by assumption, the matrix $\langle \boldsymbol{A}, F \rangle$ is reduced, we conclude that $a = b$. ⊠

Corollary 1.25. *If* $\vdash$ *is an assertional logic and* $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*(\vdash)$, *then* $F = \{1\}$.

Proof. By assumption, $\vdash$ is the assertional logic of some class K of algebras. Then $\vdash$ is also the logic induced by the class $\{\langle \boldsymbol{A}, \{1\} \rangle : \boldsymbol{A} \in \mathsf{K}\}$ of matrices, as detailed in Example 1.2. Consequently, we can apply Proposition 1.24, obtaining that $|F| \leqslant 1$, for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*(\vdash)$. Furthermore, the definition of an assertional logic ensures that $\emptyset \vdash 1$. Therefore, every deductive filter of $\vdash$ on an algebra $\boldsymbol{A}$ should contain the element 1. It follows that $F = \{1\}$, for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*(\vdash)$. ⊠

CHAPTER 2

Protoalgebraic logics

In this section, we will focus on the logics for which the Leibniz congruence can be defined by means of a set $\Delta(x, y, \vec{z})$ of formulas (cf. Proposition 1.13). In this case, the variables in the sequence $\vec{z}$ are sometimes called *parameters*.

Definition 2.1. A logic $\vdash$ is said to be *protoalgebraic* if there exists a set $\Delta(x, y, \vec{z})$ of formulas such that, for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and $a, b \in A$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \Longleftrightarrow \Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in A.$$

In this case, we say that Δ is a set of *equivalence formulas* for $\vdash$.

Protoalgebraic logics were introduced in [10] and [26, 27] (see also [13]) and constitute the core of abstract algebraic logic. Their theory is enshrined in the monograph [28].*

2.1 Sets of equivalence formulas

Given a set $\Delta(x, y, \vec{z})$ of formulas and $\varphi, \psi \in Fm$, we write

$$\Delta\langle\varphi, \psi\rangle \coloneqq \{\delta(\varphi, \psi, \vec{\gamma}) : \delta \in \Delta(x, y, \vec{z}) \text{ and } \vec{\gamma} \in Fm\}.$$

Sets of equivalence formulas can be characterized as follows [28, Thm. 1.2.4]:

Theorem 2.2. *A set $\Delta(x, y, \vec{z}) \subseteq Fm$ is a set of equivalence formulas for a logic $\vdash$ iff*

$$\emptyset \vdash \Delta\langle x, x\rangle \quad x, \Delta\langle x, y\rangle \vdash y \quad \Delta\langle x_1, y_1\rangle, \dots, \Delta\langle x_n, y_n\rangle \vdash \Delta\langle f(x_1, \dots, x_n), f(y_1, \dots, y_n)\rangle,$$

for every n-ary connective f of $\vdash$.

Proof. We begin by proving the implication from left to right. Suppose that $\Delta(x, y, \vec{z})$ is a set of equivalence formulas for $\vdash$. We will show that

$$\Delta\langle x_1, y_1\rangle, \dots, \Delta\langle x_n, y_n\rangle \vdash \Delta\langle f(x_1, \dots, x_n), f(y_1, \dots, y_n)\rangle, \tag{2.1}$$

*An intriguing variant of the notion of a protoalgebraic logic emerged recently in [59, 60].

for every n-ary connective f of $\vdash$. To this end, consider the theory

$$\Gamma := \mathrm{Cn}_{\vdash}(\Delta\langle x_1, y_1\rangle \cup \cdots \cup \Delta\langle x_n, y_n\rangle).$$

By the definition of Γ, we have

$$\Delta(x_1, y_1, \vec{\gamma}) \cup \cdots \cup \Delta(x_n, y_n, \vec{\gamma}) \subseteq \Gamma, \text{ for every } \vec{\gamma} \in Fm.$$

Since, by Proposition 1.7, $\langle \boldsymbol{Fm}, \Gamma\rangle$ is a model of $\vdash$ and, by assumption, $\Delta(x, y, \vec{z})$ is a set of equivalence formulas for $\vdash$, this implies

$$\langle x_1, y_1\rangle, \dots, \langle x_n, y_n\rangle \in \boldsymbol{\Omega}\Gamma.$$

As $\boldsymbol{\Omega}\Gamma$ is a congruence of $\boldsymbol{Fm}$, this yields

$$\langle f(x_1, \dots, x_n), f(y_1, \dots, y_n)\rangle \in \boldsymbol{\Omega}\Gamma.$$

Lastly, since $\langle \boldsymbol{Fm}, \Gamma\rangle$ is a model of $\vdash$ and $\Delta(x, y, \vec{z})$ a set of equivalence formulas for $\vdash$, we conclude that

$$\Delta(f(x_1, \dots, x_n), f(y_1, \dots, y_n), \vec{\gamma}) \subseteq \Gamma, \text{ for every } \vec{\gamma} \in \Gamma.$$

By the definition of Γ, this amounts to the validity of Condition (2.1).

The proof that $\emptyset \vdash \Delta\langle x, x\rangle$ is analogous and, therefore, omitted. Thus, it only remains to prove that $x, \Delta\langle x, y\rangle \vdash y$. Consider the theory

$$\Gamma := \mathrm{Cn}_{\vdash}(\{x\} \cup \Delta\langle x, y\rangle).$$

Since $\langle \boldsymbol{Fm}, \Gamma\rangle$ is a model of $\vdash$ and $\Delta(x, y, \vec{z})$ a set of equivalence formulas for $\vdash$, from $\Delta\langle x, y\rangle \subseteq \Gamma$ it follows $\langle x, y\rangle \in \boldsymbol{\Omega}\Gamma$. Moreover, the definition of Γ ensures that $x \in \Gamma$. Consequently,

$$x \in \Gamma \quad \text{and} \quad \langle x, y\rangle \in \boldsymbol{\Omega}\Gamma.$$

Since the congruence $\boldsymbol{\Omega}\Gamma$ is compatible with Γ, this yields $y \in \Gamma$. By the definition of Γ, we conclude that $x, \Delta\langle x, y\rangle \vdash y$, as desired.

Then we turn to prove the implication from right to left in the statement. The proof proceeds through a series of observations:

Claim 2.3. *For every formula $\varphi(x_1, \dots, x_n, \vec{\varepsilon})$,*

$$\Delta\langle x_1, y_1\rangle, \dots, \Delta\langle x_n, y_n\rangle \vdash \Delta\langle \varphi(x_1, \dots, x_n, \vec{\varepsilon}), \varphi(y_1, \dots, y_n, \vec{\varepsilon})\rangle.$$

Proof of the Claim. By induction on the construction of φ, using the assumption that $\emptyset \vdash \Delta\langle x, x\rangle$ and that, for every n-ary connective f of $\vdash$,

$$\Delta\langle x_1, y_1\rangle, \dots, \Delta\langle x_n, y_n\rangle \vdash \Delta\langle f(x_1, \dots, x_n), f(y_1, \dots, y_n)\rangle. \qquad \boxtimes$$

Claim 2.4. *We have*

$$\Delta\langle x, y\rangle \vdash \Delta\langle y, x\rangle \quad \textit{and} \quad \Delta\langle x_1, x_2\rangle, \Delta\langle x_2, x_3\rangle \vdash \Delta\langle x_1, x_3\rangle.$$

Proof of the Claim. To prove that $\Delta\langle x, y\rangle \vdash \Delta\langle y, x\rangle$, consider $\delta(x, y, \vec{z}) \in \Delta(x, y, \vec{z})$ and $\vec{\gamma} \in Fm$. We need to show that $\Delta\langle x, y\rangle \vdash \delta(y, x, \vec{\gamma})$. From Claim 2.3 it follows $\Delta\langle x, y\rangle \vdash \Delta\langle \delta(x, x, \vec{\gamma}), \delta(y, x, \vec{\gamma})\rangle$. Moreover, by the assumption that $\emptyset \vdash \Delta\langle x, x\rangle$ and $\delta(x, y, \vec{z}) \in \Delta$, we have $\emptyset \vdash \delta(x, x, \vec{\gamma})$. Thus,

$$\Delta\langle x, y\rangle \vdash \delta(x, x, \vec{\gamma}), \Delta\langle \delta(x, x, \vec{\gamma}), \delta(y, x, \vec{\gamma})\rangle.$$

Together with the assumption that $x, \Delta\langle x, y\rangle \vdash y$, this yields $\Delta\langle x, y\rangle \vdash \delta(y, x, \vec{\gamma})$, as desired.

Then we turn to prove that $\Delta\langle x_1, x_2\rangle, \Delta\langle x_2, x_3\rangle \vdash \Delta\langle x_1, x_3\rangle$. Consider $\delta(x, y, \vec{z}) \in \Delta(x, y, \vec{z})$ and $\vec{\gamma} \in Fm$. We need to show that $\Delta\langle x_1, x_2\rangle, \Delta\langle x_2, x_3\rangle \vdash \delta(x_1, x_3, \vec{\gamma})$. From $\Delta\langle x, y\rangle \vdash \Delta\langle y, x\rangle$ it follows $\Delta\langle x_1, x_2\rangle \vdash \Delta\langle x_2, x_1\rangle$. Furthermore, by Claim 2.3, we have $\Delta\langle x_2, x_1\rangle, \Delta\langle x_2, x_3\rangle \vdash \Delta\langle \delta(x_2, x_2, \vec{\gamma}), \delta(x_1, x_3, \vec{\gamma})\rangle$, whence

$$\Delta\langle x_1, x_2\rangle, \Delta\langle x_2, x_3\rangle \vdash \Delta\langle \delta(x_2, x_2, \vec{\gamma}), \delta(x_1, x_3, \vec{\gamma})\rangle.$$

Lastly, from the assumption that $\emptyset \vdash \Delta\langle x, x\rangle$ and $\delta(x, y, \vec{z}) \in \Delta$ it follows $\emptyset \vdash \delta(x_2, x_2, \vec{\gamma})$. Consequently,

$$\Delta\langle x_1, x_2\rangle, \Delta\langle x_2, x_3\rangle \vdash \delta(x_2, x_2, \vec{\gamma}), \Delta\langle \delta(x_2, x_2, \vec{\gamma}), \delta(x_1, x_3, \vec{\gamma})\rangle.$$

Together with $x, \Delta\langle x, y\rangle \vdash y$, this yields $\Delta\langle x_1, x_2\rangle, \Delta\langle x_2, x_3\rangle \vdash \delta(x_1, x_3, \vec{\gamma})$. $\boxtimes$

Now, consider a model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$. We need to prove that

$$\boldsymbol{\Omega}^{\boldsymbol{A}} F = \{\langle a, b\rangle \in A \times A : \Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in A\}.$$

To this end, consider the relation

$$\theta := \{\langle a, b\rangle \in A \times A : \Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in A\}.$$

By the assumption and Claim 2.4, we have

$$\begin{aligned} \emptyset &\vdash \Delta\langle x, x\rangle \\ \Delta\langle x, y\rangle &\vdash \Delta\langle y, x\rangle \\ \Delta\langle x_1, x_2\rangle, \Delta\langle x_2, x_3\rangle &\vdash \Delta\langle x_1, x_3\rangle \\ \Delta\langle x_1, y_1\rangle, \dots, \Delta\langle x_n, y_n\rangle &\vdash \Delta\langle f(x_1, \dots, x_n), f(y_1, \dots, y_n)\rangle, \end{aligned}$$

for every n-ary connective f. We will use these facts to prove that θ is a congruence of $\boldsymbol{A}$. As an exemplification, we shall detail the proof that θ is transitive. Suppose that $\langle a_1, a_2\rangle, \langle a_2, a_3\rangle \in \theta$. Then

$$\Delta^{\boldsymbol{A}}(a_1, a_2, \vec{c}) \cup \Delta^{\boldsymbol{A}}(a_2, a_3, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in A. \tag{2.2}$$

We will prove that

$$\Delta^{\boldsymbol{A}}(a_1, a_3, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in A. \tag{2.3}$$

To this end, consider $\delta(x, y, z_1, \dots, z_n) \in \Delta(x, y, \vec{z})$ and $c_1, \dots, c_n \in A$. Moreover, let $f \colon \boldsymbol{Fm} \to \boldsymbol{A}$ be any homomorphism such that

$$f(x_1) = a_1 \qquad f(x_2) = a_2 \qquad f(x_3) = a_3 \qquad f(z_1) = c_1 \quad \dots \quad f(z_n) = c_n,$$

where x_1, x_2, and x_3 are three variables distinct from $z_1, \ldots, z_n$. From Condition (2.2) and above display it follows

$$f[\Delta\langle x_1, x_2\rangle \cup \Delta\langle x_2, x_3\rangle] \subseteq F.$$

Since $\Delta\langle x_1, x_2\rangle, \Delta\langle x_2, x_3\rangle \vdash \Delta\langle x_1, x_3\rangle$ and $\langle \boldsymbol{A}, F\rangle$ is a model of $\vdash$, this implies

$$\delta^{\boldsymbol{A}}(a_1, a_3, c_1, \ldots, c_n) = f(\delta(x_1, x_3, z_1, \ldots, z_n)) \in f[\Delta\langle x_1, x_3\rangle] \subseteq F,$$

thus establishing Condition (2.3). By the definition of θ, this amounts to $\langle a_1, a_3\rangle \in \theta$. Hence, we conclude that θ is transitive. The remaining part of the proof that θ is a congruence of $\boldsymbol{A}$ proceeds analogously and, therefore, is omitted.

Lastly, we will prove that θ is compatible with F. To this end, consider a pair of elements $a, b \in A$ such that $a \in F$ and $\langle a, b\rangle \in \theta$. By the definition of θ, we have

$$\{a\} \cup \Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in A.$$

Then consider any homomorphism $f\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $f(x) = a$ and $f(y) = b$. Clearly, we have $f[\{x\} \cup \Delta\langle x, y\rangle] \subseteq F$. Since we assumed that $x, \Delta\langle x, y\rangle \vdash y$ and that $\langle \boldsymbol{A}, F\rangle$ is a model of $\vdash$, this implies $b \in F$. Hence, θ is compatible with F. Since, by definition, $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the largest congruence of $\boldsymbol{A}$ compatible with F, we conclude that $\theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F$.

To prove the other inclusion, consider a pair $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$. To show that $\langle a, b\rangle \in \theta$, it suffices to prove that $\Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F$, for every $\vec{c} \in A$. To this end, given $\delta(x, y, \vec{z}) \in \Delta(x, y, \vec{z})$ and $\vec{c} \in A$, we define a unary polynomial function $p(x)$ of $\boldsymbol{A}$ letting

$$p(x) := \delta^{\boldsymbol{A}}(x, b, \vec{c}).$$

Since $\emptyset \vdash \Delta\langle x, x\rangle$ and $\delta(x, y, \vec{z}) \in \Delta$, we have $\emptyset \vdash \delta(x, x, \vec{z})$. As $\langle \boldsymbol{A}, F\rangle$ is a model of $\vdash$, this implies

$$p(b) = \delta^{\boldsymbol{A}}(b, b, \vec{c}) \in F.$$

Since $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$, we can apply Proposition 1.13, obtaining

$$\delta^{\boldsymbol{A}}(a, b, \vec{c}) = p(a) \in F$$

as well. Hence, we conclude that $\Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F$, for every $\vec{c} \in A$. ⊠

As a logic is protoalgebraic iff it possesses a set of equivalence formulas, Theorem 2.2 can be viewed as a syntactic characterization of protoalgebraic logics. Furthermore, it implies the following:

Corollary 2.5. *Every extension of a protoalgebraic logic is protoalgebraic with the same set of equivalence formulas.*

Example 2.6. The definitions of K_g and IPC guarantee the validity of the rules

$$\emptyset \rhd x \leftrightarrow x \quad x, x \leftrightarrow y \rhd y \quad x_1 \leftrightarrow y_1, \ldots, x_n \leftrightarrow y_n \rhd f(x_1, \ldots, x_n) \leftrightarrow f(y_1, \ldots, y_n),$$

for each of their n-ary connectives f, where $x \leftrightarrow y$ is a shorthand for $(x \to y) \wedge (y \to x)$. Therefore, K_g and IPC are both protoalgebraic with set of equivalence formulas $\{x \leftrightarrow y\}$, by Theorem 2.2.

The case of K_ℓ is slightly different, as this logic does not validate the rule $x \leftrightarrow y \rhd \Box x \leftrightarrow \Box y$. On the other hand, letting

$$\Delta(x, y) := \{\Box^n(x \leftrightarrow y) : n \in \mathbb{N}\},$$

it is easy to see the following rules are in valid in K_ℓ:

$$\emptyset \rhd \Delta(x, x) \quad x, \Delta(x, y) \rhd y \quad \Delta(x_1, y_1), \ldots, \Delta(x_n, y_n) \rhd \Delta(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)),$$

for every n-ary connective f. Consequently, K_ℓ is protoalgebraic as well, but with the infinite set of equivalence formulas, namely, $\Delta(x, y)$.

Notice the parameters $\vec{z}$ do not occur in the sets of equivalence formulas $\Delta(x, y, \vec{z})$ of K_g, K_ℓ, and IPC. A case in which the presence of the parameters $\vec{z}$ is necessary will be discussed in Example 3.7. ⊠

Sets of equivalence formulas are instrumental in describing the reduced models of concrete logics, as we proceed to illustrate for the case of K_g, K_ℓ, and IPC.

Example 2.7. First, we will prove that

$$\mathsf{Mod}^*(\mathsf{K}_g) = \{\langle \boldsymbol{A}, \{1\}\rangle : \boldsymbol{A} \text{ is a modal algebra}\}.$$

Consider a matrix $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathsf{K}_g)$. Recall from Example 1.5 that K_g is the logic induced by a class of matrices whose algebraic reducts are modal algebras. Since the class of modal algebras can be defined by means of equations, Proposition 1.23 implies that $\boldsymbol{A}$ is also a modal algebra. Furthermore, from Corollary 1.25 it follows that $F = \{1\}$.

Then we turn to prove the inclusion from right to left in the above display. Consider a matrix $\langle \boldsymbol{A}, \{1\}\rangle$, where $\boldsymbol{A}$ is a modal algebra. Since $\{1\}$ is an open lattice filter of $\boldsymbol{A}$, it is also a filter of K_g, by Example 1.12. Hence, $\langle \boldsymbol{A}, \{1\}\rangle$ is a model of K_g. Therefore, it only remains to prove that $\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\} = \mathrm{id}_A$. Since $\{x \leftrightarrow y\}$ a set of equivalence formulas for K_g and $\langle \boldsymbol{A}, \{1\}\rangle$ a model of K_g, we have

$$\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\} = \{\langle a, b\rangle \in A \times A : a \leftrightarrow b = 1\}.$$

As modal algebras satisfy the sentence $\forall xy(x \leftrightarrow y \approx 1 \iff x \approx y)$, the right hand side of the above display is the identity relation, as desired.

An analogous argument shows that

$$\mathsf{Mod}^*(\mathsf{IPC}) = \{\langle \boldsymbol{A}, \{1\}\rangle : \boldsymbol{A} \text{ is a Heyting algebra}\}.$$

Therefore, we turn to prove the following [61, Thm. II.4]:

$$\begin{aligned}\mathsf{Mod}^*(\mathsf{K}_\ell) = \{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} &\text{ is a modal algebra, } F \text{ a lattice filter, and} \\ &\{1\} \text{ is the only open lattice filter contained in } F\}.\end{aligned} \tag{2.4}$$

Let $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathsf{K}_\ell)$. As for the case of K_g, Proposition 1.23 guarantees that $\boldsymbol{A}$ is a modal algebra. Together with the fact that F is a deductive filter of K_ℓ, this yields that F is a lattice filter of $\boldsymbol{A}$, as explained in Example 1.9. Since $\{1\} \subseteq F$ (because F is a lattice filter) and $\{1\}$ is an open lattice filter, it only remains to show that if $G \subseteq F$ is an open lattice filter, then $G = \{1\}$. Let $G \subseteq F$ be an open lattice filter. Then consider the congruence

$$\theta_G := \{\langle a, b\rangle \in A \times A : a \leftrightarrow b \in G\}$$

of $\boldsymbol{A}$ associated with G in Example 2.6 and recall that $G = 1/\theta_G$. We will prove that θ_G is compatible with F. To this end, let $a \in F$ and $\langle a, b\rangle \in \theta_G$, i.e., $a \leftrightarrow b \in G$. Since $G \subseteq F$, this yields $a, a \leftrightarrow b \in F$. As $\langle \boldsymbol{A}, F\rangle$ is a model of K_ℓ and $x, x \leftrightarrow y \vdash_{\mathsf{K}_\ell} y$, we obtain $b \in F$, as desired. Since $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is the largest congruence of $\boldsymbol{A}$ compatible with F, we get $\theta_G \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{id}_A$, where the last equality follows from the assumption that the matrix $\langle \boldsymbol{A}, F\rangle$ is reduced. Lastly, from $\theta_G = \mathrm{id}_A$ and $1/\theta_G = G$ it follows $G = \{1\}$.

Then we turn to prove the inclusion from right to left in Condition (2.4). Let $\langle \boldsymbol{A}, F\rangle$ be a matrix such that $\boldsymbol{A}$ is a modal algebra and F a lattice filter such that $\{1\}$ is the only open lattice filter contained in F. In view of Example 1.9, the matrix $\langle \boldsymbol{A}, F\rangle$ is a model of K_ℓ. Therefore, it only remains to show that $\boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{id}_A$. To this end, recall from Example 2.6 that

$$\Delta(x, y) := \{\Box^n(x \leftrightarrow y) : n \in \mathbb{N}\}$$

is a set of equivalence formulas for K_ℓ, whence

$$\boldsymbol{\Omega}^{\boldsymbol{A}} F = \{\langle a, b\rangle \in A \times A : \Delta^{\boldsymbol{A}}(a, b) \subseteq F\}. \tag{2.5}$$

We will prove that the set

$$\begin{aligned} G := \{c \in A : c = 1 \text{ or } &\Box^{k_1}(a_1 \leftrightarrow b_1) \wedge \cdots \wedge \Box^{k_n}(a_n \leftrightarrow b_n) \leqslant c, \\ &\text{for some } n, k_1, \dots, k_n \in \mathbb{N} \text{ and } a_1, b_1, \dots, a_n, b_n \in A \\ &\text{s.t. } \Delta^{\boldsymbol{A}}(a_m, b_m) \subseteq F, \text{ for every } m \leqslant n\} \end{aligned}$$

is an open lattice filter of $\boldsymbol{A}$ contained in F. Clearly, G is a lattice filter. To prove that it is open, consider $c \in G$. If $c = 1$, then $\Box c = \Box 1 = 1 \in G$ and we are done. Then we consider the case where there are $n, k_1, \dots, k_n \in \mathbb{N}$ and $a_1, b_1, \dots, a_n, b_n \in A$ such that

$$\Box^{k_1}(a_1 \leftrightarrow b_1) \wedge \cdots \wedge \Box^{k_n}(a_n \leftrightarrow b_n) \leqslant c \text{ and } \Delta^{\boldsymbol{A}}(a_m, b_m) \subseteq F, \text{ for every } m \leqslant n. \tag{2.6}$$

Since the operation $\Box$ is order preserving and commutes with finite meets in $\boldsymbol{A}$, we have

$$\Box^{k_1+1}(a_1 \leftrightarrow b_1) \wedge \cdots \wedge \Box^{k_n+1}(a_n \leftrightarrow b_n) = \Box(\Box^{k_1}(a_1 \leftrightarrow b_1) \wedge \cdots \wedge \Box^{k_n}(a_n \leftrightarrow b_n)) \leqslant \Box c.$$

Therefore, we conclude that $\Box c \in G$. Thus, G is an open lattice filter of $\boldsymbol{A}$. To prove that $G \subseteq F$, consider $c \in G$. Since $1 \in F$, we may assume, without loss of generality, that $c \neq 1$. Therefore, there are $n, k_1, \dots, k_n \in \mathbb{N}$ and $a_1, b_1, \dots, a_n, b_n \in A$ for which Condition (2.6) holds. The definition of $\Delta(x, y)$ ensures that $\Box^{k_m}(a_m \leftrightarrow b_m) \in \Delta^{\boldsymbol{A}}(a_m, b_m)$, for every $m \leqslant n$. Together with the right hand side of Condition (2.6), this yields $\Box^{k_m}(a_m \leftrightarrow b_m) \in F$, for every $m \leqslant n$. By the left hand side of Condition (2.6) and the assumption that F is a lattice filter, this implies $c \in F$, as desired.

We are now ready to prove that $\boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{id}_A$. Consider $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$. By Condition (2.5), we have $\Delta^{\boldsymbol{A}}(a, b) \subseteq F$. Therefore, the definition of G guarantees that $\Delta^{\boldsymbol{A}}(a, b) \subseteq G$. Furthermore, since G is an open lattice filter of $\boldsymbol{A}$ contained into F, the assumptions imply $G = \{1\}$. Thus, $\Delta^{\boldsymbol{A}}(a, b) \subseteq G = \{1\}$, i.e.,

$$\Box^n(a \leftrightarrow b) = 1, \text{ for every } n \in \mathbb{N}.$$

Since modal algebras satisfy the sentence $\forall xy(x \leftrightarrow y \approx 1 \Longrightarrow x \approx y)$, this implies $a = b$, whence $\boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{id}_A$. ⊠

2.2 A syntactic description

Protoalgebraic logics admit the following even simpler syntactic description which, however, does not guarantee that $\Delta(x, y)$ is a set of equivalence formulas [13, Thm. 13.2]:[†]

Theorem 2.8. *A logic $\vdash$ is protoalgebraic iff there exists a set $\Delta(x, y)$ of formulas such that*

$$\emptyset \vdash \Delta(x, x) \;\; \textit{and} \;\; x, \Delta(x, y) \vdash y.$$

Proof. Suppose first that $\vdash$ is protoalgebraic with set of equivalence formulas $\Delta(x, y, \vec{z})$. In view of Theorem 2.2, we have that

$$\emptyset \vdash \Delta\langle x, x\rangle \;\; \text{and} \;\; x, \Delta\langle x, y\rangle \vdash y. \tag{2.7}$$

Then let σ be the substitution that sends all the variables other than y to x and leaves y untouched and consider the set

$$\Delta^{+}(x, y) := \sigma[\Delta(x, y, \vec{z})].$$

From Condition (2.7) and substitution invariance it follows

$$\emptyset \vdash \Delta^{+}(x, x) \;\; \text{and} \;\; x, \Delta^{+}(x, y) \vdash y.$$

Then we turn to prove the right to left implication in the statement. Suppose that there exists a set $\Delta(x, y)$ of formulas such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Then let

$$\Delta^{+}(x, y, \vec{z}) := \{\varphi \in Fm : \emptyset \vdash \sigma_{y \mapsto x}(\varphi)\},$$

where $\sigma_{y \mapsto x}$ is the substitution that sends y to x and leaves every other variable untouched. From $\emptyset \vdash \Delta(x, x)$ and the definition of $\Delta^{+}(x, y, \vec{z})$ it follows $\Delta(x, y) \subseteq \Delta^{+}(x, y, \vec{z})$. Together with the assumption that $x, \Delta(x, y) \vdash y$, this yields

$$x, \Delta^{+}(x, y, \vec{z}) \vdash y. \tag{2.8}$$

We will show that $\Delta^{+}(x, y, \vec{z})$ is a set of equivalence formulas for $\vdash$ and, therefore, that $\vdash$ is protoalgebraic.

To this end, consider a model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and $a, b \in A$. We need to prove that

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \Delta^{+\boldsymbol{A}}(a, b, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in A. \tag{2.9}$$

Suppose first that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ and consider $\varphi(x, y, \vec{z}) \in \Delta^{+}$ and $\vec{c} \in A$. The definition of Δ^{+} ensures that $\emptyset \vdash \varphi(x, x, \vec{z})$. As $\langle \boldsymbol{A}, F\rangle$ is a model of $\vdash$, this yields $\varphi^{\boldsymbol{A}}(a, a, \vec{c}) \in F$. Then consider the unary polynomial function $p(x) := \varphi^{\boldsymbol{A}}(a, x, \vec{c})$ of $\boldsymbol{A}$. Since $p(a) = \varphi^{\boldsymbol{A}}(a, a, \vec{c}) \in F$ and, by assumption, $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$, we can apply Proposition 1.13, obtaining

$$\varphi^{\boldsymbol{A}}(a, b, \vec{c}) = p(b) \in F.$$

Hence, we conclude that $\Delta^{+\boldsymbol{A}}(a, b, \vec{c}) \subseteq F$ for every $\vec{c}$, as desired.

[†]The definition of a protoalgebraic logic given in [13] differs from ours. However, the equivalence between the two definitions is proven is the same paper [13, Thm. 13.10].

Then we turn to prove the implication from right to left in Condition (2.9). Accordingly, suppose that $\Delta^{+A}(a,b,\vec{c}) \subseteq F$, for every $\vec{c} \in A$. We need to show that $\langle a,b \rangle \in \boldsymbol{\Omega}^{A}F$. In view of Proposition 1.13, it suffices to prove that

$$p(a) \in F \Longleftrightarrow p(b) \in F,$$

for every unary polynomial function p of $\boldsymbol{A}$. Then consider a unary polynomial function p of $\boldsymbol{A}$. By symmetry, it will be enough to prove the left to right implication in the above display. Accordingly suppose that $p(a) \in F$. We may assume that the sequence $\vec{z}$ in the expression $\Delta^{+}(x,y,\vec{z})$ contains all the variables other than x and y. Consequently, there are a formula $\varphi(x,\vec{z})$ and a sequence $\vec{c} \in A$ such that $p(x) = \varphi^{\boldsymbol{A}}(x,\vec{c})$.

Claim 2.9. *We have* $\Delta^{+\boldsymbol{A}}(p(a),p(b),\vec{c}) \subseteq F$.

Proof of the Claim. Consider $d \in \Delta^{+\boldsymbol{A}}(p(a),p(b),\vec{c})$. Then there exists $\psi(x,y,\vec{z}) \in \Delta^{+}$ such that

$$d = \psi^{\boldsymbol{A}}(p(a),p(b),\vec{c}).$$

Furthermore, from $\psi(x,y,\vec{z}) \in \Delta^{+}$ it follows $\emptyset \vdash \psi(x,x,\vec{z})$. By substitution invariance, this yields

$$\emptyset \vdash \psi(\varphi(x,\vec{z}),\varphi(x,\vec{z}),\vec{z}).$$

Together with the definition of Δ^{+} and the assumption that $\vec{z}$ contains all the variables other than x and y, this implies

$$\psi(\varphi(x,\vec{z}),\varphi(y,\vec{z}),\vec{z}) \in \Delta^{+}.$$

Therefore, from the assumption that $\Delta^{+\boldsymbol{A}}(a,b,\vec{c}) \subseteq F$ it follows

$$\psi^{\boldsymbol{A}}(\varphi^{\boldsymbol{A}}(a,\vec{c}),\varphi^{\boldsymbol{A}}(b,\vec{c}),\vec{c}) \in F.$$

Since $p(a) = \varphi^{\boldsymbol{A}}(a,\vec{c})$ and $p(b) = \varphi^{\boldsymbol{A}}(b,\vec{c})$, we conclude that $d \in F$. ⊠

Together with the assumption that $p(a) \in F$, the Claim yields

$$\{p(a)\} \cup \Delta^{+\boldsymbol{A}}(p(a),p(b),\vec{c}) \subseteq F.$$

By Condition (2.8) and the assumption that $\langle \boldsymbol{A}, F \rangle$ is a model of $\vdash$, this implies $p(b) \in F$. ⊠

Remark 2.10. Let $\sigma_{y \mapsto x}$ be the substitution that sends y to x and leaves every other variable untouched. The above argument reveals that if $\vdash$ is a protoalgebraic logic, then

$$\Delta(x,y,\vec{z}) := \{\varphi \in Fm : \emptyset \vdash \sigma_{y \mapsto x}(\varphi)\}$$

is always a set of equivalence formulas for $\vdash$. Because of this, the above set has been called the *fundamental set* of $\vdash$ [51]. ⊠

Remark 2.11. As a consequence of Theorem 2.8, every logic $\vdash$ possessing an implication $\to$ such that

$$\emptyset \vdash x \to x \quad \text{and} \quad x, x \to y \vdash y$$

is protoalgebraic, as witnessed by the set $\Delta = \{x \to y\}$. Because of this, most familiar logics are protoalgebraic. ⊠

As we shall see, however, there are exceptions to this rule. A formula φ is said to be a *theorem* of a logic $\vdash$ when $\emptyset \vdash \varphi$. We will prove that every nonpathological protoalgebraic logic has theorems. To this end, we shall isolate some limit cases: a logic $\vdash$ is said to be *almost inconsistent* if, for every set $\Gamma \cup \{\varphi\}$ of formulas,

$$\Gamma \vdash \varphi \iff \Gamma \neq \emptyset.$$

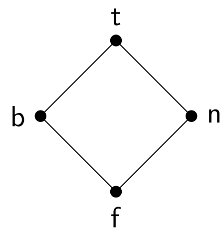
Notice that almost inconsistent logics lack theorems by definition. On the other hand, in view of by Theorem 2.8, they are protoalgebraic as witnessed by the set $\Delta(x, y) := \emptyset$.

Corollary 2.12. *Every protoalgebraic logic that is not almost inconsistent has theorems.*

Proof. Let $\vdash$ be a protoalgebraic logic without theorems. By Theorem 2.8, there exists a set $\Delta(x, y)$ of formulas such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. As $\emptyset \vdash \Delta(x, x)$ and $\vdash$ lacks theorems, the set $\Delta(x, y)$ must be empty. Therefore, from $x, \Delta(x, y) \vdash y$ it follows $x \vdash y$. By substitution invariance, this implies $\Gamma \vdash \varphi$, for every $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \neq \emptyset$. Since $\vdash$ lacks theorems, this means that $\vdash$ is almost inconsistent. ⊠

This allows us to spot interesting logics that fail to be protoalgebraic.

Example 2.13. Let $\boldsymbol{A} = \langle A; \wedge, \vee, \neg \rangle$ be the algebra that comprises the lattice $\langle A; \wedge, \vee \rangle$ depicted below



and the unary operation $\neg$ given by

$$\neg \mathsf{t} = \mathsf{f} \qquad \neg \mathsf{f} = \mathsf{t} \qquad \neg \mathsf{b} = \mathsf{b} \qquad \neg \mathsf{n} = \mathsf{n}.$$

The *Belnap-Dunn logic* BD [4, 5] is defined as

$$\begin{aligned} \Gamma \vdash_{\mathsf{BD}} \varphi \iff & \text{for every homomorphism } f \colon \boldsymbol{Fm} \to \boldsymbol{A}, \\ & \text{if } f[\Gamma] \subseteq \{\mathsf{t}, \mathsf{b}\}, \text{ then } f(\varphi) \subseteq \{\mathsf{t}, \mathsf{b}\}. \end{aligned}$$

It is easy to see that $x \nvdash_{\mathsf{BD}} y$, whence BD is not almost inconsistent. On the other hand, BD lacks theorems, because the map $f \colon \boldsymbol{Fm} \to \boldsymbol{A}$ that sends every formula to n is a homomorphism such that $f[\emptyset] = \emptyset \subseteq \{\mathsf{t}, \mathsf{b}\}$ and $f(\varphi) = \mathsf{n} \notin \{\mathsf{t}, \mathsf{b}\}$, for every formula φ. By Corollary 2.12, we conclude that the logic BD is not protoalgebraic [37, Thm. 2.11]. ⊠

2.3 The Leibniz operator

We denote the congruence lattice of an algebra $\boldsymbol{A}$ by $\mathsf{Con}(\boldsymbol{A})$. Given a logic $\vdash$, the *Leibniz operator* is the map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ defined by the rule

$$F \longmapsto \boldsymbol{\Omega}^{\boldsymbol{A}} F.$$

Protoalgebraic logics can be characterized in terms of the behavior of the Leibniz operator as follows [13, Thm. 13.10]:

Theorem 2.14. *The following conditions are equivalent for a logic $\vdash$:*

(i) *The logic $\vdash$ is protoalgebraic;*

(ii) *The Leibniz operator $\boldsymbol{\Omega}\colon \mathcal{T}h(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is order preserving;*

(iii) *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is order preserving, for every algebra $\boldsymbol{A}$.*

Proof. (i)$\Rightarrow$(iii): Let $\vdash$ be protoalgebraic with set of equivalence formulas $\Delta(x, y, \vec{z})$. Then consider an algebra $\boldsymbol{A}$ and let $F, G \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$ be such that $F \subseteq G$. We need to prove that $\boldsymbol{\Omega}^{\boldsymbol{A}}F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$. To this end, consider a pair $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$. Since $\Delta(x, y, \vec{z})$ is a set of equivalence formulas for $\vdash$ and $\langle \boldsymbol{A}, F\rangle$ a model of $\vdash$, we have

$$\Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in F.$$

Together with the assumption that $F \subseteq G$, this yields

$$\Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq G, \text{ for every } \vec{c} \in F.$$

As $\Delta(x, y, \vec{z})$ is a set of equivalence formulas for $\vdash$ and $\langle \boldsymbol{A}, G\rangle$ a model of $\vdash$, we conclude that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}G$, as desired.

By Proposition 1.7, the implication (iii)$\Rightarrow$(ii) is straightforward. Therefore we turn to prove the implication (ii)$\Rightarrow$(i). Consider the fundamental set $\Delta(x, y, \vec{z})$ of $\vdash$, defined in Remark 2.10. We begin with the following observation:

Claim 2.15. *The set Δ is a theory of $\vdash$ such that $\langle x, y\rangle \in \boldsymbol{\Omega}\Delta$.*

Proof of the Claim. The definition of Δ guarantees that $\Delta = \sigma_{y\mapsto x}^{-1}[\mathrm{Cn}_{\vdash}(\emptyset)]$. Together with Corollary 1.11, this implies that Δ is a theory of $\vdash$.

In view of Proposition 1.13, to prove that $\langle x, y\rangle \in \boldsymbol{\Omega}\Delta$, it suffices to show that

$$p(x) \in \Delta \Longleftrightarrow p(y) \in \Delta,$$

for every unary polynomial function p of $\boldsymbol{Fm}$. Accordingly, consider a unary polynomial function p of $\boldsymbol{Fm}$. Then there exists formula $\varphi(x, \vec{z})$ and a sequence of formulas $\vec{\gamma}(x, y, \vec{z})$ such that $p(\psi) = \varphi(\psi, \vec{\gamma}(x, y, \vec{z}))$, for every $\psi \in Fm$. We need to show that

$$\varphi(x, \vec{\gamma}(x, y, \vec{z})) \in \Delta \Longleftrightarrow \varphi(y, \vec{\gamma}(x, y, \vec{z})) \in \Delta.$$

We have that

$$\sigma_{y\mapsto x}(\varphi(x, \vec{\gamma}(x, y, \vec{z}))) = \varphi(x, \vec{\gamma}(x, x, \vec{z})) = \sigma_{y\mapsto x}(\varphi(y, \vec{\gamma}(x, y, \vec{z}))).$$

Together with the definition of Δ, this implies

$$\begin{aligned}\varphi(x, \vec{\gamma}(x, y, \vec{z})) \in \Delta &\Longleftrightarrow \emptyset \vdash \sigma_{y\mapsto x}(\varphi(x, \vec{\gamma}(x, y, \vec{z})))\\ &\Longleftrightarrow \emptyset \vdash \sigma_{y\mapsto x}(\varphi(y, \vec{\gamma}(x, y, \vec{z})))\\ &\Longleftrightarrow \varphi(y, \vec{\gamma}(x, y, \vec{z})) \in \Delta\end{aligned}$$

as desired. ⊠

By the Claim, $\varDelta$ is a theory of $\vdash$. Since $\mathrm{Cn}_{\vdash}(\{x\} \cup \varDelta)$ is also a theory of $\vdash$, the assumption that the Leibniz operator $\boldsymbol{\Omega}\colon \mathcal{T}h(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is order preserving guarantees that $\boldsymbol{\Omega}\varDelta \subseteq \boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\{x\} \cup \varDelta)$. As, by the Claim, it holds $\langle x, y\rangle \in \boldsymbol{\Omega}\varDelta$, we conclude that

$$\langle x, y\rangle \in \boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\{x\} \cup \varDelta).$$

Since $\boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\{x\} \cup \varDelta)$ is compatible with $\mathrm{Cn}_{\vdash}(\{x\} \cup \varDelta)$, this implies that $y \in \mathrm{Cn}_{\vdash}(\{x\} \cup \varDelta)$, that is, $x, \varDelta(x, y, \vec{z}) \vdash y$. Moreover, the definition of $\varDelta$ ensures that $\emptyset \vdash \varDelta(x, x, \vec{z})$. Let then σ be the substitution that sends every variable other than y to x and leaves y untouched. By substitution invariance, we obtain

$$\emptyset \vdash \Delta^{+}(x, x) \;\text{ and }\; x, \Delta^{+}(x, y) \vdash y,$$

where $\Delta^{+}(x, y) := \sigma[\Delta]$. Hence, with an application of Theorem 2.8, we conclude that $\vdash$ is protoalgebraic. ☒

Remark 2.16. In the proof of Claim 2.15, it was shown that the fundamental set $\varDelta(x, y, \vec{z})$ of an arbitrary logic $\vdash$ satisfies the following property:

$$\varDelta(x, y, \vec{z}) \text{ is a theory of } \vdash \text{ such that } \langle x, y\rangle \in \boldsymbol{\Omega}\varDelta(x, y, \vec{z}). \tag{2.10}$$

This fact will be used repeatedly in what follows.

Theorem 2.14 is instrumental in disproving that concrete logics are protoalgebraic.

Example 2.17. Let IPC^{-} be the *implication-free fragment* of IPC, i.e., the fragment of IPC in the signature $\langle \wedge, \vee, \neg\rangle$. We will prove that IPC^{-} fails to be protoalgebraic [11, Thm. 5.13]. Suppose the contrary, with a view to contradiction. Then consider the algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \neg\rangle$, where $\langle A; \wedge, \vee\rangle$ is the lattice with order $0 < c < a < 1$ and $\neg$ the unary operation defined by

$$\neg 1 = \neg a = \neg c = 0 \;\text{ and }\; \neg 0 = 1.$$

By inspection, one sees that $\boldsymbol{A}$ has precisely five congruences, namely, the identity relation id_A, the total relation $A \times A$, and

$$\begin{aligned} \theta_1 &:= \text{the equivalence relation whose blocks are } \{1, a\}, \{c\}, \{0\}; \\ \theta_2 &:= \text{the equivalence relation whose blocks are } \{1\}, \{a, c\}, \{0\}; \\ \theta_3 &:= \text{the equivalence relation whose blocks are } \{1, a, c\}, \{0\}. \end{aligned}$$

As a consequence, the largest congruence of $\boldsymbol{A}$ compatible with $\{1\}$ (resp. $\{a, 1\}$) is θ_2 (resp. θ_1). By the definition of the Leibniz congruence, this guarantees that

$$\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\} = \theta_2 \;\text{ and }\; \boldsymbol{\Omega}^{\boldsymbol{A}}\{a, 1\} = \theta_1.$$

Other other hand, $\{1\} \subseteq \{a, 1\}$ and $\theta_2 \nsubseteq \theta_1$. By Theorem 2.14, to prove that IPC^{-} is not protoalgebraic, it suffices to show that $\{1\}$ and $\{1, a\}$ are deductive filters of IPC^{-} on $\boldsymbol{A}$.

To this end, observe that $\boldsymbol{A}$ can be expanded with an implication connective $\to$ that turns it into a Heyting algebra $\boldsymbol{A}^{+}$, by setting

$$p \to q := \begin{cases} 1 & \text{if } p \leqslant q, \\ q & \text{otherwise,} \end{cases}$$

for every $p, q \in A$. Recall from Example 1.9 that the deductive filters of IPC on a Heyting algebra $\boldsymbol{B}$ are precisely the lattice filters of $\boldsymbol{B}$. Consequently, $\{1\}$ and $\{1, a\}$ are deductive filters of IPC on the Heyting algebra $\boldsymbol{A}^{+}$. It follows immediately that they are also deductive filters of IPC^{-} on $\boldsymbol{A}$, as desired. ☒

2.4 A model theoretic description

A matrix $\langle \boldsymbol{A}, F\rangle$ is a *submatrix* of a matrix $\langle \boldsymbol{B}, G\rangle$ when $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$ and $F = A \cap G$. In this case, we write $\langle \boldsymbol{A}, F\rangle \leqslant \langle \boldsymbol{B}, G\rangle$. Furthermore, the *direct product* of a family $\{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$ of matrices is the matrix

$$\langle \prod_{i\in I} \boldsymbol{A}_i, \prod_{i \in I} F_i\rangle.$$

Lastly, a matrix $\langle \boldsymbol{A}, F\rangle$ is said to be a *subdirect product* of $\{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$ when

$$\langle \boldsymbol{A}, F\rangle \leqslant \langle \prod_{i\in I} \boldsymbol{A}_i, \prod_{i \in I} F_i\rangle$$

and the canonical projection $\pi_i \colon \boldsymbol{A} \to \boldsymbol{A}_i$ is surjective, for every $i \in I$. We denote the class operator of closure under subdirect products of matrices by $\mathbb{P}_{\text{SD}}$.

While the class of *arbitrary* models of a logic is always closed under $\mathbb{P}_{\text{SD}}$, the same closure property for the class of the *reduced* models amounts to protoalgebraicity [13, Thm. 9.3]:

Theorem 2.18. *A logic $\vdash$ is protoalgebraic iff $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{P}_{\text{SD}}$.*

Proof. Suppose first that $\vdash$ is protoalgebraic and let $\langle \boldsymbol{A}, F\rangle$ be a subdirect product of a family $\{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$ of matrices in $\mathsf{Mod}^*(\vdash)$. As we mentioned, the class of arbitrary models of $\vdash$ is closed under $\mathbb{P}_{\text{SD}}$, whence $\langle \boldsymbol{A}, F\rangle$ is also a model of $\vdash$. Therefore, it only remains to prove that the matrix $\langle \boldsymbol{A}, F\rangle$ is reduced.

To this end, consider two distinct elements $a, b \in A$. We need to show that $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{A}} F$. Since $\boldsymbol{A}$ is a subalgebra of $\prod_{i\in I} \boldsymbol{A}_i$, from $a \neq b$ it follows $a(i) \neq b(i)$, for some $i \in I$. Since, by assumption, the matrix $\langle \boldsymbol{A}_i, F_i\rangle$ is reduced, we have $\langle a(i), b(i)\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{A}_i} F_i$. Then let $\Delta(x, y, \vec{z})$ be the set of equivalence formulas witnessing the protoalgebraicity of $\vdash$. Since $\langle \boldsymbol{A}_i, F_i\rangle$ is a model of $\vdash$ and $\langle a(i), b(i)\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{A}_i} F_i$, there exists $\vec{c} \in A_i$ such that

$$\Delta^{\boldsymbol{A}_i}(a(i), b(i), \vec{c}) \nsubseteq F_i.$$

As the canonical projection $\pi_i \colon \boldsymbol{A} \to \boldsymbol{A}_i$ is surjective, there exists $\vec{e} \in A$ such that $\pi_i(\vec{e}) = \vec{c}$. Therefore, from the above display and the assumption that $F = A \cap \prod_{i\in I} F_i$ it follows

$$\Delta^{\boldsymbol{A}}(a, b, \vec{e}) \nsubseteq F.$$

Since $\langle \boldsymbol{A}, F\rangle$ is a model of $\vdash$ and $\Delta(x, y, \vec{z})$ a set of equivalence formulas for $\vdash$, we conclude that $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{A}} F$, as desired.

Then we turn to prove the implication from right to left in the statement. Suppose that $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{P}_{\text{SD}}$. In view to Theorem 2.14, in order to prove that $\vdash$ is protoalgebraic, it suffices to show that that the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is order preserving, for every algebra $\boldsymbol{A}$. Accordingly, consider an algebra $\boldsymbol{A}$ and $F, G \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$ such that $F \subseteq G$. Our aim is to establish the inclusion $\boldsymbol{\Omega}^{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G$.

Let

$$f \colon \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}} F \times \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}} G$$

be the homomorphism defined by the rule

$$a \longmapsto \langle a/\boldsymbol{\Omega}^{\boldsymbol{A}} F, a/\boldsymbol{\Omega}^{\boldsymbol{A}} G\rangle.$$

We consider the matrix

$$\langle \boldsymbol{B}, H\rangle := \langle f[\boldsymbol{A}], f[A] \cap (F/\boldsymbol{\Omega}^{\boldsymbol{A}}F \times G/\boldsymbol{\Omega}^{\boldsymbol{A}}G)\rangle,$$

where $f[\boldsymbol{A}]$ is the subalgebra of $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F \times \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}G$ with universe $f[A]$.

Claim 2.19. *The map $f\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, H\rangle$ is a strict surjective homomorphism.*

Proof of the Claim. The fact that the homomorphism f is surjective follows from the equality $\boldsymbol{B} = f[\boldsymbol{A}]$. To prove that f is strict, consider $a \in F$. It suffices to show that

$$\begin{aligned} a \in F &\Longleftrightarrow (a/\boldsymbol{\Omega}^{\boldsymbol{A}}F \in F/\boldsymbol{\Omega}^{\boldsymbol{A}}F \text{ and } a/\boldsymbol{\Omega}^{\boldsymbol{A}}G \in G/\boldsymbol{\Omega}^{\boldsymbol{A}}G)\\ &\Longleftrightarrow \langle a/\boldsymbol{\Omega}^{\boldsymbol{A}}F, a/\boldsymbol{\Omega}^{\boldsymbol{A}}G\rangle \in F/\boldsymbol{\Omega}^{\boldsymbol{A}}F \times G/\boldsymbol{\Omega}^{\boldsymbol{A}}G\\ &\Longleftrightarrow f(a) \in F/\boldsymbol{\Omega}^{\boldsymbol{A}}F \times G/\boldsymbol{\Omega}^{\boldsymbol{A}}G\\ &\Longleftrightarrow f(a) \in f[A] \cap (F/\boldsymbol{\Omega}^{\boldsymbol{A}}F \times G/\boldsymbol{\Omega}^{\boldsymbol{A}}G)\\ &\Longleftrightarrow f(a) \in H. \end{aligned}$$

The left to right implication in the first equivalence above holds because $F \subseteq G$, while the implication from right to left holds because $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is compatible with F. The second and the fourth equivalences are straightforward, the third holds by the definition of f, and the fifth by that of H. ⊠

Now, the definition of $\langle \boldsymbol{B}, H\rangle$ guarantees that it is a subdirect product of $\langle \boldsymbol{A}, F\rangle^*$ and $\langle \boldsymbol{A}, G\rangle^*$. Moreover, $\langle \boldsymbol{A}, F\rangle^*, \langle \boldsymbol{A}, G\rangle^* \in \mathsf{Mod}^*(\vdash)$, by Theorem 1.21, whence

$$\langle \boldsymbol{B}, H\rangle \in \mathbb{P}_{\text{SD}}(\mathsf{Mod}^*(\vdash)).$$

Since, by assumption, $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{P}_{\text{SD}}$, this implies $\langle \boldsymbol{B}, H\rangle \in \mathsf{Mod}^*(\vdash)$ and, therefore, $\boldsymbol{\Omega}^{\boldsymbol{B}}H = \mathrm{id}_B$. Together with the Claim and Corollary 1.15, this yields

$$\boldsymbol{\Omega}^{\boldsymbol{A}}F = \mathsf{Ker}(f).$$

To prove that $\boldsymbol{\Omega}^{\boldsymbol{A}}F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$, consider a pair $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$. In view of the above display, $f(a) = f(b)$. By the definition of f, this means that

$$\langle a/\boldsymbol{\Omega}^{\boldsymbol{A}}F, a/\boldsymbol{\Omega}^{\boldsymbol{A}}G\rangle = \langle b/\boldsymbol{\Omega}^{\boldsymbol{A}}F, b/\boldsymbol{\Omega}^{\boldsymbol{A}}G\rangle.$$

Consequently, $a/\boldsymbol{\Omega}^{\boldsymbol{A}}G = b/\boldsymbol{\Omega}^{\boldsymbol{A}}G$, that is, $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}G$. ⊠

2.5 The correspondence property

The Correspondence Theorem of universal algebra [18, Thm. II.6.20] states that if $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then the congruence lattice $\mathsf{Con}(\boldsymbol{B})$ is isomorphic to the sublattice of $\mathsf{Con}(\boldsymbol{A})$ consisting of the congruences of $\boldsymbol{A}$ that extend $\mathsf{Ker}(f)$. As we shall see, the existence of a similar isomorphism characterizes protoalgebraic logics.

Given a logic $\vdash$, an algebra $\boldsymbol{A}$, and a subset $F \subseteq A$, we denote the sublattice of $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ consisting of the deductive filters extending F by $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^F$. When $\boldsymbol{A}$ is $\boldsymbol{Fm}$, we will write $\mathcal{T}h(\vdash)^F$ instead of $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^F$.

Definition 2.20. A logic $\vdash$ is said to have the *correspondence property* when, for every strict surjective homomorphism $f\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ between models of $\vdash$, the direct image map

$$f[-]\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{F} \to \mathsf{Fi}_{\vdash}(\boldsymbol{B})^{G}$$

is a well-defined lattice isomorphism.

Our aim is to prove the following [13, Thm. 7.6 & Cor. 7.7] (see also [26, Lem. 2.10]:

Theorem 2.21. *A logic is protoalgebraic iff it has the correspondence property.*

Proof. Consider a logic $\vdash$. Suppose first that $\vdash$ is protoalgebraic and let $f\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ be a strict surjective homomorphism between models of $\vdash$.

Claim 2.22. *For every $H \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{F}$, we have $H = f^{-1}[f[H]]$.*

Proof of the Claim. Since $\vdash$ is protoalgebraic, by Theorem 2.8, there exists a set $\Delta(x, y)$ of formulas such that

$$\emptyset \vdash \Delta(x, x) \;\text{ and }\; x, \Delta(x, y) \vdash y.$$

Then consider $H \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{F}$. As $H \subseteq f^{-1}[f[H]]$ always holds, we turn to prove the reverse inclusion. Accordingly, let $a \in f^{-1}[f[H]]$. Then there exists $b \in H$ such that $f(b) = f(a)$. Since $\langle \boldsymbol{B}, G\rangle$ is a model of $\vdash$ and $\emptyset \vdash \Delta(x, x)$, this yields

$$f[\Delta^{\boldsymbol{A}}(b, a)] = \Delta^{\boldsymbol{B}}(f(b), f(a)) = \Delta^{\boldsymbol{B}}(f(a), f(a)) \subseteq G.$$

Together with the assumption that $f\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is a strict homomorphism, this implies $\Delta^{\boldsymbol{A}}(b, a) \subseteq F$. As we assumed that $F \subseteq H$ and $b \in H$, we obtain

$$\{b\} \cup \Delta^{\boldsymbol{A}}(b, a) \subseteq H.$$

Since $\langle \boldsymbol{A}, H\rangle$ is a model of $\vdash$ and $x, \Delta(x, y) \vdash y$, we conclude that $a \in H$. ⊠

Now, we turn to prove that the direct image map

$$f[-]\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{F} \to \mathsf{Fi}_{\vdash}(\boldsymbol{B})^{G} \tag{2.11}$$

is a lattice isomorphism. To show that it is well defined, let $H \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{F}$. Since $f\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is a strict surjective homomorphism, we have $f[F] = G$. Together with $F \subseteq H$, this yields $G = f[F] \subseteq f[H]$. Therefore, it only remains to prove that $f[H]$ is a deductive filter of $\vdash$ on $\boldsymbol{B}$. To this end, consider $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash \varphi$ and a homomorphism $g\colon \boldsymbol{Fm} \to \boldsymbol{B}$ such that $g[\Gamma] \subseteq f[H]$. By Lemma 1.17, there exists a homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $g = f \circ h$. Consequently, from the assumption that $g[\Gamma] \subseteq f[H]$ it follows $f[h[\Gamma]] \subseteq f[H]$. This yields $h[\Gamma] \subseteq f^{-1}[f[H]]$, which, by the Claim, amounts to $h[\Gamma] \subseteq H$. Since $\Gamma \vdash \varphi$ and $\langle \boldsymbol{A}, H\rangle$ is a model of $\vdash$, it follows that $h(\varphi) \in H$. Clearly, this implies $g(\varphi) = f(h(\varphi)) \in f[H]$. Hence, we conclude that $f[H]$ is a deductive filter of $\vdash$ on $\boldsymbol{B}$. This establishes that the map $f[-]$ in Condition (2.11) is well defined.

Notice that $f[-]$ is order preserving, by definition. We will prove that it is also order reflecting. To this end, consider $H_1, H_2 \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{F}$ such that $f[H_1] \subseteq f[H_2]$. Clearly, $f^{-1}[f[H_1]] \subseteq f^{-1}[f[H_2]]$, whence the Claim implies $H_1 \subseteq H_2$. Therefore, $f[-]$ is an order embedding. To prove that it is an order isomorphism, it only remains to show that it is surjective. Accordingly, consider $H \in \mathsf{Fi}_{\vdash}(\boldsymbol{B})^G$. Proposition 1.10 guarantees that $f^{-1}[H] \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})^F$ and the surjectivity of $f\colon \boldsymbol{A} \to \boldsymbol{B}$ implies $f[f^{-1}[H]] = H$.

Then we turn to prove the implication from right to left in the statement. In view of Theorem 2.14, it will be enough to show that the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is order preserving. Consider two theories Γ and Σ such that $\Gamma \subseteq \Sigma$. Since by definition $\boldsymbol{\Omega}\Sigma$ is the largest congruence of $\boldsymbol{Fm}$ compatible with Σ, it suffices to show that $\boldsymbol{\Omega}\Gamma$ is compatible with Σ. To this end, consider a pair φ and ψ of formulas such that $\varphi \in \Sigma$ and $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega}\Gamma$. We need to prove that $\psi \in \Sigma$.

Then let $p_\Gamma\colon \langle \boldsymbol{Fm}, \Gamma \rangle \to \langle \boldsymbol{Fm}, \Gamma \rangle^*$ be the strict surjective homomorphism defined in Proposition 1.16. Since, by Proposition 1.7, the matrices $\langle \boldsymbol{Fm}, \Gamma \rangle$ and $\langle \boldsymbol{Fm}, \Gamma \rangle^*$ are models of $\vdash$, we can apply the correspondence property, obtaining that the map

$$p_\Gamma[-]\colon \mathcal{T}h(\vdash)^\Gamma \to \mathsf{Fi}_{\vdash}(\boldsymbol{Fm}/\boldsymbol{\Omega}\Gamma)^{\Gamma/\boldsymbol{\Omega}\Gamma} \tag{2.12}$$

is a lattice isomorphism. We will prove that

$$p_\Gamma[\mathrm{Cn}_{\vdash}(\Gamma \cup \{\psi\})] \subseteq p_\Gamma[\Sigma]. \tag{2.13}$$

Consider a formula γ such that $\Gamma, \psi \vdash \gamma$. From $\Gamma \cup \{\varphi\} \subseteq \Sigma$ and $p_\Gamma(\varphi) = p_\Gamma(\psi)$ (the latter, because we assumed that $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega}\Gamma$) it follows

$$p_\Gamma[\Gamma \cup \{\psi\}] \subseteq p_\Gamma[\Sigma].$$

Since Σ is a theory of $\vdash$ extending Γ and the map $p_\Gamma[-]$ in Condition (2.12) is well defined, the set $p_\Gamma[\Sigma]$ is a deductive filter of $\vdash$ on $\boldsymbol{Fm}/\boldsymbol{\Omega}\Gamma$. Together with the above display and $\Gamma, \psi \vdash \gamma$, this yields $p_\Gamma(\gamma) \in p_\Gamma[\Sigma]$, thus establishing Condition (2.13). Lastly, since the map $p_\Gamma[-]$ in Condition (2.12) is a lattice isomorphism and $\mathrm{Cn}_{\vdash}(\Gamma \cup \{\psi\}), \Sigma \in \mathsf{Th}(\vdash)^\Gamma$, Condition (2.13) yields

$$\psi \in \mathrm{Cn}_{\vdash}(\Gamma \cup \{\psi\}) \subseteq \Sigma. \qquad \boxtimes$$

2.6 The parametrized local deduction theorem

One of the weakest forms of deduction theorem considered in algebraic logic is the following:

Definition 2.23. A logic $\vdash$ is said to have the *parametrized local deduction theorem* (PLDT, for short) when there exists a family Φ of sets $\Sigma(x, y, \vec{z})$ of formulas such that, for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$,

$$\Gamma, \varphi \vdash \psi \iff \text{there exist } \Sigma(x, y, \vec{z}) \in \Phi \text{ and } \vec{\gamma} \in Fm \text{ s.t. } \Gamma \vdash \Sigma(\varphi, \psi, \vec{\gamma}).$$

Notably, the validity of this kind of deduction theorem characterizes protoalgebraic logics [30, pg. 387] (see also [31]):

Theorem 2.24. *A logic is protoalgebraic iff it has the parametrized local deduction theorem.*

Proof. Consider a logic $\vdash$. Suppose first that it has the PLDT, witnessed by a family Φ of sets of formulas. Since $x \vdash x$, the implication from left to right in the definition of the PLDT ensures the existence of some $\Sigma(x, y, \vec{z}) \in \Phi$ and $\vec{\gamma} \in Fm$ such that

$\emptyset \vdash \Sigma(x, x, \vec{\gamma})$. Then let σ be the substitution that sends every variable other than y to x and leaves y untouched. Moreover, consider the set

$$\Delta(x, y) := \Sigma(x, y, \sigma(\vec{\gamma})).$$

We will prove that

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y. \tag{2.14}$$

Let σ_x be the substitution that sends all the variables to x. From $\emptyset \vdash \Sigma(x, x, \vec{\gamma})$ and substitution invariance it follows $\emptyset \vdash \Sigma(\sigma_x(x), \sigma_x(x), \sigma_x(\vec{\gamma}))$. By the definition of σ and σ_x, it holds $\sigma_x = \sigma_x \circ \sigma$. Utilizing this fact in the last equality below, we obtain

$$\Delta(x, x) = \sigma_x[\Delta(x, y)] = \Sigma(\sigma_x(x), \sigma_x(x), \sigma_x(\sigma(\vec{\gamma}))) = \Sigma(\sigma_x(x), \sigma_x(x), \sigma_x(\vec{\gamma})).$$

Since the right hand side of the above display is a set of theorems, we conclude that $\emptyset \vdash \Delta(x, x)$.

Lastly, observe that $\Sigma(x, y, \sigma(\vec{\gamma})) \vdash \Sigma(x, y, \sigma(\vec{\gamma}))$ and $\Sigma(x, y, \vec{z}) \in \Phi$. Therefore, the implication from right to left in the definition of the PLDT yields $x, \Sigma(x, y, \sigma(\vec{\gamma})) \vdash y$, i.e., $x, \Delta(x, y) \vdash y$. This establishes Condition (2.14). By Theorem 2.8, we conclude that $\vdash$ is protoalgebraic.

Then we turn to prove the implication from left to right in the statement. Suppose that $\vdash$ is protoalgebraic. We will show that the family

$$\Phi := \{\Sigma(x, y, \vec{z}) \in \mathsf{Th}(\vdash) : x, \Sigma(x, y, \vec{z}) \vdash y\}$$

witnesses the PLDT for $\vdash$. To this end, consider $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$. We need to prove that

$$\Gamma, \varphi \vdash \psi \iff \text{there exist } \Sigma(x, y, \vec{z}) \in \Phi \text{ and } \vec{\gamma} \in Fm \text{ s.t. } \Gamma \vdash \Sigma(\varphi, \psi, \vec{\gamma}).$$

Suppose first that there exist $\Sigma(x, y, \vec{z}) \in \Phi$ and $\vec{\gamma} \in Fm$ such that $\Gamma \vdash \Sigma(\varphi, \psi, \vec{\gamma})$. As $\Sigma(x, y, \vec{z}) \in \Phi$, the definition of Φ guarantees that $x, \Sigma(x, y, \vec{z}) \vdash y$. By substitution invariance, this yields $\varphi, \Sigma(\varphi, \psi, \vec{\gamma}) \vdash \psi$. Since $\Gamma \vdash \Sigma(\varphi, \psi, \vec{\gamma})$, we conclude that $\Gamma, \varphi \vdash \psi$.

Therefore, it only remains to prove left to right implication in the above display. Suppose that $\Gamma, \varphi \vdash \psi$. Then let σ be any surjective substitution such that

$$\sigma(x) = \varphi \text{ and } \sigma(y) = \psi.$$

Let also $\Sigma(x, y, \vec{z}) := \sigma^{-1}[\mathrm{Cn}_{\vdash}(\Gamma)]$. The definition of $\Sigma(x, y, \vec{z})$ ensures that

$$\sigma \colon \langle \boldsymbol{Fm}, \Sigma(x, y, \vec{z})\rangle \to \langle \boldsymbol{Fm}, \mathrm{Cn}_{\vdash}(\Gamma)\rangle$$

is a strict homomorphism which, moreover, is surjective, by assumption. In addition, the matrix $\langle \boldsymbol{Fm}, \mathrm{Cn}_{\vdash}(\Gamma)\rangle$ is a model of $\vdash$, by Proposition 1.7. The same holds for the matrix $\langle \boldsymbol{Fm}, \Sigma(x, y, \vec{z})\rangle = \langle \boldsymbol{Fm}, \sigma^{-1}[\mathrm{Cn}_{\vdash}(\Gamma)]\rangle$, by Proposition 1.10. Therefore, the map in the above display is a strict surjective homomorphism between models of $\vdash$. In view of Theorem 2.21, the assumption that $\vdash$ is protoalgebraic guarantees that the direct image map

$$\sigma[-] \colon \mathsf{Fi}_{\vdash}(\boldsymbol{Fm})^{\Sigma} \to \mathsf{Fi}_{\vdash}(\boldsymbol{Fm})^{\mathrm{Cn}_{\vdash}(\Gamma)} \tag{2.15}$$

is a lattice isomorphism.

Claim 2.25. *We have* $x, \Sigma \vdash y$.

Proof of the Claim. We will use repeatedly the fact that the theories of $\vdash$ are precisely the deductive filters of $\vdash$ on $\boldsymbol{Fm}$, i.e., Proposition 1.7.

Consider the theory $\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)$. Since the map $\sigma[-]$ in Condition (2.15) is well defined,

$$\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)] \text{ is a theory containing } \mathrm{Cn}_{\vdash}(\mathit{\Gamma}).$$

Since we assumed that $\sigma(x) = \varphi$, we have $\varphi \in \sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)]$. Together with $\mathit{\Gamma}, \varphi \vdash \psi$ and the above display, this yields $\psi \in \sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)]$. As we assumed that $\psi = \sigma(y)$, we conclude that

$$y \in \sigma^{-1}[\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)]]. \tag{2.16}$$

On the other hand, since $\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)]$ is a theory, from Corollary 1.11 it follows that so is $\sigma^{-1}[\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)]]$. Furthermore, the surjectivity of σ ensures that

$$\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)] = \sigma[\sigma^{-1}[\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)]]].$$

Since the map $\sigma[-]$ in Condition (2.15) is injective and the theories $\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)$ and $\sigma^{-1}[\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)]]$ extend Σ, this yields

$$\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma) = \sigma^{-1}[\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)]].$$

By Condition (2.16), this implies $y \in \mathrm{Cn}_{\vdash}(\{x\} \cup \Sigma)$. ⊠

Now, from the Claim and the definition of Φ it follows $\Sigma \in \Phi$. Moreover, recall that $\Sigma = \sigma^{-1}[\mathrm{Cn}_{\vdash}(\mathit{\Gamma})]$, by definition. Therefore, we obtain

$$\sigma[\Sigma] = \sigma[\sigma^{-1}[\mathrm{Cn}_{\vdash}(\mathit{\Gamma})]] \subseteq \mathrm{Cn}_{\vdash}(\mathit{\Gamma}),$$

that is, $\mathit{\Gamma} \vdash \Sigma(\sigma(x), \sigma(y), \sigma(\vec{z}))$. Since, by assumption, $\sigma(x) = \varphi$ and $\sigma(y) = \psi$, this amounts to $\mathit{\Gamma} \vdash \Sigma(\varphi, \psi, \sigma(\vec{z}))$. Consequently, letting $\vec{\gamma} := \sigma(\vec{z})$, we are done. ⊠

Recall from Example 2.6 that the logics K_g, K_ℓ, and IPC are all protoalgebraic. In view of Theorem 2.24, they have the parametrized local deduction theorem, as we proceed to illustrate.

Example 2.26. The logics K_ℓ and IPC have the standard deduction theorem given by

$$\mathit{\Gamma}, \varphi \vdash \psi \iff \mathit{\Gamma} \vdash \varphi \to \psi.$$

Therefore, the family $\Phi = \{\{x \to y\}\}$ witnesses the parametrized local deduction theorem for them.

On the other hand, the deduction theorem for K_g takes the following form:

$$\mathit{\Gamma}, \varphi \vdash_{\mathsf{K}_g} \psi \iff \mathit{\Gamma} \vdash_{\mathsf{K}_g} (\varphi \wedge \Box\varphi \wedge \cdots \wedge \Box^n \varphi) \to \psi, \text{ for some } n \in \mathbb{N}.$$

Therefore, the parametrized local deduction theorem for K_g is witnessed by the family

$$\Phi := \{\{(x \wedge \Box x \wedge \cdots \wedge \Box^n x) \to y\} : n \in \mathbb{N}\}.$$ ⊠

However, none of the deduction theorems in the previous example features the parameters $\vec{z}$. A case where they are needed is the following.

Example 2.27. An *FL-algebra* is a structure $\boldsymbol{A} = \langle A; \wedge, \vee, \cdot, \to, \leftarrow, 0, 1\rangle$ that comprises a lattice $\langle A; \wedge, \vee\rangle$, a monoid $\langle \boldsymbol{A}; \cdot, 1\rangle$, a constant 0, and two binary operations $\to, \leftarrow$ that satisfy the following generalization of the residuation law: for every $a, b, c \in A$,

$$a \cdot b \leqslant c \iff b \leqslant a \to c \iff a \leqslant c \leftarrow b.$$

The *full Lambek calculus* FL is the logic defined as

$$\begin{aligned} \Gamma \vdash_{\mathsf{FL}} \varphi \iff &\text{for every FL-algebra } \boldsymbol{A} \text{ and homomorphism } f \colon \boldsymbol{Fm} \to \boldsymbol{A}, \\ &\text{if } 1 \leqslant f(\gamma) \text{ for every } \gamma \in \Gamma, \text{ then } 1 \leqslant f(\varphi). \end{aligned}$$

The axiomatic extensions of FL have been called *substructural logics* [45, 72, 85].

In order to describe the parametrized local deduction theorem for FL, we define

$$\lambda(x, z) := (z \to (x \cdot z)) \wedge 1 \ \text{ and } \ \rho(x, z) := ((z \cdot x) \leftarrow z) \wedge 1.$$

An *iterated conjugate* is a formula $\varepsilon(x, z_{m_1}, \ldots, z_{m_n})$ of the form

$$\delta_n(\delta_{n-1}(\ldots \delta_3(\delta_2(\delta_1(x, z_{m_1}), z_{m_2}), z_{m_3}) \ldots), z_{m_n}),$$

where $n \in \mathbb{Z}^+$ and $\delta_k \in \{\lambda, \rho\}$ for every $k \leqslant n$.

The deduction theorem for FL takes the following form:

$$\Gamma, \varphi \vdash_{\mathsf{FL}} \psi \iff \Gamma \vdash_{\mathsf{FL}} (\varepsilon_1(\varphi, \vec{\gamma}) \cdots \varepsilon_n(\varphi, \vec{\gamma})) \to \psi,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are iterated conjugates and $\vec{\gamma} \in Fm$. Therefore, the family

$$\Phi = \{\{(\varepsilon_1(x, \vec{z}) \cdot \ldots \cdot \varepsilon_n(x, \vec{z})) \to y\} : n \in \mathbb{N} \text{ and each } \varepsilon_m \text{ is an iterated conjugate}\}$$

witnesses the parametrized local deduction theorem for FL (see [46, 45]). ⊠

Remark 2.28. Variants of the parametrized local deduction theorem of increasing strength as well as their algebraic counterparts have been thoroughly investigated in algebraic logic (see, e.g., [12, 14, 26, 27, 80]). ⊠

CHAPTER 3

Equivalential logics

Recall from Example 2.6 that the sets $\Delta(x, y, \vec{z})$ of equivalence formulas for K_g, K_ℓ, and IPC have no parameters $\vec{z}$. This motivates the following:

Definition 3.1. A logic is said to be *equivalential* when it has a set of equivalence formulas in variables x and y only, i.e., when there exists a set $\Delta(x, y)$ of formulas such that, for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and $a, b \in A$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \Longleftrightarrow \Delta^{\boldsymbol{A}}(a, b) \subseteq F.$$

Equivalential logics were introduced in [76] and their theory was developed in [24, 89]. In view of Theorem 2.2, they can be described as follows [24, Cor. I.6 & Thm. I.11]:

Theorem 3.2. *A logic $\vdash$ is equivalential iff there exists a set $\Delta(x, y)$ of formulas such that*

$$\emptyset \vdash \Delta(x, x) \quad x, \Delta(x, y) \vdash y \quad \Delta(x_1, y_1), \dots, \Delta(x_n, y_n) \vdash \Delta(f(x_1, \dots, x_n), f(y_1, \dots, y_n)),$$

for every n-ary connective f of $\vdash$.

Corollary 3.3. *Every extension of an equivalential logic is equivalential with the same set of equivalence formulas.*

3.1 The Leibniz operator

Given a logic $\vdash$ and an algebra $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is said to *commute with endomorphisms* when

$$\boldsymbol{\Omega}^{\boldsymbol{A}} f^{-1}[F] = f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}} F],$$

for every $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$ and endomorphism f of $\boldsymbol{A}$. When $\boldsymbol{A} = \boldsymbol{Fm}$ and the above condition holds, we say that $\boldsymbol{\Omega} \colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ *commutes with substitutions.*

Equivalential logics can be characterized in terms of the behavior of the Leibniz operator as follows [51, Thm. 4.5]:

Theorem 3.4. *The following conditions are equivalent for a logic $\vdash$:*

(i) *The logic* $\vdash$ *is equivalential;*

(ii) *The Leibniz operator* $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ *is order preserving and commutes with substitutions;*

(iii) *The Leibniz operator* $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ *is order preserving and commutes with endomorphisms, for every algebra* $\boldsymbol{A}$*.*

In this case, the Leibniz operator commutes also with arbitrary homomorphisms, in the sense that $\boldsymbol{\Omega}^{\boldsymbol{A}} f^{-1}[F] = f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} F]$*, for every homomorphism* $f\colon \boldsymbol{A} \to \boldsymbol{B}$ *and* $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{B})$*.*

Proof. (i)⇒(iii): We will prove that the Leibniz operator commutes with arbitrary homomorphisms. To this end, let $f\colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism and $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{B})$. Moreover, let $\Delta(x, y)$ be a set of equivalence formulas for $\vdash$. We will show that, for every $a, b \in B$,

$$\begin{aligned} \langle a, b \rangle \in f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} F] &\iff \langle f(a), f(b) \rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}} F \\ &\iff \Delta^{\boldsymbol{B}}(f(a), f(b)) \subseteq F \\ &\iff f[\Delta^{\boldsymbol{A}}(a, b)] \subseteq F \\ &\iff \Delta^{\boldsymbol{A}}(a, b) \subseteq f^{-1}[F] \\ &\iff \langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} f^{-1}[F]. \end{aligned}$$

The first, third, and fourth equivalences above are straightforward. Furthermore, recall that F is a deductive filter of $\vdash$ on $\boldsymbol{B}$. Therefore, the matrix $\langle \boldsymbol{B}, F \rangle$ is a model of $\vdash$ and, by Proposition 1.10, so is $\langle \boldsymbol{A}, f^{-1}[F] \rangle$. Together with the assumption that $\Delta(x, y)$ is a set of equivalence formulas for $\vdash$, this yields the second and fifth equivalences. In view of the above display, we conclude that $\boldsymbol{\Omega}^{\boldsymbol{A}} f^{-1}[F] = f^{-1}[\boldsymbol{\Omega}^{\boldsymbol{B}} F]$, as desired.

Now, consider an algebra $\boldsymbol{A}$. In view of the above discussion, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ commutes with endomorphisms. In view of Theorem 2.14, it is also order preserving.

By Proposition 1.7, the implication (iii)⇒(ii) is straightforward. Therefore, we turn to prove the implication (ii)⇒(i). Since the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is order preserving, Theorem 2.14 guarantees that $\vdash$ is protoalgebraic. Consequently, the fundamental set

$$\Delta(x, y, \vec{z}) \coloneqq \{\varphi \in Fm : \emptyset \vdash \sigma_{y \mapsto x}(\varphi)\}$$

is a set of equivalence formulas for $\vdash$, by Remark 2.10. Then let σ be the substitution that sends every variable other than y to x and leaves y untouched. We will prove that

$$\Delta^{+}(x, y) \coloneqq \sigma[\Delta(x, y, \vec{z})]$$

is also a set of equivalence formulas for $\vdash$ and, therefore, that $\vdash$ is equivalential.

Since $\Delta(x, y, \vec{z})$ is a set of equivalence formulas for $\vdash$, it suffices to show that, for every model $\langle \boldsymbol{A}, F \rangle$ of $\vdash$ and $a, b \in A$,

$$\Delta^{+\boldsymbol{A}}(a, b) \subseteq F \iff \Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in A.$$

By the definition of a model of $\vdash$, in turn, the above equivalence is a consequence of the demand that

$$\Delta(x, y, \vec{z}) \vdash \Delta^{+}(x, y) \text{ and } \Delta^{+}(x, y) \vdash \Delta(x, y, \vec{z}). \tag{3.1}$$

Therefore, to conclude the proof, it will be enough to establish the condition above.

To prove that $\Delta(x, y, \vec{z}) \vdash \Delta^+(x, y)$, consider $\varphi(x, y) \in \Delta^+(x, y)$. By the definition of $\Delta^+(x, y)$, there exists $\psi(x, y, z_1, \dots, z_n) \in \Delta(x, y, \vec{z})$ such that

$$\varphi = \psi(x, y, x, \dots, x).$$

Moreover, the assumption that $\psi \in \Delta(x, y, \vec{z})$ guarantees that $\emptyset \vdash \psi(x, x, z_1, \dots, z)$. Together with substitution invariance and the above display, this yields $\emptyset \vdash \varphi(x, x)$. As $\varphi(x, x) = \sigma_{y \mapsto x}(\varphi(x, y)) = \sigma_{y \mapsto x}(\varphi)$, we obtain $\emptyset \vdash \sigma_{y \mapsto x}(\varphi)$. By the definition of $\Delta(x, y, \vec{z})$, we conclude that $\varphi \in \Delta(x, y, \vec{z})$ and, therefore, $\Delta(x, y, \vec{z}) \vdash \varphi$.

In order to prove that $\Delta^+(x, y) \vdash \Delta(x, y, \vec{z})$, we begin by observing that

$$\Delta(x, y, \vec{z}) \subseteq \sigma^{-1}[\sigma[\Delta(x, y, \vec{z})]] \subseteq \sigma^{-1}[\mathrm{Cn}_\vdash(\sigma[\Delta(x, y, \vec{z})])] = \sigma^{-1}[\mathrm{Cn}_\vdash(\Delta^+(x, y))].$$

As the sets $\Delta(x, y, \vec{z})$ and $\sigma^{-1}[\mathrm{Cn}_\vdash(\Delta^+(x, y))]$ are theories (the first by Condition (2.10) and the second by Corollary 1.11), the above display and the assumption that the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is order preserving and commutes with substitutions imply

$$\boldsymbol{\Omega}\Delta(x, y, \vec{z}) \subseteq \boldsymbol{\Omega}\sigma^{-1}[\mathrm{Cn}_\vdash(\Delta^+(x, y))] = \sigma^{-1}[\boldsymbol{\Omega}\mathrm{Cn}_\vdash(\Delta^+(x, y))].$$

Therefore, from Condition (2.10) it follows $\langle x, y \rangle \in \sigma^{-1}[\boldsymbol{\Omega}\mathrm{Cn}_\vdash(\Delta^+(x, y))]$, i.e.,

$$\langle x, y \rangle = \langle \sigma(x), \sigma(y) \rangle \in \boldsymbol{\Omega}\mathrm{Cn}_\vdash(\Delta^+(x, y)).$$

Now, consider a formula $\varphi(x, y, \vec{z}) \in \Delta(x, y, \vec{z})$. In view of the above display,

$$\langle \varphi(x, x, \vec{z}), \varphi(x, y, \vec{z}) \rangle \in \boldsymbol{\Omega}\mathrm{Cn}_\vdash(\Delta^+(x, y)).$$

Since $\varphi(x, y, \vec{z}) \in \Delta(x, y, \vec{z})$, the definition of $\Delta(x, y, \vec{z})$ ensures that $\emptyset \vdash \varphi(x, x, \vec{z})$, whence $\varphi(x, x, \vec{z}) \in \mathrm{Cn}_\vdash(\Delta^+(x, y))$. Therefore, the above display implies $\varphi(x, y, \vec{z}) \in \mathrm{Cn}_\vdash(\Delta^+(x, y))$, because the congruence $\boldsymbol{\Omega}\mathrm{Cn}_\vdash(\Delta^+(x, y))$ is compatible with $\mathrm{Cn}_\vdash(\Delta^+(x, y))$. Hence, we conclude that $\Delta^+(x, y) \vdash \varphi(x, y, \vec{z})$. ⊠

3.2 A model theoretic description

We denote the class operators of closure under submatrices and direct products, respectively, by $\mathbb{S}$ and $\mathbb{P}$. While the class of *arbitrary* models of a logic is always closed under $\mathbb{S}$ and $\mathbb{P}$, the same closure property for the class of the *reduced* models amounts to equivalentiality [13, Thm. 13.12(ii)]:

Theorem 3.5. *A logic $\vdash$ is equivalential iff* $\mathsf{Mod}^*(\vdash)$ *is closed under $\mathbb{S}$ and $\mathbb{P}$.*

Proof. Suppose first that $\vdash$ is equivalential. Clearly, $\vdash$ is also protoalgebraic. By Theorem 2.18, the class $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{P}_{\text{SD}}$. Since direct products are a special case of subdirect products, we conclude that $\mathsf{Mod}^*(\vdash)$ is also closed under $\mathbb{P}$. Therefore, it only remains to show that $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{S}$. Accordingly, consider $\langle \boldsymbol{B}, G \rangle \in \mathsf{Mod}^*(\vdash)$ and $\langle \boldsymbol{A}, F \rangle \leqslant \langle \boldsymbol{B}, G \rangle$. As the class of arbitrary models of $\vdash$ is closed under $\mathbb{S}$, the matrix $\langle \boldsymbol{A}, F \rangle$ is also a model of $\vdash$. Then consider a set $\Delta(x, y)$ of equivalence formulas for $\vdash$. In order to prove that $\langle \boldsymbol{A}, F \rangle$ is reduced, it suffices to show that, for every $a, b \in A$,

$$\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \Delta^{\boldsymbol{A}}(a, b) \subseteq F \iff \Delta^{\boldsymbol{B}}(a, b) \subseteq G \iff \langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}} G \iff a = b.$$

The first and the third equivalences above hold because $\langle \boldsymbol{A}, F \rangle$ and $\langle \boldsymbol{B}, G \rangle$ are models of $\vdash$ and $\Delta(x, y)$ is a set of equivalence formulas for $\vdash$, the second because $\langle \boldsymbol{A}, F \rangle$ is a submatrix of $\langle \boldsymbol{B}, G \rangle$ and $a, b \in A$, and the fifth because, by assumption, $\langle \boldsymbol{B}, G \rangle$ is reduced.

Then we turn to prove the implication from right to left in the statement. Since subdirect products are a special case of submatrices of direct products, the assumptions guarantee that $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{P}_{\text{SD}}$. By Theorem 2.18, this means that the logic $\vdash$ is protoalgebraic. Consequently, the fundamental set

$$\Delta(x, y, \vec{z}) := \{\varphi \in Fm : \emptyset \vdash \sigma_{y \mapsto x}(\varphi)\}$$

is a set of equivalence formulas for $\vdash$, by Remark 2.10.

Let $Fm(x, y)$ be the set of formulas in variables x and y only and $\boldsymbol{Fm}(x, y)$ the corresponding algebra. We will prove that

$$\Delta^+(x, y) := \Delta(x, y, \vec{z}) \cap Fm(x, y)$$

is a set of equivalence formulas for $\vdash$ and, therefore, that $\vdash$ is equivalential.

Claim 3.6. *We have* $\langle x, y \rangle \in \boldsymbol{\Omega} \mathrm{Cn}_{\vdash}(\Delta^+(x, y))$.

Proof of the Claim. Recall that $\Delta = \Delta(x, y, \vec{z})$ and $\Delta^+ = \Delta^+(x, y)$. We will make use of the fact that Δ is a theory of $\vdash$ such that $\langle x, y \rangle \in \boldsymbol{\Omega}\Delta$ (see Condition (2.10)).

First, notice that

$$\Delta \cap Fm(x, y) = \mathrm{Cn}_{\vdash}(\Delta^+) \cap Fm(x, y). \tag{3.2}$$

The inclusion from left to right holds, by the definition of Δ^+. To prove the reverse inclusion, consider $\varphi \in \mathrm{Cn}_{\vdash}(\Delta^+) \cap Fm(x, y)$. Clearly, $\Delta^+ \vdash \varphi$. Since, by the definition of Δ, we have $\Delta^+ \subseteq \Delta$, this yields $\Delta \vdash \varphi$. Hence, we conclude that $\varphi \in \Delta$, because Δ is a theory.

Then observe that $Fm(x, y)^2 \cap \boldsymbol{\Omega}\Delta$ is a congruence of $\boldsymbol{Fm}(x, y)$ and define

$$\langle \boldsymbol{A}, F \rangle := \langle \boldsymbol{Fm}(x, y)/Fm(x, y)^2 \cap \boldsymbol{\Omega}\Delta, \mathrm{Cn}_{\vdash}(\Delta^+) \cap Fm(x, y)/Fm(x, y)^2 \cap \boldsymbol{\Omega}\Delta \rangle.$$

From Condition (3.2) it follows that $\langle \boldsymbol{A}, F \rangle$ is a submatrix of $\langle \boldsymbol{Fm}, \Delta \rangle^*$. Furthermore, as Δ is a theory of $\vdash$, Corollary 1.19 ensures that $\langle \boldsymbol{Fm}, \Delta \rangle^*$ is a model of $\vdash$. In addition, this model is reduced, by Proposition 1.16, whence $\langle \boldsymbol{Fm}, \Delta \rangle^* \in \mathsf{Mod}^*(\vdash)$. Therefore, from the assumption that $\mathsf{Mod}^*(\vdash)$ is closed under $\mathbb{S}$ and $\langle \boldsymbol{A}, F \rangle \leqslant \langle \boldsymbol{Fm}, \Delta \rangle^*$ it follows that $\langle \boldsymbol{A}, F \rangle$ is also a reduced model of $\vdash$. As a consequence,

$$\boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{id}_A. \tag{3.3}$$

Now, observe that the congruence $Fm(x, y)^2 \cap \boldsymbol{\Omega}\Delta$ is compatible with $\Delta^+ = \Delta \cap Fm(x, y)$. Because of that, the canonical surjective homomorphism

$$p \colon \langle \boldsymbol{Fm}(x, y), \mathrm{Cn}_{\vdash}(\Delta^+) \cap Fm(x, y) \rangle \to \langle \boldsymbol{A}, F \rangle,$$

defined by the rule $p(\varphi) = \varphi / Fm(x, y)^2 \cap \boldsymbol{\Omega}\Delta$, is strict. By Lemma 1.14 and Condition (3.3), this yields

$$\boldsymbol{\Omega}^{\boldsymbol{Fm}(x,y)}(\mathrm{Cn}_{\vdash}(\Delta^+) \cap Fm(x, y)) = p^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}} F] = p^{-1}[\mathrm{id}_A] = \mathsf{Ker}(p) = Fm(x, y)^2 \cap \boldsymbol{\Omega}\Delta.$$

An analogous argument shows that

$$\boldsymbol{\Omega}^{\boldsymbol{Fm}(x,y)}(\mathrm{Cn}_{\vdash}(\Delta^{+}) \cap Fm(x,y)) = Fm(x,y)^{2} \cap \boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\Delta^{+}).$$

From the last two displays it follows $Fm(x,y)^{2} \cap \boldsymbol{\Omega}\Delta = Fm(x,y)^{2} \cap \boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\Delta^{+})$. Since, by Condition (2.10), we have $\langle x,y\rangle \in \boldsymbol{\Omega}\Delta$, this yields $\langle x,y\rangle \in \boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\Delta^{+})$. ⊠

In order to prove that $\Delta^{+}(x,y)$ is a set of equivalence formulas for $\vdash$, it suffices to show that it satisfies the conditions in Theorem 3.2.

First, consider $\varphi(x,y) \in \Delta^{+}(x,y)$. Since $\varphi(x,y) \in \Delta(x,y,\vec{z})$, the definition of $\Delta(x,y,\vec{z})$ ensures that $\emptyset \vdash \varphi(x,x)$. Thus, we conclude that $\emptyset \vdash \Delta^{+}(x,x)$.

On the other hand, we have

$$\langle x,y\rangle \in \boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\Delta^{+}(x,y)) \subseteq \boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\{x\} \cup \Delta^{+}(x,y)),$$

where the first step holds by the Claim and the second because $\vdash$ is protoalgebraic and, consequently, the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is order preserving, by Theorem 2.14. Since the congruence $\boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\{x\} \cup \Delta^{+}(x,y))$ is compatible with $\mathrm{Cn}_{\vdash}(\{x\} \cup \Delta^{+}(x,y))$, from the above display it follows $x, \Delta^{+}(x,y) \vdash y$.

Lastly, consider an n-ary connective f. The Claim implies

$$\langle x_i, y_i\rangle \in \boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\Delta^{+}(x_i,y_i)), \text{ for every } i \leqslant n.$$

Since the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is order preserving, the above display yields

$$\langle x_1,y_1\rangle, \dots, \langle x_n,y_n\rangle \in \theta,$$

where $\theta := \boldsymbol{\Omega}\mathrm{Cn}_{\vdash}(\Delta^{+}(x_1,y_1) \cup \cdots \cup \Delta^{+}(x_n,y_n))$. Then consider a formula $\delta \in \Delta^{+}(x,y)$. In view of the above display,

$$\langle \delta(f(x_1,\dots,x_n), f(x_1,\dots,x_n)), \delta(f(x_1,\dots,x_n), f(y_1,\dots,y_n))\rangle \in \theta. \tag{3.4}$$

Moreover, from $\delta(x,y) \in \Delta^{+}(x,y)$, $\emptyset \vdash \Delta^{+}(x,x)$, and substitution invariance it follows $\emptyset \vdash \delta(f(x_1,\dots,x_n), f(x_1,\dots,x_n))$, whence

$$\delta(f(x_1,\dots,x_n), f(x_1,\dots,x_n)) \in \mathrm{Cn}_{\vdash}(\Delta^{+}(x_1,y_1) \cup \cdots \cup \Delta^{+}(x_n,y_n)). \tag{3.5}$$

Since θ is, by definition, compatible with $\mathrm{Cn}_{\vdash}(\Delta^{+}(x_1,y_1) \cup \cdots \cup \Delta^{+}(x_n,y_n))$, Conditions (3.4) and (3.5) yield

$$\delta(f(x_1,\dots,x_n), f(y_1,\dots,y_n)) \in \mathrm{Cn}_{\vdash}(\Delta^{+}(x_1,y_1) \cup \cdots \cup \Delta^{+}(x_n,y_n)),$$

that is, $\Delta^{+}(x_1,y_1),\dots,\Delta^{+}(x_n,y_n) \vdash \Delta^{+}(f(x_1,\dots,x_n), f(y_1,\dots,y_n))$. ⊠

We are now ready to exhibit a protoalgebraic logic whose set $\Delta(x,y,\vec{z})$ of equivalence formulas must necessarily contain some parameters $\vec{z}$.

Example 3.7. An *ortholattice* [6, 7, 55] is a structure $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, 0, 1\rangle$ that comprises a bounded lattice $\langle A; \wedge, \vee, 0, 1\rangle$ and a unary operation $\neg$ such that, for every $a, b \in A$,

$$a \wedge \neg a = 0 \qquad a \vee \neg a = 1 \qquad a = \neg\neg a \qquad \neg(a \wedge b) = \neg a \vee \neg b.$$

We denote the class of ortholattices by OL. The *minimal orthologic* MOL is the assertional logic of OL, that is, the logic defined as

$$\Gamma \vdash_{\mathsf{MOL}} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{OL} \text{ and homomorphism } f \colon \boldsymbol{Fm} \to \boldsymbol{A}, \\ \text{if } f[\Gamma] \subseteq \{1\}, \text{ then } f(\varphi) = 1.$$

The definition of MOL guarantees that

$$\emptyset \vdash_{\mathsf{MOL}} x \vee \neg x \quad \text{and} \quad x, \neg x \vee y \vdash_{\mathsf{MOL}} y.$$

This amounts to

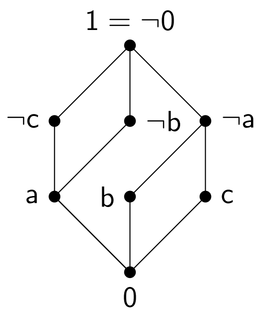
$$\emptyset \vdash_{\mathsf{MOL}} \Delta(x, x) \quad \text{and} \quad x, \Delta(x, y) \vdash_{\mathsf{MOL}} y,$$

where $\Delta(x, y) := \{\neg x \vee y\}$. By Theorem 2.8, we conclude that MOL is protoalgebraic. Consequently, the fundamental set

$$\Delta(x, y, \vec{z}) = \{\varphi \in Fm : \emptyset \vdash_{\mathsf{MOL}} \sigma_{y \mapsto x}(\varphi)\}$$

is a set of equivalence formulas for MOL (see Remark 2.10).

On the other hand, we will show that MOL lacks a set of equivalence formulas in variables x and y only, i.e., it fails to be equivalential [62] (see also [32]). In view of Theorem 3.5, it suffices to prove that the class $\mathsf{Mod}^*(\mathsf{MOL})$ is not closed under $\mathbb{S}$. To this end, let $\boldsymbol{A}$ be the ortholattice depicted below (the definition of the operation $\neg$ on $\boldsymbol{A}$ can be inferred from the labels of its elements and assumption that $\boldsymbol{A}$ satisfies the equation $x \approx \neg\neg x$).



The definition of MOL ensures that $\langle \boldsymbol{A}, \{1\} \rangle \in \mathsf{Mod}(\mathsf{MOL})$. Furthermore, by inspection, one sees that $\boldsymbol{A}$ has precisely three congruences, namely, the identity relation id_A, the total relation $A \times A$, and the equivalence relation with blocks $\{0, \neg\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ and $\{1, \mathsf{a}, \neg\mathsf{b}, \neg\mathsf{c}\}$. It follows that id_A is the largest congruence of $\boldsymbol{A}$ compatible with $\{1\}$, whence the matrix $\langle \boldsymbol{A}, \{1\} \rangle$ is reduced. Consequently, $\langle \boldsymbol{A}, \{1\} \rangle \in \mathsf{Mod}^*(\mathsf{MOL})$.

Then let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}$ with universe $\{\mathsf{a}, \neg\mathsf{a}, \mathsf{b}, \neg\mathsf{b}, 0, 1\}$. Clearly, $\langle \boldsymbol{B}, \{1\} \rangle$ is a submatrix of $\langle \boldsymbol{A}, \{1\} \rangle$. However, the matrix $\langle \boldsymbol{B}, \{1\} \rangle$ is not reduced, because the equivalence relation on B with blocks $\{0\}, \{1\}, \{\mathsf{a}, \neg\mathsf{b}\}$, and $\{\neg\mathsf{a}, \mathsf{b}\}$ is a congruence of $\boldsymbol{B}$ compatible with $\{1\}$. Hence, we conclude that $\mathsf{Mod}^*(\mathsf{MOL})$ is not closed under $\mathbb{S}$. $\boxtimes$

CHAPTER 4

Truth equational logics

While every logic has a matrix semantics (Theorem 1.1), in this section we will focus on the logics that admit an equational completeness theorem.

4.1 Equational completeness theorems

Formally speaking, an equation $\varphi \approx \psi$ is simply an ordered pair $\langle \varphi, \psi \rangle$ of formulas. Consequently, $Eq := Fm \times Fm$ is the set of equations built up from formulas in Fm.

The *equational consequence relative to a class* K *of algebras* is the consequence relation $\vDash_{\mathsf{K}}$ on Eq defined, for every $\Psi \cup \{\varphi \approx \psi\} \subseteq Eq$, as

$$\begin{aligned}\Psi \vDash_{\mathsf{K}} \varphi \approx \psi \iff & \text{for every algebra } \boldsymbol{A} \in \mathsf{K} \text{ and homomorphism } f\colon \boldsymbol{Fm} \to \boldsymbol{A},\\ & \text{if } f(\varepsilon) = f(\delta) \text{ for every } \varepsilon \approx \delta \in \Psi, \text{ then } f(\varphi) = f(\psi).\end{aligned}$$

Moreover, given a set $\tau(x)$ of equations and a set $\Gamma \cup \{\varphi\}$ of formulas, we abbreviate

$$\{\varepsilon(\varphi) \approx \delta(\varphi) \colon \varepsilon \approx \delta \in \tau\} \text{ as } \tau(\varphi), \text{ and } \bigcup_{\gamma \in \Gamma} \tau(\gamma) \text{ as } \tau[\Gamma].$$

Definition 4.1. A logic $\vdash$ is said to admit an *equational completeness theorem* when there exist a set $\tau(x)$ of equations and a class K of algebras such that, for every $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash \varphi \iff \tau[\Gamma] \vDash_{\mathsf{K}} \tau(\varphi).$$

In this case, we say that K is a τ*-algebraic semantics* (or, simply, an *algebraic semantics*) for $\vdash$.

The notion of an algebraic semantics was introduced in [11] and studied in [16, 65].

Example 4.2. If $\vdash$ is the assertional logic of a class K of algebras, then K is a τ-algebraic semantics for $\vdash$, where $\tau = \{x \approx 1\}$. Consequently, the class of modal algebras (resp. Heyting algebras) is an algebraic semantics for K_g (resp. IPC).

A nonassertional example of a logic with an algebraic semantics is FL. This is because, by definition, the class of FL-algebras is a τ-algebraic semantics for FL where

$\tau = \{x \wedge 1 \approx 1\}$. To prove that FL is not assertional, consider the set $A := \{0, 1, \top\}$ and let $\neg\colon A \to A$ be the map defined as

$$\neg\top = 0 \qquad \neg 0 = \top \qquad \neg 1 = 1.$$

Moreover, let $\boldsymbol{A}$ be the FL-algebra with universe is A, whose lattice reduct is the chain $0 < 1 < \top$ and in which, for every $p, q \in A$,

$$p \cdot q := \begin{cases} 0 & \text{if } 0 \in \{p, q\}, \\ \max\{p, q\} & \text{otherwise;} \end{cases} \qquad p \to q = q \leftarrow p := \begin{cases} \max\{\neg p, q\} & \text{if } p \leqslant q, \\ \min\{\neg p, q\} & \text{otherwise.} \end{cases}$$

The definition of FL, guarantees that the matrix $\langle \boldsymbol{A}, \uparrow 1\rangle$ is a model of FL. Furthermore, by inspection, it is easy to see that this matrix is reduced. Consequently, $\langle \boldsymbol{A}, \uparrow 1\rangle \in \mathsf{Mod}^*(\mathsf{FL})$. Since $\uparrow 1 = \{1, \top\}$ is a two-element set, Corollary 1.25 implies that FL is not assertional. $\boxtimes$

In order to describe the relation between algebraic and matrix semantics, given a set $\tau(x, \vec{z})$ of equations and an algebra $\boldsymbol{A}$, we let

$$\tau(\boldsymbol{A}) := \{a \in A : \boldsymbol{A} \vDash \tau(a, \vec{c}), \text{ for every } \vec{c} \in A\}.$$

Definition 4.3. We say that *truth is parametrically equationally definable* in a class M of matrices when there exists a set $\tau(x, \vec{z})$ of equations such that $F = \tau(\boldsymbol{A})$, for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$. In this case, we say that τ *defines truth* in M. If, moreover, $\tau = \tau(x)$, we say that *truth is equationally definable* in M.

Proposition 4.4. *The following conditions hold for a logic $\vdash$, a set $\tau(x)$ of equations, and a class K of algebras:*

(i) *The class K is a τ-algebraic semantics for $\vdash$ iff $\{\langle \boldsymbol{A}, \tau(\boldsymbol{A})\rangle : \boldsymbol{A} \in \mathsf{K}\}$ is a matrix semantics for $\vdash$;*

(ii) *The logic $\vdash$ admits an equational completeness theorem iff it has a matrix semantics in which truth is equationally definable.*

Proof. Condition (i) is an immediate consequence of the definitions of an algebraic and matrix semantics. To prove Condition (ii), observe that $\vdash$ admits an equational completeness theorem iff it has a τ'-algebraic semantics K', for some set $\tau'(x)$ of equations and class K' of algebras. In view of Condition (i), the latter amounts to the demand that $\vdash$ has a matrix semantics M of the form $\{\langle \boldsymbol{A}, \tau'(\boldsymbol{A})\rangle : \boldsymbol{A} \in \mathsf{K}'\}$, that is, a matrix semantics in which truth is equationally definable. $\boxtimes$

An arbitrary equational completeness theorem is not sufficient, however, to ensure the existence of a significant relation between a logic and its algebraic semantics.

Example 4.5. Glivenko's Theorem [49] provides an interpretation of the *classical propositional calculus* CPC into IPC, stating that

$$\Gamma \vdash_{\mathsf{CPC}} \varphi \Longleftrightarrow \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathsf{IPC}} \neg\neg\varphi,$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$. Moreover, letting $\tau(x) := \{\neg\neg x \approx 1\}$, we obtain

$$\{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathsf{IPC}} \neg\neg\varphi \Longleftrightarrow \{\neg\neg\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1 \Longleftrightarrow \tau[\Gamma] \vDash_{\mathsf{HA}} \tau(\varphi),$$

where the first equivalence holds because, in view of Example 4.2, the class HA of Heyting algebras is an $\{x \approx 1\}$-algebraic semantics for IPC, while the second is straightforward.

From the two displays above it follows that HA is an algebraic semantics for CPC, although certainly not the intended one [16]. In the same spirit, it should be observed that algebraic semantics are not unique. For instance, CPC is the assertional logic of the class BA of Boolean algebras and, therefore, BA is also an algebraic semantics for CPC (and, obviously, a more appropriate one than HA). ⊠

In order to strengthen the relation between a logic and its algebraic semantics, we will restrict our attention to the following kind of logics:

Definition 4.6. A logic $\vdash$ is said to be *truth equational* when truth is equationally definable in $\mathsf{Mod}^*(\vdash)$.

In view of Theorem 1.21 and Proposition 4.4(ii), every truth equational logic admits an equational completeness theorem.

Example 4.7. By Corollary 1.25, every assertional logic is truth equational, as witnessed by the set of equations $\tau(x) = \{x \approx 1\}$. In particular, both K_g, IPC, and MOL are truth equational. Another such example is the implication-free fragment IPC^- of IPC, which is the assertional logic of the class of the implication-free reducts of Heyting algebras. ⊠

The theory of truth equational logics was first developed in [79] and it was extended subsequently to accommodate for parameters as follows [64]:

Definition 4.8. A logic $\vdash$ is said to be *parametrically truth equational* when truth is parametrically equationally definable in $\{\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\vdash) : F \neq \emptyset\}$.

If $\vdash^+$ is an extension of a logic $\vdash$, then $\mathsf{Mod}^*(\vdash^+) \subseteq \mathsf{Mod}^*(\vdash)$. As a consequence, we obtain the following:

Proposition 4.9. *Extensions of (parametrically) truth equational logics are still (parametrically) truth equational with the same set of equations.*

The definition of a (parametrically) truth equational logic can be rephrased in terms of the behavior of the Leibniz congruence, as we proceed to illustrate.

Remark 4.10. Let $\vdash$ be a logic. We will show that a set $\tau(x)$ of equations defines truth in $\mathsf{Mod}^*(\vdash)$ iff

$$a \in F \Longleftrightarrow \tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F, \tag{4.1}$$

for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $a \in A$. Notice that the inclusion $\tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F$ above makes sense, because equations are ordered pairs of formulas.

Suppose first that $\tau(x)$ defines truth in $\mathsf{Mod}^*(\vdash)$. Then consider $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $a \in A$. To prove that Condition (4.1) holds, it suffices to show that

$$a \in F \Longleftrightarrow a/\boldsymbol{\Omega}^{\boldsymbol{A}} F \in F/\boldsymbol{\Omega}^{\boldsymbol{A}} F \Longleftrightarrow \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}} F \vDash \tau(a/\boldsymbol{\Omega}^{\boldsymbol{A}} F) \Longleftrightarrow \tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F.$$

The first of the equivalences above holds because $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is compatible with F and the third is straightforward. Lastly, the second follows from the assumption that τ defines truth in $\mathsf{Mod}^*(\vdash)$ and the observation that, by Theorem 1.21, $\langle \boldsymbol{A}, F\rangle^* \in \mathsf{Mod}^*(\vdash)$.

Now, suppose that there exists a set $\tau(x)$ of equations witnessing the validity of Condition (4.1), for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $a \in A$. Then consider $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{*}(\vdash)$ and $a \in A$. By assumption, $a \in F$ iff $\tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F$. Furthermore, as the matrix $\langle \boldsymbol{A}, F\rangle$ is reduced, we have $\tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F$ iff $\tau^{\boldsymbol{A}}(a) \subseteq \mathrm{id}_A$, i.e., $\boldsymbol{A} \vDash \tau(a)$. Hence, we conclude that $\tau(x)$ defines truth in $\mathsf{Mod}^{*}(\vdash)$.

A similar argument shows that a set $\tau(x, \vec{z})$ of equations defines truth in $\{\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{*}(\vdash) : F \neq \emptyset\}$ iff

$$a \in F \Longleftrightarrow \tau^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F, \text{ for every } \vec{c} \in A,$$

for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ such that $F \neq \emptyset$ and $a \in A$.

The reason why the definition of a parametrically truth equational logic makes reference only to the reduced models $\langle \boldsymbol{A}, F\rangle$ such that $F \neq \emptyset$ (as opposed to arbitrary reduced models) is that dropping this requirement yields an equivalent characterization of truth equational logics:

Proposition 4.11. *Let $\vdash$ be a logic. Then truth is equationally definable in* $\mathsf{Mod}^{*}(\vdash)$ *iff it is parametrically equationally definable.*

Proof. The implication from left to right is straightforward. To prove the reverse implication, let $\tau(x, \vec{z})$ be a set of equations that defines truth in $\mathsf{Mod}^{*}(\vdash)$. By Remark 4.10, for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $a \in A$, it holds

$$a \in F \Longleftrightarrow \tau^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F, \text{ for every } \vec{c} \in A. \tag{4.2}$$

Then let σ be the substitution that sends every variable to x. We will prove that

$$\tau^{+}(x) := \bigcup\{\sigma(\tau(x, \vec{\gamma})) : \vec{\gamma} \in Fm\}$$

defines truth in $\mathsf{Mod}^{*}(\vdash)$. In view of Remark 4.10, it suffices to show that, for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $a \in A$,

$$a \in F \Longleftrightarrow \tau^{+\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F. \tag{4.3}$$

Then let $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $a \in A$. Suppose first that $a \in F$ and consider an equation $\varepsilon \approx \delta \in \tau^{+}(x)$. By the definition of σ and τ^{+}, there exist $\varphi(x, \vec{z}) \approx \psi(x, \vec{z}) \in \tau(x, \vec{z})$ and $\vec{\gamma}(x, \vec{z}) \in Fm$ such that

$$\varepsilon = \varphi(x, \vec{\gamma}(x, \vec{x})) \text{ and } \delta = \psi(x, \vec{\gamma}(x, \vec{x})), \tag{4.4}$$

where $\vec{x}$ is the sequence $\langle x, x, x \ldots\rangle$. Then let $\vec{c} := \vec{\gamma}^{\boldsymbol{A}}(a, \vec{a})$, where $\vec{a}$ is the sequence $\langle a, a, a \ldots\rangle$. We have

$$\langle \varepsilon^{\boldsymbol{A}}(a), \delta^{\boldsymbol{A}}(a)\rangle = \langle \varphi^{\boldsymbol{A}}(a, \vec{\gamma}^{\boldsymbol{A}}(a, \vec{a})), \psi^{\boldsymbol{A}}(a, \vec{\gamma}^{\boldsymbol{A}}(a, \vec{a}))\rangle = \langle \varphi^{\boldsymbol{A}}(a, \vec{c}), \psi^{\boldsymbol{A}}(a, \vec{c})\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F,$$

where the first equality follows from Condition (4.4), the second from the definition of $\vec{c}$, and the third step from the assumption that $a \in F$ and $\varphi \approx \psi \in \tau$ and Condition (4.2). From the above display it follows $\tau^{+\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F$, as desired.

Then we turn to prove the implication from right to left in Condition (4.3). Suppose that $\tau^{+\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F$ and let $f \colon \boldsymbol{Fm} \to \boldsymbol{A}$ be any homomorphism such that $f(x) = a$. By the definition of τ^{+}, we have

$$\bigcup\{f(\sigma(\tau(x, \vec{\gamma}))) : \vec{\gamma} \in Fm\} = f(\tau^{+}(x)) = \tau^{+\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F.$$

Consequently, $\tau(x, \vec{\gamma}) \subseteq (f \circ \sigma)^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}} F]$, for every $\vec{\gamma} \in Fm$. Since the composition $f \circ \sigma \colon \langle \boldsymbol{Fm}, (f \circ \sigma)^{-1}[F] \rangle \to \langle \boldsymbol{A}, F \rangle$ is a strict homomorphism, we can apply Lemma 1.14, obtaining

$$\tau(x, \vec{\gamma}) \subseteq (f \circ \sigma)^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}} F] \subseteq \boldsymbol{\Omega}(f \circ \sigma)^{-1}[F], \text{ for every } \vec{\gamma} \in Fm.$$

As, by Proposition 1.10, $(f \circ \sigma)^{-1}[F]$ is a deductive filter of $\vdash$ on $\boldsymbol{Fm}$, the matrix $\langle \boldsymbol{Fm}, (f \circ \sigma)^{-1}[F] \rangle$ is a model of $\vdash$. Therefore, we can apply Condition (4.2) to the above display, obtaining $x \in (f \circ \sigma)^{-1}[F]$, that is, $a = f(x) = f(\sigma(x)) \in F$. ⊠

From the above result we deduce [64, Cor. 3.10]:

Corollary 4.12. *A logic is truth equational iff it is parametrically truth equational and has theorems.*

Proof. Let $\vdash$ be a logic and $\mathbf{1}$ the trivial algebra. As $\mathbf{1}$ has only one congruence, namely, the identity relation, the matrix $\langle \mathbf{1}, \emptyset \rangle$ is reduced. Furthermore, the definition of a deductive filter implies that $\langle \mathbf{1}, \emptyset \rangle$ is a model of $\vdash$ iff the logic $\vdash$ lacks theorems. Consequently,

$$\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}^*(\vdash) \iff \vdash \text{ lacks theorems.} \tag{4.5}$$

Now, suppose that $\vdash$ is truth equational and observe that $\vdash$ is, obviously, parametrically truth equational as well. Suppose, with a view to contradiction that $\vdash$ lacks theorems. Then let $\tau(x)$ be a set of equations that defines truth in $\mathsf{Mod}^*(\vdash)$. In view of the above display and the assumption that $\vdash$ lacks theorems, we have $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}^*(\vdash)$ and, therefore, $\tau(\mathbf{1}) = \emptyset$. On the other hand, since the trivial algebra satisfies every equation, we obtain $\tau(\mathbf{1}) \neq \emptyset$, a contradiction.

Then we turn to prove the implication from right to left in the statement. Let φ be a theorem of $\vdash$. Clearly, every deductive filter of $\vdash$ on an algebra $\boldsymbol{A}$ contains every interpretations of φ and, therefore, is nonempty. As a consequence,

$$\mathsf{Mod}^*(\vdash) = \{\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*(\vdash) : F \neq \emptyset\}.$$

Since, by assumption, $\vdash$ is parametrically truth equational, this implies that truth is parametrically equationally definable in $\mathsf{Mod}^*(\vdash)$. By Proposition 4.11, we conclude that $\vdash$ is truth equational. ⊠

In view of Corollary 4.12, the logics that are genuinely *parametrically* truth equational should lack theorems.

Example 4.13. Let $\boldsymbol{A}$ be the algebra defined in Example 2.13. The *exactly true logic* ETL [73] is the logic induced by the matrix $\langle \boldsymbol{A}, \{\mathsf{t}\} \rangle$. An argument analogous to the one detailed in Example 2.13 for the case of the Belnap-Dunn logic BD shows that ETL lacks theorems. Together with Corollaries 2.12 and 4.12, this implies that ETL is neither protoalgebraic nor truth equational.

Hence, we will prove that ETL is parametrically truth equational. To this end, it will be enough to show that the following condition holds [86, Prop. 9]:

$$\{\langle \boldsymbol{B}, F \rangle \in \mathsf{Mod}^*(\mathsf{ETL}) : F \neq \emptyset\} \subseteq \{\langle \boldsymbol{B}, \{b\} \rangle : \langle B; \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}} \rangle \text{ is a lattice with maximum element } b\}. \tag{4.6}$$

For suppose that the above inclusion holds and let $\tau(x,z) := \{x \wedge z \approx z\}$. Then, for every $\langle \boldsymbol{B}, F\rangle \in \mathsf{Mod}^*(\mathsf{ETL})$ such that $F \neq \emptyset$, the structure $\langle B; \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}\rangle$ is a lattice with maximum element b and $F = \{b\}$. Consequently, for every $a \in B$,

$$\begin{aligned} a \in F &\iff a \text{ is the maximum of } \langle B; \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}\rangle \\ &\iff c \leqslant a, \text{ for every } c \in B \\ &\iff a \wedge^{\boldsymbol{B}} c = c, \text{ for every } c \in B \\ &\iff \boldsymbol{B} \vDash \tau(a,c), \text{ for every } c \in B. \end{aligned}$$

It follows that $\tau(x,z)$ defines truth in $\{\langle \boldsymbol{B}, F\rangle \in \mathsf{Mod}^*(\mathsf{ETL}) : F \neq \emptyset\}$, whence ETL is parametrically truth equational.

Then we turn to prove Condition (4.6). Consider a matrix $\langle \boldsymbol{B}, F\rangle \in \mathsf{Mod}^*(\mathsf{ETL})$ such that $F \neq \emptyset$. Since ETL is the logic induced by $\langle \boldsymbol{A}, \{\mathsf{t}\}\rangle$, we can apply Proposition 1.23, obtaining that $\langle B; \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}\rangle$ is a lattice. By the same token, Proposition 1.24 ensures that $|F| \leqslant 1$. Since we assumed that F is nonempty, this yields that F is a singleton. Furthermore, as the rule $x \wedge y \rhd x$ is valid in ETL, the deductive filters of this logic on $\boldsymbol{B}$ must be upsets in the lattice order of $\langle B; \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}\rangle$. As $\langle \boldsymbol{B}, F\rangle$ is a model of ETL, this implies that F is a one-element upset. Hence, the unique element of F must be the maximum of the lattice $\langle B; \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}\rangle$. ☒

4.2 The Leibniz operator

Given a logic $\vdash$ and an algebra $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is said to be *completely order reflecting* when

$$\text{if } \bigcap_{F \in \mathcal{F}} \boldsymbol{\Omega}^{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G, \text{ then } \bigcap \mathcal{F} \subseteq G,$$

for every $\mathcal{F} \cup \{G\} \subseteq \mathsf{Fi}_{\vdash}(\boldsymbol{A})$. Similarly, we say that $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is *almost completely order reflecting* when the above display holds, for every $\mathcal{F} \cup \{G\} \subseteq \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \smallsetminus \{\emptyset\}$.

Parametrically truth equational logics can be characterized in terms of the behavior of the Leibniz operator as follows [64, Thm. 3.9]:

Theorem 4.14. *The following conditions are equivalent for a logic $\vdash$:*

(i) *The logic $\vdash$ is parametrically truth equational;*

(ii) *The Leibniz operator $\boldsymbol{\Omega} \colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is almost completely order reflecting;*

(iii) *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is almost completely order reflecting, for every algebra $\boldsymbol{A}$.*

Proof. (i)⇒(iii): Suppose that $\vdash$ is parametrically truth equational. In view of Remark 4.10, there exists a set $\tau(x, \vec{z})$ of equations such that

$$a \in F \iff \tau^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F, \text{ for every } \vec{c} \in A, \tag{4.7}$$

for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ such that $F \neq \emptyset$ and $a \in A$. Then consider an algebra $\boldsymbol{A}$ and a family $\mathcal{F} \cup \{G\} \subseteq \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \smallsetminus \{\emptyset\}$ such that

$$\bigcap_{F \in \mathcal{F}} \boldsymbol{\Omega}^{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G. \tag{4.8}$$

We need to prove that $\bigcap \mathcal{F} \subseteq G$.

To this end, let $a \in \bigcap \mathcal{F}$ and consider $F \in \mathcal{F}$. Since $\mathcal{F}$ is a family of nonempty deductive filters of $\vdash$ on $\boldsymbol{A}$, the matrix $\langle \boldsymbol{A}, F \rangle$ is a model of $\vdash$ such that $F \neq \emptyset$. Therefore, we can apply Condition (4.7) to the assumption that $a \in \bigcap \mathcal{F} \subseteq F$, obtaining

$$\tau^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F, \text{ for every } \vec{c} \in A.$$

As a consequence,

$$\tau^{\boldsymbol{A}}(a, \vec{c}) \subseteq \bigcap_{F \in \mathcal{F}} \boldsymbol{\Omega}^{\boldsymbol{A}} F, \text{ for every } \vec{c} \in A.$$

Together with Condition (4.8), this yields $\tau^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G$, for every $\vec{c} \in A$. Since G is a nonempty deductive filter of $\vdash$ on $\boldsymbol{A}$, the matrix $\langle \boldsymbol{A}, G \rangle$ is a model of $\vdash$ such that $G \neq \emptyset$. Therefore, Condition (4.7) yields $a \in G$.

As usual, the implication (iii)$\Rightarrow$(ii) is straightforward. Therefore, we turn to prove the implication (ii)$\Rightarrow$(i). Recall that equations are ordered pairs of formulas. Therefore, the following is a set of equations:

$$\tau(x, \vec{z}) := \bigcap \{ \boldsymbol{\Omega} \Gamma : \Gamma \in \mathsf{Th}(\vdash) \text{ and } x \in \Gamma \}.$$

We will prove that τ defines truth in $\{ \langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{*}(\vdash) : F \neq \emptyset \}$ and, therefore, that $\vdash$ is parametrically truth equational.

In view of Remark 4.10, it suffices to show that

$$a \in F \Longleftrightarrow \tau^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F, \text{ for every } \vec{c} \in A, \tag{4.9}$$

for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}(\vdash)$ such that $F \neq \emptyset$ and $a \in A$. Accordingly, consider $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}(\vdash)$ such that $F \neq \emptyset$ and $a \in A$.

Suppose first that $a \in F$ and consider a sequence $\vec{c} \in A$. Then let $\varepsilon \approx \delta \in \tau(x, \vec{z})$. We will prove that, for every formula $\varphi(x, \vec{y})$,

$$x, \varphi(\varepsilon(x, \vec{z}), \vec{y}) \vdash \varphi(\delta(x, \vec{z}), \vec{y}) \text{ and } x, \varphi(\delta(x, \vec{z}), \vec{y}) \vdash \varphi(\varepsilon(x, \vec{z}), \vec{y}). \tag{4.10}$$

By symmetry, it will be enough to show that $x, \varphi(\varepsilon, \vec{y}) \vdash \varphi(\delta, \vec{y})$. To this end, consider the theory

$$\Gamma := \mathrm{Cn}_{\vdash}(\{ x, \varphi(\varepsilon, \vec{y}) \}).$$

Since $x \in \Gamma$, the assumption that $\varepsilon \approx \delta \in \tau$ and the definition of τ ensure that $\langle \varepsilon, \delta \rangle \in \boldsymbol{\Omega} \Gamma$. As $\boldsymbol{\Omega} \Gamma$ is a congruence of $\boldsymbol{Fm}$, this yields $\langle \varphi(\varepsilon, \vec{y}), \varphi(\delta, \vec{y}) \rangle \in \boldsymbol{\Omega} \Gamma$. Therefore, since $\boldsymbol{\Omega} \Gamma$ is compatible with Γ and, by construction, $\varphi(\varepsilon, \vec{y}) \in \Gamma$, we conclude that $\varphi(\delta, \vec{y}) \in \Gamma = \mathrm{Cn}_{\vdash}(\{ x, \varphi(\varepsilon, \vec{y}) \})$, i.e., $x, \varphi(\varepsilon, \vec{y}) \vdash \varphi(\delta, \vec{y})$.

From Condition (4.10) and the assumption that F is a deductive filter of $\vdash$ on $\boldsymbol{A}$ containing a it follows that, for every $\vec{e} \in A$,

$$\varphi^{\boldsymbol{A}}(\varepsilon^{\boldsymbol{A}}(a, \vec{c}), \vec{e}) \in F \Longleftrightarrow \varphi^{\boldsymbol{A}}(\delta^{\boldsymbol{A}}(a, \vec{c}), \vec{e}) \in F.$$

Consequently, for every unary polynomial function p of $\boldsymbol{A}$,

$$p(\varepsilon^{\boldsymbol{A}}(a, \vec{c})) \in F \Longleftrightarrow p(\delta^{\boldsymbol{A}}(a, \vec{c})) \in F.$$

By Proposition 1.13, this amounts to $\langle \varepsilon^{\boldsymbol{A}}(a, \vec{c}), \delta^{\boldsymbol{A}}(a, \vec{c}) \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$. Hence, we conclude that $\tau^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F$, as desired.

Then we turn to prove the implication from right to left in Condition (4.9). Suppose that

$$\tau^{A}(a,\vec{c}) \subseteq \boldsymbol{\Omega}^{A}F, \text{ for every } \vec{c} \in A.$$

Recall that F is nonempty, by assumption. Then let b be an element of F and $f\colon \boldsymbol{Fm} \to \boldsymbol{A}$ any homomorphism such that $f(x) = a$ and $f(y) = b$. From the above display it follows $\tau^{A}(f(x), f(\vec{z})) \subseteq \boldsymbol{\Omega}^{A}F$, that is,

$$\tau(x,\vec{z}) \subseteq f^{-1}[\boldsymbol{\Omega}^{A}F].$$

As the map $f\colon \langle \boldsymbol{Fm}, f^{-1}[F]\rangle \to \langle \boldsymbol{A}, F\rangle$ is a strict homomorphism, we can apply Lemma 1.14 to the above display, obtaining $\tau(x,\vec{z}) \subseteq \boldsymbol{\Omega} f^{-1}[F]$. By the definition of $\tau(x,\vec{z})$, this amounts to

$$\bigcap\{\boldsymbol{\Omega}\Gamma : \Gamma \in \mathsf{Th}(\vdash) \text{ and } x \in \Gamma\} = \tau(x,\vec{z}) \subseteq \boldsymbol{\Omega} f^{-1}[F].$$

By Propositions 1.7 and 1.10, the set $f^{-1}[F]$ is a theory. Furthermore, $f^{-1}[F]$ is nonempty, since we assumed that $f(y) = b \in F$. On the other hand, every theory Γ containing x is also nonempty. As a consequence, we can apply the assumption that the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is almost completely order reflecting to the above display, obtaining

$$\bigcap\{\Gamma \in \mathsf{Th}(\vdash) : x \in \Gamma\} \subseteq f^{-1}[F].$$

Since the left hand side of the above display is $\mathrm{Cn}_{\vdash}(\{x\})$, this yields $x \in f^{-1}[F]$, whence $a = f(x) \in F$. ⊠

Truth equational logics admit a very similar description [79, Thm. 28]:

Theorem 4.15. *The following conditions are equivalent for a logic $\vdash$:*

(i) *The logic $\vdash$ is truth equational;*

(ii) *The Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is completely order reflecting;*

(iii) *The Leibniz operator $\boldsymbol{\Omega}^{A}\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is completely order reflecting, for every algebra $\boldsymbol{A}$.*

Proof. (i)⇒(iii): Suppose that $\vdash$ is truth equational and consider an algebra $\boldsymbol{A}$. As $\vdash$ is also parametrically truth equational, Theorem 4.14 implies that the Leibniz operator $\boldsymbol{\Omega}^{A}\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is almost completely order reflecting. Therefore, to conclude the proof it suffices to show that every deductive filter of $\vdash$ on $\boldsymbol{A}$ is nonempty. But this is a consequence of the fact that, in view of Corollary 4.12, every truth equational logic (and $\vdash$, in particular) has theorems.

As usual, the implication (iii)⇒(ii) is straightforward. To prove the implication (ii)⇒(i), observe that the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is also almost completely order reflecting. Therefore, $\vdash$ is parametrically truth equational, by Theorem 4.14. Therefore, in view of Corollary 4.12, to prove that $\vdash$ is truth equational, it suffices to show that it has theorems. Suppose the contrary, with a view to contradiction. Then $\emptyset$ is a theory of $\vdash$. Observe that the total congruence $Fm \times Fm$ of $\boldsymbol{Fm}$ is compatible with $\emptyset$ and Fm. Therefore, $\boldsymbol{\Omega} Fm = Fm \times Fm = \boldsymbol{\Omega}\emptyset$. Since $\boldsymbol{Fm}$ and $\emptyset$ are theories of $\vdash$ (the first, by definition, and the second, because $\vdash$ lacks theorems), we can apply the assumption that Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is completely order reflecting, obtaining $Fm \subseteq \emptyset$ which is false, because Fm contains all the variables. ⊠

4.3 Implicit definability

We shall now focus on the classes of matrices that are uniquely determined by their algebraic reducts:

Definition 4.16. We say that *truth is implicitly definable* in a class M of matrices when, for every $\langle \boldsymbol{A}, F \rangle, \langle \boldsymbol{A}, G \rangle \in \mathsf{M}$, it holds $F = G$.

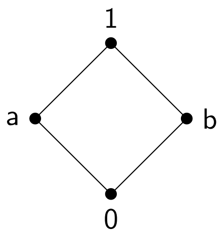
Remark 4.17. If truth is parametrically equationally definable in a class M of matrices, then it is also implicitly definable in M. This is because, if a set $\tau(x, \vec{z})$ of equations defines truth in M, then $F = \tau(\boldsymbol{A}) = G$, for every $\langle \boldsymbol{A}, F \rangle, \langle \boldsymbol{A}, G \rangle \in \mathsf{M}$. ⊠

This observation is often instrumental in showing that concrete logics fail to be parametrically truth equational.

Example 4.18. We will prove that the logic K_ℓ is not parametrically truth equational (essentially [11, Cor. 5.6]). To this end, let $\boldsymbol{A}$ be the modal algebra obtained by endowing the four-element Boolean algebra with a unary operation $\Box$ defined, for every $a \in A$, as

$$\Box a := \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\boldsymbol{A}$ has the structure of a four-element Boolean algebra, its lattice reduct is the following:



In particular, the upsets $\uparrow\mathsf{a}$ and $\uparrow\mathsf{b}$ are distinct lattice filters of $\boldsymbol{A}$. Furthermore, $\{1\}$ is the only open lattice filter contained in them, because the unique open lattice filters of $\boldsymbol{A}$ are $\{1\}$ and the total set A. In view of Condition (2.4), this implies $\langle \boldsymbol{A}, \uparrow\mathsf{a} \rangle, \langle \boldsymbol{A}, \uparrow\mathsf{b} \rangle \in \mathsf{Mod}^*(\mathsf{K}_\ell)$. Thus, truth is not implicitly definable in $\{\langle \boldsymbol{B}, G \rangle \in \mathsf{Mod}^*(\mathsf{K}_\ell) : G \neq \emptyset\}$. By Remark 4.17, neither is it parametrically equationally definable, whence the logic K_ℓ fails to be parametrically truth equational. ⊠

A similar argument gives the example of a logic that is neither protoalgebraic nor parametrically truth equational.

Example 4.19. Recall from Example 2.13 that the logic BD is not protoalgebraic. We will show it also fails to be parametrically truth equational. In view of Remark 4.17, it suffices to show that truth is not implicitly definable in $\{\langle \boldsymbol{B}, G \rangle \in \mathsf{Mod}^*(\mathsf{BD}) : G \neq \emptyset\}$. To this end, let $\boldsymbol{A}$ be the algebra introduced in 2.13. The definition of BD ensures that BD is the logic induced by the matrix $\langle \boldsymbol{A}, \{\mathsf{t}, \mathsf{b}\} \rangle$. Then let $f \colon \boldsymbol{A} \to \boldsymbol{A}$ be the homomorphism defined as

$$f(\mathsf{t}) = \mathsf{t} \qquad f(\mathsf{f}) = \mathsf{f} \qquad f(\mathsf{n}) = \mathsf{b} \qquad f(\mathsf{b}) = \mathsf{n}.$$

Since f is a homomorphism and $\{\mathsf{t}, \mathsf{b}\}$ a deductive filter of BD on $\boldsymbol{A}$ (the latter, because BD is induced by the matrix $\langle \boldsymbol{A}, \{\mathsf{t}, \mathsf{b}\} \rangle$), Proposition 1.10 implies that $\{\mathsf{t}, \mathsf{n}\} =$

$f^{-1}[\{\mathsf{t},\mathsf{b}\}]$ is also a deductive filter of BD on $\boldsymbol{A}$. Consequently, both $\langle \boldsymbol{A}, \{\mathsf{t},\mathsf{b}\}\rangle$ and $\langle \boldsymbol{A}, \{\mathsf{t},\mathsf{n}\}\rangle$ are models of BD. Furthermore, they are reduced, because the only congruences of $\boldsymbol{A}$ are the identity relation id_A and the total relation $A \times A$ and, therefore, $\boldsymbol{\Omega}^{\boldsymbol{A}}F = \mathrm{id}_A$, for every proper nonempty $F \subseteq A$. Thus, $\langle \boldsymbol{A}, \{\mathsf{t},\mathsf{b}\}\rangle, \langle \boldsymbol{A}, \{\mathsf{t},\mathsf{n}\}\rangle \in \mathsf{Mod}^*(\mathsf{BD})$. Hence, we conclude that truth is not implicitly definable in $\{\langle \boldsymbol{B}, G\rangle \in \mathsf{Mod}^*(\mathsf{BD}) : G \neq \emptyset\}$, as desired. $\boxtimes$

The implicit definability of truth in the classes of the form $\mathsf{Mod}^*(\vdash)$ can be characterized in terms of the behavior of the Leibniz operator:

Proposition 4.20. *Let $\vdash$ be a logic. Then truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$ iff the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is injective, for every algebra $\boldsymbol{A}$.*

Proof. Suppose first that truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$. Consider an algebra $\boldsymbol{A}$ and $F, G \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}F = \boldsymbol{\Omega}^{\boldsymbol{A}}G$. Let $\theta := \boldsymbol{\Omega}^{\boldsymbol{A}}F = \boldsymbol{\Omega}^{\boldsymbol{A}}G$. Then

$$\langle \boldsymbol{A}, F\rangle^* = \langle \boldsymbol{A}/\theta, F/\theta\rangle \text{ and } \langle \boldsymbol{A}, G\rangle^* = \langle \boldsymbol{A}/\theta, G/\theta\rangle.$$

Furthermore, from $F, G \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$ it follows that the matrices $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{A}, G\rangle$ are models of $\vdash$. By Theorem 1.21, this yields $\langle \boldsymbol{A}, F\rangle^*, \langle \boldsymbol{A}, G\rangle^* \in \mathsf{Mod}^*(\vdash)$. Together with the above display, this implies

$$\langle \boldsymbol{A}/\theta, F/\theta\rangle, \langle \boldsymbol{A}/\theta, G/\theta\rangle \in \mathsf{Mod}^*(\vdash).$$

Since, by assumption, truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$, we obtain $F/\theta = G/\theta$.

We will prove that $F = G$. By symmetry, it suffices to show that $F \subseteq G$. Accordingly, let $a \in F$. From $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}F = \boldsymbol{\Omega}^{\boldsymbol{A}}G$ and $F/\theta = G/\theta$ it follows

$$a/\boldsymbol{\Omega}^{\boldsymbol{A}}G = a/\boldsymbol{\Omega}^{\boldsymbol{A}}F \in F/\boldsymbol{\Omega}^{\boldsymbol{A}}F = F/\theta = G/\theta = G/\boldsymbol{\Omega}^{\boldsymbol{A}}G.$$

As $\boldsymbol{\Omega}^{\boldsymbol{A}}G$ is compatible with G, the above display implies $a \in G$. Hence, we conclude that $F \subseteq G$, as desired.

Then we turn to prove the implication from right to left in the statement. Consider two matrices $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{A}, G\rangle \in \mathsf{Mod}^*(\vdash)$. As $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{A}, G\rangle$ are reduced, we have $\boldsymbol{\Omega}^{\boldsymbol{A}}F = \mathrm{id}_A = \boldsymbol{\Omega}^{\boldsymbol{A}}G$. Furthermore, as they are models of $\vdash$, it holds that $F, G \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$. Since, by assumption, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is injective, we conclude that $F = G$. $\boxtimes$

In general, the demand that the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is injective, for every algebra $\boldsymbol{A}$, is not equivalent to the injectivity of $\boldsymbol{\Omega} \colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ [64, Sec. 6]. However, for protoalgebraic logics this is indeed the case and both conditions are equivalent to truth equationality [32, Thm. 3.6 & 3.8]:

Theorem 4.21. *The following conditions are equivalent for a protoalgebraic logic $\vdash$:*

(i) *The logic $\vdash$ is truth equational;*

(ii) *The Leibniz operator $\boldsymbol{\Omega} \colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is injective;*

(iii) *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is injective, for every algebra $\boldsymbol{A}$.*

Proof. (i)⇒(iii): Immediate from Remark 4.17 and Proposition 4.20.

As usual the implication (iii)⇒(ii) is straightforward. Therefore, we turn to prove the implication (ii)⇒(i). Suppose that $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is injective. In view of Theorem 4.15, to prove that $\vdash$ is truth equational, it suffices to show that $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is completely order reflecting. To this end, consider $\mathcal{T} \cup \{\Sigma\} \subseteq \mathsf{Th}(\vdash)$ such that

$$\bigcap_{\Gamma \in \mathcal{T}} \boldsymbol{\Omega}\Gamma \subseteq \boldsymbol{\Omega}\Sigma. \tag{4.11}$$

We need to prove that $\bigcap \mathcal{T} \subseteq \Sigma$, that is, $\bigcap \mathcal{T} = \Sigma \cap \bigcap \mathcal{T}$.

Since the sets $\bigcap \mathcal{T}$ and Σ are both theories and, by assumption, the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is injective, it will be enough to show that $\boldsymbol{\Omega} \bigcap \mathcal{T} = \boldsymbol{\Omega}(\Sigma \cap \bigcap \mathcal{T})$. Furthermore, as $\vdash$ is protoalgebraic, the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is order preserving, by Theorem 2.14. Since $\bigcap \mathcal{T}$ and Σ are theories such that $\Sigma \cap \bigcap \mathcal{T} \subseteq \bigcap \mathcal{T}$, this yields $\boldsymbol{\Omega}(\Sigma \cap \bigcap \mathcal{T}) \subseteq \boldsymbol{\Omega} \bigcap \mathcal{T}$. Therefore, it only remains to prove that

$$\boldsymbol{\Omega} \bigcap \mathcal{T} \subseteq \boldsymbol{\Omega}(\Sigma \cap \bigcap \mathcal{T}).$$

We will prove that the congruence $\boldsymbol{\Omega} \bigcap \mathcal{T}$ of $\boldsymbol{Fm}$ is compatible with $\Sigma \cap \bigcap \mathcal{T}$. Since $\boldsymbol{\Omega}(\Sigma \cap \bigcap \mathcal{T})$ is the largest such congruence, this will establish the above display. Accordingly, let $\varphi, \psi \in Fm$ be such that $\varphi \in \Sigma \cap \bigcap \mathcal{T}$ and $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega} \bigcap \mathcal{T}$. As $\boldsymbol{\Omega} \bigcap \mathcal{T}$ is compatible with $\bigcap \mathcal{T}$, from $\varphi \in \Sigma \cap \bigcap \mathcal{T} \subseteq \bigcap \mathcal{T}$ and $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega} \bigcap \mathcal{T}$ it follows

$$\psi \in \bigcap \mathcal{T}. \tag{4.12}$$

On the other hand, since $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is order preserving and $\mathcal{T} \subseteq \mathsf{Th}(\vdash)$, we have

$$\boldsymbol{\Omega} \bigcap \mathcal{T} \subseteq \bigcap_{\Gamma \in \mathcal{T}} \boldsymbol{\Omega}\Gamma.$$

Together with the assumption that $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega} \bigcap \mathcal{T}$ and Condition (4.11), this yields $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega}\Sigma$. Since $\boldsymbol{\Omega}\Sigma$ is compatible with Σ and we assumed that $\varphi \in \Sigma \cap \bigcap \mathcal{T} \subseteq \Sigma$, we obtain $\psi \in \Sigma$. By Condition (4.12), we conclude that $\psi \in \Sigma \cap \bigcap \mathcal{T}$. ⊠

Logics that are both protoalgebraic and truth equational have been called *weakly algebraizable* in [32]. In view of Theorem 2.14 and 4.21, they can be characterized in terms of the behavior of the Leibniz operator as follows (cf. Theorem 5.20):

Corollary 4.22. *The following conditions are equivalent for a logic* $\vdash$*:*

(i) *The logic* $\vdash$ *is weakly algebraizable;*

(ii) *The Leibniz operator* $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ *is order preserving and injective;*

(iii) *The Leibniz operator* $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ *is order preserving and injective, for every algebra* $\boldsymbol{A}$*.*

We close this section with a definability theorem [32, Thm. 3.9]* that is specific to algebraic logic, in the sense that it cannot be derived from similar model theoretic results such as Beth definability theorem:

*For a generalization of this fact beyond the framework of protoalgebraic logics, see [79, Cor. 2.9].

Theorem 4.23. *Let $\vdash$ be a protoalgebraic logic. Truth is implicitly definable in* $\mathsf{Mod}^*(\vdash)$ *iff it is equationally definable.*

Proof. In view of Remark 4.17, it suffices to prove the left to right implication. Then suppose that truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$. By Proposition 4.20, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}(\boldsymbol{A})$ is injective, for every algebra $\boldsymbol{A}$. Since we assumed that $\vdash$ is protoalgebraic, we can apply Theorem 4.21, obtaining that $\vdash$ is truth equational, i.e., that truth is equationally definable in $\mathsf{Mod}^*(\vdash)$. ⊠

CHAPTER 5

Algebraizable logics

In this section we will focus on a family of logics whose connection to algebra is particularly strong.

5.1 Generalized quasivarieties

We denote the class operators of closure under isomorphic copies, subalgebras, and direct products of algebras by $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. In what follows, we will make repeated use of the next technical observation:

Lemma 5.1. *For every algebra $\boldsymbol{A}$ and $\{\theta_i : i \in I\} \cup \{\phi\} \subseteq \mathsf{Con}(\boldsymbol{A})$,*

$$\text{if } \theta = \bigcap_{i \in I} \theta_i, \text{ then } \boldsymbol{A}/\phi \in \mathbb{ISP}(\{\boldsymbol{A}/\theta_i : i \in I\}).$$

Proof. Suppose that $\theta = \bigcap_{i \in I} \theta_i$ and consider the map

$$f \colon \boldsymbol{A}/\theta \to \prod_{i \in I} \boldsymbol{A}/\theta_i$$

defined by the rule

$$a/\theta \longmapsto \langle a/\theta_i : i \in I \rangle.$$

We will prove that f is an embedding.

Accordingly, consider an n-ary connective g and $a_1, \dots, a_n \in A$. For every $j \in I$, we have

$$\begin{aligned}
f(g^{\boldsymbol{A}/\theta}(a_1/\phi, \dots, a_n/\phi))(j) &= f(g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta)(j) \\
&= g^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta_j \\
&= g^{\boldsymbol{A}/\theta_j}(a_1/\theta_j, \dots, a_n/\theta_j) \\
&= g^{\boldsymbol{A}/\theta_j}(f(a_1/\theta)(j), \dots, f(a_n/\theta)(j)) \\
&= g^{\prod_{i \in I} \boldsymbol{A}/\theta_i}(f(a_1/\theta), \dots, f(a_n/\theta))(j).
\end{aligned}$$

The first equality above holds by the definition of $\boldsymbol{A}/\theta$, the second and the fourth by the definition of f, the third by the definition of $\boldsymbol{A}/\theta_j$, and the fifth by the definition of $\prod_{i \in I} \boldsymbol{A}/\theta_i$. Hence, f is a homomorphism.

To prove that it is injective, consider a pair $a, b \in A$ such that $a/\theta \neq b/\theta$. Then $\langle a, b\rangle \notin \theta$. Since we assumed that $\theta = \bigcap_{i \in I} \theta_i$, there exists $i \in I$ such that $\langle a, b\rangle \notin \theta_i$. Therefore,

$$f(a/\theta)(i) = a/\theta_i \neq b/\theta_i = f(b/\theta)(i),$$

whence $f(a/\theta) \neq f(b/\theta)$. Thus, we conclude that f is injective and, therefore, an embedding. ⊠

Given a class K of algebras, let

$$\mathbb{U}(\mathsf{K}) := \{\boldsymbol{A} : \text{every countably generated subalgebra of } \boldsymbol{A} \text{ belongs to } \mathsf{K}\}.$$

Definition 5.2. A class of similar algebras closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{U}$ is said to be a *generalized quasivariety*.

Remark 5.3. Two generalized quasivarieties K and W coincide iff they have the same countably generated members. For suppose that K and W have the same countably generated members and consider $\boldsymbol{A} \in \mathsf{K}$. Since K is closed under $\mathbb{S}$, the countably generated subalgebras of $\boldsymbol{A}$ belong to K and, by assumption, to W as well. As W is closed under $\mathbb{U}$, this yields $\boldsymbol{A} \in \mathsf{W}$. Hence, we conclude that $\mathsf{K} \subseteq \mathsf{W}$. The reverse inclusion is proved similarly. ⊠

We will show that generalized quasivarieties are the classes axiomatizable by the following kind of infinitary formulas [9, Thm. 8.1]:

Definition 5.4. A *generalized quasiequation* is an expression Φ of the form

$$\Big(\mathop{\&}_{i \in I} \varphi_i \approx \psi_i\Big) \Longrightarrow \varepsilon \approx \delta, \tag{5.1}$$

where $\{\varphi_i \approx \psi_i : i \in I\} \cup \{\varepsilon \approx \delta\} \subseteq Eq$. An algebra $\boldsymbol{A}$ *satisfies* Φ, in symbols $\boldsymbol{A} \vDash \Phi$, when, for every homomorphism $f \colon \boldsymbol{Fm} \to \boldsymbol{A}$,

$$\text{if } f(\varphi_i) = f(\psi_i) \text{ for all } i \in I, \text{ then } f(\varepsilon) = f(\delta).$$

Notice that, while the set of indexes I in the above definition can be arbitrarily large, the variables occurring in Φ belong to Fm and, therefore, their number is countable.

Remark 5.5. Generalized quasiequations and relative equational consequences are related as follows: a class K of algebras satisfies the generalized quasiequation in Condition (5.1) iff $\{\varphi_i \approx \psi_i : i \in I\} \vDash_{\mathsf{K}} \varepsilon \approx \delta$. ⊠

Theorem 5.6. *A class of algebras is a generalized quasivariety iff it can be axiomatized by a set of generalized quasiequations.*

Proof. The implication from right to left follows from the fact that the satisfaction of generalized quasiequations is preserved by the class operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{U}$.

To prove the converse, consider a generalized quasivariety K and let K^+ be the class of algebras axiomatized by the generalized quasiequations satisfied by K. To conclude the proof, it suffices to show that $\mathsf{K} = \mathsf{K}^+$.

As K^+ is also a generalized quasivariety (by the implication from right to left in the statement), it will be enough to prove that K and K^+ have the same countably generated members (see Remark 5.3). On the one hand, the definition of K^+ guarantees that $\mathsf{K} \subseteq \mathsf{K}^+$. Then consider some $\boldsymbol{A} \in \mathsf{K}^+$ generated by a countable set G. Since G is countable, there exists a surjective map $f\colon \mathit{Var} \to G$, where Var is the set of variables of Fm. By the universal property of the algebra of formulas $\boldsymbol{Fm}$, there exists a unique homomorphism $f^*\colon \boldsymbol{Fm} \to \boldsymbol{A}$ that extends f. In addition, f^* is surjective, because it is onto the set G of generators of $\boldsymbol{A}$. Thus,

$$\boldsymbol{A} \cong \boldsymbol{Fm}/\mathsf{Ker}(f^*). \tag{5.2}$$

Now, recall that equations are ordered pairs of formulas. In particular, $\mathsf{Ker}(f^*)$ is a set of equations. Bearing this in mind, we associate a generalized quasiequation

$$\Phi_{\varphi,\psi} := \Big(\mathop{\&}\mathsf{Ker}(f^*)\Big) \Longrightarrow \varphi \approx \psi$$

with every $\langle \varphi, \psi \rangle \in \mathit{Eq} \smallsetminus \mathsf{Ker}(f^*)$. On the one hand, from the definition of $\mathsf{Ker}(f^*)$ it follows $f^*(\varepsilon) = f^*(\delta)$, for every $\langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*)$. On the other hand, the assumption that $\langle \varphi, \psi \rangle \in \mathit{Eq} \smallsetminus \mathsf{Ker}(f^*)$ yields $f^*(\varphi) \neq f^*(\psi)$. Hence, $\boldsymbol{A}$ does not satisfy $\Phi_{\varphi,\psi}$. Since $\boldsymbol{A} \in \mathsf{K}^+$ and K^+ is axiomatized by the generalized quasiequations satisfied by K, this yields $\mathsf{K} \nvDash \Phi_{\varphi,\psi}$. Therefore, there exist an algebra $\boldsymbol{B}_{\varphi,\psi} \in \mathsf{K}$ and a homomorphism $g_{\varphi,\psi}\colon \boldsymbol{Fm} \to \boldsymbol{B}_{\varphi,\psi}$ such that

$$g_{\varphi,\psi}(\varepsilon) = g_{\varphi,\psi}(\delta), \text{ for all } \langle \varepsilon, \delta \rangle \in \mathsf{Ker}(f^*), \text{ and } g_{\varphi,\psi}(\varphi) \neq g_{\varphi,\psi}(\psi). \tag{5.3}$$

Consequently, for every $\langle \varphi, \psi \rangle \in \mathit{Eq} \smallsetminus \mathsf{Ker}(f^*)$,

$$\mathsf{Ker}(f^*) \subseteq \mathsf{Ker}(g_{\varphi,\psi}) \text{ and } \langle \varphi, \psi \rangle \notin \mathsf{Ker}(g_{\varphi,\psi}).$$

It follows that

$$\mathsf{Ker}(f^*) = \bigcap\{\mathsf{Ker}(g_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in \mathit{Eq} \smallsetminus \mathsf{Ker}(f^*)\}.$$

By Lemma 5.1, this implies

$$\boldsymbol{Fm}/\mathsf{Ker}(f^*) \in \mathbb{ISP}\{\boldsymbol{Fm}/\mathsf{Ker}(g_{\varphi,\psi}) : \langle \varphi, \psi \rangle \in \mathit{Eq} \smallsetminus \mathsf{Ker}(f^*)\}.$$

Now, recall that each $g_{\varphi,\psi}$ is a homomorphism from $\boldsymbol{Fm}$ to $\boldsymbol{B}_{\varphi,\psi}$, whence $\boldsymbol{Fm}/\mathsf{Ker}(g_{\varphi,\psi}) \in \mathbb{IS}(\boldsymbol{B}_{\varphi,\psi})$. As, by assumption, the various $\boldsymbol{B}_{\varphi,\psi}$ belong to K, we obtain $\boldsymbol{Fm}/\mathsf{Ker}(g_{\varphi,\psi}) \in \mathbb{IS}(\mathsf{K})$. Together with the above display, this yields $\boldsymbol{Fm}/\mathsf{Ker}(f^*) \in \mathbb{ISPIS}(\mathsf{K})$. Since K is a generalized quasivariety, it is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$ and, therefore, $\mathbb{ISPIS}(\mathsf{K}) = \mathsf{K}$. As a consequence,

$$\boldsymbol{Fm}/\mathsf{Ker}(f^*) \in \mathbb{ISPIS}(\mathsf{K}) = \mathsf{K}.$$

Together with Condition (5.2) and the assumption that K is closed under $\mathbb{I}$, this yields $\boldsymbol{A} \in \mathbb{I}(\mathsf{K}) = \mathsf{K}$, as desired. ⊠

Since generalized quasivarieties need not be closed under arbitrary homomorphic images, the next concept is often useful.

Definition 5.7. Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of algebras. A congruence θ of $\boldsymbol{A}$ is said to be a K-*congruence* if $\boldsymbol{A}/\theta \in \mathsf{K}$. When ordered under the inclusion relation, the set of K-congruences of $\boldsymbol{A}$ becomes a poset which we denote by $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.

Proposition 5.8. *If K is a generalized quasivariety, then $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a complete lattice in which meets are intersections, for every algebra $\boldsymbol{A}$.*

Proof. Consider an algebra $\boldsymbol{A}$. In order to prove that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a complete lattice, it is enough to show that the poset $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ has arbitrary meets. To this end, we will prove that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is closed under arbitrary intersections. This will suffice, as in this case the meets in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ coincide with intersections.

Accordingly, consider a family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. In view of Lemma 5.1, we have

$$\boldsymbol{A}/\bigcap_{i\in I}\theta_i \in \mathbb{ISP}(\{\boldsymbol{A}/\theta_i : i \in I\}).$$

Furthermore, the assumption that $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ ensures that each $\boldsymbol{A}/\theta_i$ belongs to K. Together with the above display, this yields $\boldsymbol{A}/\bigcap_{i\in I}\theta_i \in \mathbb{ISP}(\mathsf{K})$. As K is a generalized quasivariety, it is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. Consequently, $\boldsymbol{A}/\bigcap_{i\in I}\theta_i \in \mathbb{ISP}(\mathsf{K}) = \mathsf{K}$. Hence, we conclude that $\bigcap_{i\in I}\theta_i \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. ⊠

Lastly, the following observation will be needed later on (cf. Proposition 1.10).

Proposition 5.9. *Let K be a generalized quasivariety. If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism and $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$, then $f^{-1}[\theta] \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.*

Proof. Let $p_\theta\colon \boldsymbol{B} \to \boldsymbol{B}/\theta$ be the canonical homomorphism defined by the rule $p_\theta(b) := b/\theta$. Moreover, let $p_\theta[f[\boldsymbol{A}]]$ be the subalgebra of $\boldsymbol{B}/\theta$ with universe $p_\theta[f[A]]$. Since the composition $p_\theta \circ f\colon \boldsymbol{A} \to p_\theta[f[\boldsymbol{A}]]$ is a surjective homomorphism, we have

$$\boldsymbol{A}/\mathsf{Ker}(p_\theta \circ f) \cong p_\theta[f[\boldsymbol{A}]].$$

As $p_\theta[f[\boldsymbol{A}]] \in \mathbb{S}(\boldsymbol{B}/\theta)$ and, by assumption, $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$, we have $p_\theta[f[\boldsymbol{A}]] \in \mathbb{S}(\mathsf{K}) \subseteq \mathsf{K}$, where the last inclusion follows from the assumption that K is a generalized quasivariety. Together with the above display and the assumption that K is closed under $\mathbb{I}$, this yields $\mathsf{Ker}(p_\theta \circ f) \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.

Therefore, to conclude the proof, it suffices to show that $f^{-1}[\theta] = \mathsf{Ker}(p_\theta \circ f)$. To this end, consider $a, b \in A$. We have

$$\langle a,b\rangle \in f^{-1}[\theta] \Longleftrightarrow \langle f(a), f(b)\rangle \in \theta \Longleftrightarrow p_\theta(f(a)) = p_\theta(f(b)) \Longleftrightarrow \langle a,b\rangle \in \mathsf{Ker}(p_\theta \circ f).$$

Hence, we conclude that $f^{-1}[\theta] = \mathsf{Ker}(p_\theta \circ f)$. ⊠

5.2 Algebraization

Given a set $\Delta(x,y)$ of formulas and a set Ψ of equations, we let

$$\Delta[\Psi] := \bigcup\{\Delta(\varphi,\psi) : \varphi \approx \psi \in \Psi\}.$$

Definition 5.10. A logic $\vdash$ is *algebraizable* when there exist a set $\tau(x)$ of equations, a set $\Delta(x,y)$ of formulas, and a generalized quasivariety K such that

$$\Gamma \vdash \varphi \Longleftrightarrow \tau[\Gamma] \vDash_{\mathsf{K}} \tau(\varphi) \tag{Alg1}$$

$$\Psi \vDash_{\mathsf{K}} \varepsilon \approx \delta \Longleftrightarrow \Delta[\Psi] \vdash \Delta(\varepsilon,\delta) \tag{Alg2}$$

$$x \vdash \Delta[\tau(x)] \ \text{ and } \ \Delta[\tau(x)] \vdash x \tag{Alg3}$$

$$x \approx y \vDash_{\mathsf{K}} \tau[\Delta(x,y)] \ \text{ and } \ \tau[\Delta(x,y)] \vDash_{\mathsf{K}} x \approx y, \tag{Alg4}$$

for every set $\Gamma \cup \{\varphi\}$ of formulas and every set $\Psi \cup \{\varepsilon \approx \delta\}$ of equations. In this case, K is said to be an *equivalent algebraic semantics* for $\vdash$. In addition, we say that τ, Δ, and K *witness* the algebraization of the logic $\vdash$.

Algebraizable logics were introduced in the seminal monograph [11]. Condition (Alg1) in their definition states that K is a τ-algebraic semantics for $\vdash$, i.e., $\vdash$ can be interpreted into $\vDash_{\mathsf{K}}$ by means of the set $\tau(x)$ of equations that allows us to translate sets Γ of formulas into sets $\tau[\Gamma]$ of equations. Condition (Alg2) states that this interpretation can be reversed, in the sense that $\vDash_{\mathsf{K}}$ can also be interpreted into $\vdash$ by means of the set $\Delta(x,y)$ of formulas that allows us to translate sets Ψ of equations into sets $\Delta[\Psi]$ of formulas. Lastly, Conditions (Alg3) and (Alg4) require that these two interpretations are inverses of each other up to provability equivalence. Because of this, the definition of an algebraizable logic essentially states that the consequence relations $\vdash$ and $\vDash_{\mathsf{K}}$ are *equivalent* [9], as witnessed by the translations $\tau(x)$ and $\Delta(x,y)$.*

The definition of an algebraizable logic can be made more concise, as we proceed to illustrate.

Proposition 5.11. *The following conditions are equivalent for a logic* $\vdash$*:*

(i) $\vdash$ *is algebraizable;*

(ii) *There exist a set* $\tau(x)$ *of equations, a set* $\Delta(x,y)$ *of formulas, and a generalized quasivariety* K *that satisfy Conditions* (Alg1) *and* (Alg4)*;*

(iii) *There exist a set* $\tau(x)$ *of equations, a set* $\Delta(x,y)$ *of formulas, and a generalized quasivariety* K *that satisfy Conditions* (Alg2) *and* (Alg3)*.*

In this case, τ*,* Δ*, and* K *witness the algebraization of* $\vdash$*.*

Proof. The implication (i)$\Rightarrow$(ii) is straightforward. To prove (ii)$\Rightarrow$(iii), suppose that Conditions (Alg1) and (Alg4) hold. Then observe that Condition (Alg4) implies

$$\varepsilon \approx \delta \vDash_{\mathsf{K}} \tau[\Delta(\varepsilon,\delta)] \ \text{ and } \ \tau[\Delta(\varepsilon,\delta)] \vDash_{\mathsf{K}} \varepsilon \approx \delta,$$

for every $\varepsilon \approx \delta \in \tau$. Since $\tau[\Delta[\tau(x)]] = \bigcup\{\tau[\Delta(\varepsilon,\delta)] : \varepsilon \approx \delta \in \tau\}$, this yields

$$\tau(x) \vDash_{\mathsf{K}} \tau[\Delta[\tau(x)]] \ \text{ and } \ \tau[\Delta[\tau(x)]] \vDash_{\mathsf{K}} \tau(x).$$

By Condition (Alg1), we conclude that $x \vdash \Delta[\tau(x)]$ and $\Delta[\tau(x)] \vdash x$, i.e., that Condition (Alg3) holds.

Then consider a set of equations $\Psi \cup \{\varepsilon \approx \delta\}$. We have

$$\Psi \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff \tau[\Delta[\Psi]] \vDash_{\mathsf{K}} \tau[\Delta(\varepsilon,\delta)] \iff \Delta[\Psi] \vdash \Delta(\varepsilon,\delta),$$

where the first equivalence follows from Condition (Alg4) and the second from Condition (Alg1). In view of the above display, Condition (Alg2) holds as well, as desired.

(iii)$\Rightarrow$(i): Suppose that Conditions (Alg2) and (Alg3) hold. An argument analogous to the one detailed in the proof of the implication (ii)$\Rightarrow$(iii) establishes Conditions (Alg1) and (Alg4). Hence, we conclude that τ, Δ, and K witness the algebraization of $\vdash$. ☒

Example 5.12. We will prove that the logic K_g is algebraizable with equivalent algebraic semantics the class MA of modal algebras and

$$\tau(x) := \{x \approx 1\} \ \text{ and } \ \Delta(x,y) := \{x \leftrightarrow y\}.$$

*For logics that are equivalent to consequence relations on the set of inequalities, see [83].

Since MA is a generalized quasivariety, in view of Proposition 5.11, it suffices to show that Conditions (Alg1) and (Alg4) hold. On the one hand, Condition (Alg1) holds because MA is a τ-algebraic semantics for K_g, as detailed in Example 4.2. On the other hand, Condition (Alg4) amounts to

$$x \approx y \vDash_{\mathsf{MA}} x \leftrightarrow y \approx 1 \quad \text{and} \quad x \leftrightarrow y \approx 1 \vDash_{\mathsf{MA}} x \approx y,$$

which is easily seen to be true.

A similar argument shows that IPC is algebraizable with equivalent algebraic semantics the class of Heyting algebras and the sets τ and Δ defined above. Lastly, the logic FL is algebraizable with equivalent algebraic semantics the generalized quasivariety FA of FL-algebras and

$$\tau(x) := \{x \wedge 1 \approx 1\} \quad \text{and} \quad \Delta(x, y) := \{x \leftrightarrow y\}.$$

This is because FA is a τ-algebraic semantics for FL, as explained in Example 4.2, and

$$x \approx y \vDash_{\mathsf{FA}} (x \leftrightarrow y) \wedge 1 \approx 1 \quad \text{and} \quad (x \leftrightarrow y) \wedge 1 \approx 1 \vDash_{\mathsf{FA}} x \approx y. \qquad \boxtimes$$

Given a logic $\vdash$, we denote the class of the algebraic reducts of the matrices in $\mathsf{Mod}^*(\vdash)$ by $\mathsf{Alg}^*(\vdash)$. More precisely,

$$\mathsf{Alg}^*(\vdash) := \{\boldsymbol{A} : \text{there exists } F \subseteq A \text{ s.t. } \langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\vdash)\}.$$

Theorem 5.13. *The following conditions hold for a logic $\vdash$, a set $\tau(x)$ of equations, a set $\Delta(x, y)$ of formulas, and a generalized quasivariety* K*:*

(i) *If τ, Δ, and* K *witness the algebraization of $\vdash$, then τ defines truth in $\mathsf{Mod}^*(\vdash)$, Δ is a set of equivalence formulas for $\vdash$, and $\mathsf{K} = \mathsf{Alg}^*(\vdash)$;*

(ii) *If τ defines truth in $\mathsf{Mod}^*(\vdash)$ and Δ is a set of equivalence formulas for $\vdash$, then τ, Δ, and $\mathsf{Alg}^*(\vdash)$ witness the algebraization of $\vdash$.*

Proof. (i): We begin by proving that Δ is a set of equivalence formulas for $\vdash$. To this end, it will be enough to show that Δ satisfies the conditions in Theorem 3.2. First, observe that

$$\emptyset \vDash_{\mathsf{K}} x \approx x \quad \text{and} \quad x_1 \approx y_1, \dots, x_n \approx y_n \vDash_{\mathsf{K}} f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n),$$

for every n-ary connective f. Together with Condition (Alg2), this yields

$$\emptyset \vdash \Delta(x, x) \quad \text{and} \quad \Delta(x_1, y_1), \dots, \Delta(x_n, y_n) \vdash \Delta(f(x_1, \dots, x_n), f(y_1, \dots, y_n)),$$

for every n-ary connective f. Moreover, we have

$$x, \Delta(x, y) \vdash y \iff \tau(x), \tau[\Delta(x, y)] \vDash_{\mathsf{K}} \tau(y) \iff \tau(x), x \approx y \vDash_{\mathsf{K}} \tau(y),$$

where the first equivalence holds by Condition (Alg1) and the second by Condition (Alg4). Since the right hand side of the above display always holds, we conclude that $x, \Delta(x, y) \vdash y$. Hence, Δ is a set of equivalence formulas for $\vdash$, as desired.

Then we turn to prove that τ defines truth in $\mathsf{Mod}^*(\vdash)$. Accordingly, consider $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $a \in A$. We have

$$a \in F \iff \Delta^{\boldsymbol{A}}[\tau^{\boldsymbol{A}}(a)] \subseteq F \iff \tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F,$$

where the first equivalence holds by Condition (Alg3) and the assumption that $\langle \boldsymbol{A}, F\rangle$ is a model of $\vdash$, while the second follows from the fact that Δ is a set of equivalence formulas for $\vdash$ and $\langle \boldsymbol{A}, F\rangle$ a model of $\vdash$. In view of Remark 4.10, the above display ensures that τ defines truth in $\mathsf{Mod}^*(\vdash)$.

It only remains to show that $\mathsf{K} = \mathsf{Alg}^*(\vdash)$. We begin by proving the inclusion $\mathsf{K} \subseteq \mathsf{Alg}^*(\vdash)$. To this end, recall that K is a τ-algebraic semantics for $\vdash$, by Condition (Alg1). Therefore, $\{\langle \boldsymbol{A}, \tau(\boldsymbol{A})\rangle : \boldsymbol{A} \in \mathsf{K}\}$ is a class of models of $\vdash$, by Proposition 4.4(i). Then consider $\boldsymbol{A} \in \mathsf{K}$. We will prove that $\langle \boldsymbol{A}, \tau(\boldsymbol{A})\rangle \in \mathsf{Mod}^*(\vdash)$ and, therefore, that $\boldsymbol{A} \in \mathsf{Alg}^*(\vdash)$. Since the matrix $\langle \boldsymbol{A}, \tau(\boldsymbol{A})\rangle$ is a model of $\vdash$, it suffices to show that it is reduced. Accordingly, consider $a, b \in A$. We have

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\tau(\boldsymbol{A}) \iff \Delta^{\boldsymbol{A}}(a, b) \subseteq \tau(\boldsymbol{A}) \iff \boldsymbol{A} \vDash \tau[\Delta(a, b)] \iff a = b,$$

where the fist equivalence holds because Δ is a set of equivalence formulas for $\vdash$ and $\langle \boldsymbol{A}, \tau(\boldsymbol{A})\rangle$ a model of $\vdash$, the second is straightforward, and the third holds by Condition (Alg4) and the assumption that $\boldsymbol{A} \in \mathsf{K}$. In view of the above display, the matrix $\langle \boldsymbol{A}, \tau(\boldsymbol{A})\rangle$ is reduced, as desired.

To prove the inclusion $\mathsf{Alg}^*(\vdash) \subseteq \mathsf{K}$, consider $\boldsymbol{A} \in \mathsf{Alg}^*(\vdash)$. Since τ defines truth in $\mathsf{Mod}^*(\vdash)$ and $\boldsymbol{A} \in \mathsf{Alg}^*(\vdash)$, the matrix $\langle \boldsymbol{A}, \tau(\boldsymbol{A})\rangle$ is a reduced model of $\vdash$. As Δ is a set of equivalence formulas for $\vdash$, this implies that, for every $a, b \in A$,

$$a = b \iff \Delta^{\boldsymbol{A}}(a, b) \subseteq \tau(\boldsymbol{A}). \tag{5.4}$$

Recall from Theorem 5.6 that the generalized quasivariety K is axiomatized by a set of generalized quasiequations. Therefore, in order to prove that $\boldsymbol{A} \in \mathsf{K}$, it suffices to show that $\boldsymbol{A}$ satisfies all the generalized quasiequations satisfied by K. To this end, suppose that

$$\mathsf{K} \vDash \Big(\mathop{\&}_{i \in I} \varphi_i \approx \psi_i \Big) \Longrightarrow \varepsilon \approx \delta.$$

By Remark 5.5, this amounts to $\{\varphi_i \approx \psi_i : i \in I\} \vDash_{\mathsf{K}} \varepsilon \approx \delta$. Therefore, we can apply Condition (Alg2), obtaining

$$\bigcup_{i \in I} \Delta(\varphi_i, \psi_i) \vdash \Delta(\varepsilon, \delta).$$

Then consider a homomorphism $f\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $f(\varphi_i) = f(\psi_i)$, for every $i \in I$. From Condition (5.4) it follows $f[\Delta(\varphi_i, \psi_i)] \subseteq \tau(\boldsymbol{A})$, for every $i \in I$. Since $\langle \boldsymbol{A}, \tau(\boldsymbol{A})\rangle$ is a model of $\vdash$, the above display guarantees that $f[\Delta(\varepsilon, \delta)] \subseteq \tau(\boldsymbol{A})$. Together with Condition (5.4), this implies $f(\varepsilon) = f(\delta)$. Hence, we conclude that $\boldsymbol{A} \in \mathsf{K}$. ☒

From Theorem 5.13, we deduce [50, Thm. 3]:

Corollary 5.14. *A logic is algebraizable iff it is both truth equational and equivalential.*

While a logic might have different algebraic semantics (Example 4.5), from Theorem 5.13(i) it follows that *equivalent* algebraic semantics are necessarily unique [11, Thm. 2.15]:

Corollary 5.15. *Every algebraizable logic $\vdash$ has a unique equivalent algebraic semantics, namely,* $\mathsf{Alg}^*(\vdash)$.

Lastly from Corollaries 3.3, 4.9, and 5.14 it follows:

Corollary 5.16. *Extensions of algebraizable logics are still algebraizable.*

Algebraizable logics admit also a syntactic description [11, Thm. 4.7]:

Theorem 5.17. *A logic $\vdash$ is algebraizable iff there exist a set $\tau(x)$ of equations and a set $\Delta(x, y)$ of formulas such that*

$$\emptyset \vdash \Delta(x, x) \qquad x, \Delta(x, y) \vdash y \qquad x \vdash \Delta[\tau(x)] \qquad \Delta[\tau(x)] \vdash x$$

and, for every n-ary connective f,

$$\Delta(x_1, y_1), \ldots, \Delta(x_n, y_n) \vdash \Delta(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)).$$

Proof. Suppose first that $\vdash$ is algebraizable as witnessed by a set $\tau(x)$ of equations, a set $\Delta(x, y)$ of formulas, and a generalized quasivariety K. Then we can apply Theorem 5.13(i), obtaining that $\Delta(x, y)$ is a set of equivalence formulas for $\vdash$. By Theorem 3.2, this implies

$$\emptyset \vdash \Delta(x, x) \quad x, \Delta(x, y) \vdash y \quad \Delta(x_1, y_1), \ldots, \Delta(x_n, y_n) \vdash \Delta(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)),$$

for every n-ary connective f of $\vdash$. Furthermore, Theorem 5.13(i) ensures also that $\tau(x)$ defines truth in $\mathsf{Mod}^*(\vdash)$. Then, for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*(\vdash)$ and $a \in A$,

$$a \in F \iff \tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \Delta^{\boldsymbol{A}}[\tau^{\boldsymbol{A}}(a)] \subseteq F,$$

where the first equivalence holds by Remark 4.10 and the assumption that $\tau(x)$ defines truth in $\mathsf{Mod}^*(\vdash)$ and the second because $\Delta(x, y)$ is a set of equivalence formulas for $\vdash$ and $\langle \boldsymbol{A}, F \rangle$ a model of $\vdash$. Now, recall from Theorem 1.21 that $\vdash$ is the logic induced by $\mathsf{Mod}^*(\vdash)$. Together with the above display, this implies $x \vdash \Delta[\tau(x)]$ and $\Delta[\tau(x)] \vdash x$.

Then we turn to prove the implication from right to left in the statement. From Theorem 3.2 it follows that $\Delta(x, y)$ is a set of equivalence formulas for $\vdash$ and, therefore, that $\vdash$ is equivalential. We will prove that $\tau(x)$ defines truth in $\mathsf{Mod}^*(\vdash)$. In view of Remark 4.10, it suffices to show that, for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}(\vdash)$ and $a \in A$,

$$a \in F \iff \tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F.$$

To this end, consider $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}(\vdash)$ and $a \in A$. Since we assumed that $x \vdash \Delta[\tau(x)]$ and $\Delta[\tau(x)] \vdash x$ and $\langle \boldsymbol{A}, F \rangle$ is a model of $\vdash$, we have $a \in F$ iff $\Delta^{\boldsymbol{A}}[\tau^{\boldsymbol{A}}(a)] \subseteq F$. Furthermore, as $\Delta(x, y)$ is a set of equivalence formulas for $\vdash$ and $\langle \boldsymbol{A}, F \rangle$ a model of $\vdash$, we get $\Delta^{\boldsymbol{A}}[\tau^{\boldsymbol{A}}(a)] \subseteq F$ iff $\tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F$. This establishes the above equivalence and, therefore, that τ defines truth in $\mathsf{Mod}^*(\vdash)$. As a consequence, $\vdash$ is truth equational. Since $\vdash$ is also equivalential, Corollary 5.14 implies that $\vdash$ is algebraizable. ⊠

5.3 The isomorphism theorem

The aim of this section is to establish the isomorphism theorem of algebraizable logics. To this end, we rely on the following observation:

Lemma 5.18. *For every logic $\vdash$ and algebra $\boldsymbol{A}$,*

$$\mathsf{Con}_{\mathsf{Alg}^*(\vdash)}(\boldsymbol{A}) = \{\boldsymbol{\Omega}^{\boldsymbol{A}} F : F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})\}.$$

Proof. Consider first some $\theta \in \mathsf{Con}_{\mathsf{Alg}^*(\vdash)}(\boldsymbol{A})$. We have $\boldsymbol{A}/\theta \in \mathsf{Alg}^*(\vdash)$. Therefore, there exists $F \subseteq A/\theta$ such that $\langle \boldsymbol{A}/\theta, F\rangle \in \mathsf{Mod}^*(\vdash)$. Then let $p_\theta \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ be the canonical homomorphism given by the rule $p_\theta(a) := a/\theta$ and observe that

$$p_\theta \colon \langle \boldsymbol{A}, p_\theta^{-1}[F]\rangle \to \langle \boldsymbol{A}/\theta, F\rangle$$

is a strict surjective homomorphism. Furthermore, $\boldsymbol{\Omega}^{\boldsymbol{A}/\theta}F = \mathrm{id}_{A/\theta}$, because the matrix $\langle \boldsymbol{A}/\theta, F\rangle$ is reduced. Thus, we can apply Corollary 1.15, obtaining

$$\boldsymbol{\Omega}^{\boldsymbol{A}} p_\theta^{-1}[F] = \mathsf{Ker}(p_\theta) = \theta.$$

On the other hand, since $\langle \boldsymbol{A}/\theta, F\rangle$ is a model of $\vdash$, Proposition 1.10 guarantees that $p_\theta^{-1}[F]$ is a deductive filter of $\vdash$ on $\boldsymbol{A}$. Hence, we conclude that $p_\theta^{-1}[F] \in \mathsf{Fi}_\vdash(\boldsymbol{A})$ and $\boldsymbol{\Omega}^{\boldsymbol{A}} p_\theta^{-1}[F] = \theta$. This establishes the inclusion from left to right in the statement.

To prove the reverse inclusion, consider $F \in \mathsf{Fi}_\vdash(\boldsymbol{A})$. By Theorem 1.21, we have $\langle \boldsymbol{A}, F\rangle^* \in \mathsf{Mod}^*(\vdash)$, whence $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F \in \mathsf{Alg}^*(\vdash)$ and $\boldsymbol{\Omega}^{\boldsymbol{A}}F \in \mathsf{Con}_{\mathsf{Alg}^*(\vdash)}(\boldsymbol{A})$. ⊠

Corollary 5.19. *The map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_\vdash(\boldsymbol{A}) \to \mathsf{Con}_{\mathsf{Alg}^*(\vdash)}(\boldsymbol{A})$ is well defined and surjective, for every logic $\vdash$ and algebra $\boldsymbol{A}$.*

As a consequence, we obtain a description of weakly algebraizable logics in terms of a lattice isomorphism [32, Thm. 4.8]:

Theorem 5.20. *The following conditions are equivalent for a logic $\vdash$:*

(i) *The logic $\vdash$ is weakly algebraizable;*

(ii) *The Leibniz operator $\boldsymbol{\Omega} \colon \mathsf{Th}(\vdash) \to \mathsf{Con}_{\mathsf{Alg}^*(\vdash)}(\boldsymbol{Fm})$ is a lattice isomorphism;*

(iii) *The Leibniz operator $\boldsymbol{\Omega} \colon \mathsf{Fi}_\vdash(\boldsymbol{A}) \to \mathsf{Con}_{\mathsf{Alg}^*(\vdash)}(\boldsymbol{A})$ is a lattice isomorphism, for every algebra $\boldsymbol{A}$.*

Proof. (i)⇒(iii): Consider an algebra $\boldsymbol{A}$. In view of Corollary 5.19, the map $\boldsymbol{\Omega} \colon \mathsf{Fi}_\vdash(\boldsymbol{A}) \to \mathsf{Con}_{\mathsf{Alg}^*(\vdash)}(\boldsymbol{A})$ is well defined and surjective. Therefore, it only remains to prove that it is an order embedding. To this end, recall that, by assumption, the logic $\vdash$ is weakly algebraizable, i.e., protoalgebraic and truth equational. As it is protoalgebraic, we can apply Theorem 3.4, obtaining that $\boldsymbol{\Omega} \colon \mathsf{Fi}_\vdash(\boldsymbol{A}) \to \mathsf{Con}_{\mathsf{Alg}^*(\vdash)}(\boldsymbol{A})$ is order preserving. Moreover, since $\vdash$ is truth equational, this map is also order reflecting, by Theorem 4.15.

As usual, the implication (iii)⇒(ii) is straightforward. Lastly, the implication (ii)⇒(i) follows from Corollary 4.22. ⊠

We denote the set of endomorphisms of an algebra $\boldsymbol{A}$ by $\mathsf{End}(\boldsymbol{A})$. Recall that, given a logic $\vdash$ and a generalized quasivariety K, the posets $\mathsf{Fi}_\vdash(\boldsymbol{A})$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ are complete lattices. Furthermore, in view of Propositions 1.10 and 5.9, for every $F \in \mathsf{Fi}_\vdash(\boldsymbol{A})$ and $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, it holds that

$$\text{if } \sigma \in \mathsf{End}(\boldsymbol{A}), \text{ then } \sigma^{-1}[F] \in \mathsf{Fi}_\vdash(\boldsymbol{A}) \text{ and } \sigma^{-1}[\theta] \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}).$$

Consequently, we can expand the lattices $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ with the family of unary operations $\{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}$. We denote the results of these expansions by $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+}$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$, that is,

$$\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} := \langle \mathsf{Fi}_{\vdash}(\boldsymbol{A}); \wedge, \vee, \{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}\rangle;$$
$$\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+} := \langle \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}); \wedge, \vee, \{\sigma^{-1} : \sigma \in \mathsf{End}(\boldsymbol{A})\}\rangle.$$

We say that $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+}$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ are *isomorphic*, in symbols $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \cong \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$, when there exists a lattice isomorphism $\Phi \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ such that

$$\Phi(\sigma^{-1}[F]) = \sigma^{-1}[\Phi(F)],$$

for every $\sigma \in \mathsf{End}(\boldsymbol{A})$ and $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$. When $\boldsymbol{A} = \boldsymbol{Fm}$, we will write $\mathsf{Th}(\vdash)^{+}$ instead of $\mathsf{Fi}_{\vdash}(\boldsymbol{Fm})^{+}$.

The isomorphism theorem of algebraizable logics originates in [11, Thms. 3.7(ii) & 5.1] (see also [51]):

Theorem 5.21. *The following conditions are equivalent for a logic $\vdash$ and a generalized quasi-variety K:*

(i) *The logic $\vdash$ is algebraizable with equivalent algebraic semantics K;*

(ii) $\mathsf{Th}(\vdash)^{+} \cong \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})^{+}$*;*

(iii) $\mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \cong \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$*, for every algebra $\boldsymbol{A}$.*

Furthermore, in this case, the isomorphism in Condition (iii) can be taken to be the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.

Proof. (i)⇒(iii): First, recall from Theorem 5.13(i) that

$$\mathsf{K} = \mathsf{Alg}^{*}(\vdash). \tag{5.5}$$

Furthermore, recall from Corollary 5.14 that $\vdash$ is both truth equational and equivalential and, consequently, weakly algebraizable too.

Then consider an algebra $\boldsymbol{A}$. Since $\vdash$ is weakly algebraizable, from Theorem 5.20 and the above display it follows that the map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A}) \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a well-defined lattice isomorphism. Furthermore, since $\vdash$ is equivalential, we can apply Theorem 3.4, obtaining that this map commutes with endomorphisms. Consequently, $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathsf{Fi}_{\vdash}(\boldsymbol{A})^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})^{+}$ is an isomorphism, as desired.

As usual, the implication (iii)⇒(ii) is straightforward. Therefore, we will detail only the proof of the implication (ii)⇒(i). Suppose that there exists an isomorphism

$$\Phi \colon \mathsf{Th}(\vdash)^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})^{+}.$$

Then let $\sigma_{x,y}$ be the substitution that sends all the variables other than y to x and leaves y untouched. Moreover, let $\mathrm{Cg}_{\mathsf{K}}(x,y)$ be the least K-congruence of $\boldsymbol{Fm}$ containing the pair $\langle x,y\rangle$ (which exists by Proposition 5.8). The assumption that Φ is an isomorphism and the definition of $\sigma_{x,y}$ guarantee that the following is a set of formulas in variables x and y only:

$$\Delta(x,y) := \sigma_{x,y}[\Phi^{-1}(\mathrm{Cg}_{\mathsf{K}}(x,y))].$$

The proof proceeds through a series of observations:

Claim 5.22. *For every $\Gamma \in \mathsf{Th}(\vdash)$,*

$$\Phi(\Gamma) = \{\langle \varphi, \psi \rangle \in Fm \times Fm : \Delta(\varphi, \psi) \subseteq \Gamma\}.$$

Proof of the Claim. Consider a pair φ and ψ of formulas and let $\sigma_{\varphi,\psi}$ be the substitution that sends every variable other than y to φ and that sends y to ψ. We will prove that

$$\begin{aligned}
\Delta(\varphi, \psi) \subseteq \Gamma &\iff \sigma_{\varphi,\psi}[\Delta(x, y)] \subseteq \Gamma \\
&\iff \sigma_{\varphi,\psi}[\sigma_{x,y}[\Phi^{-1}(\mathrm{Cg}_{\mathsf{K}}(x, y))]] \subseteq \Gamma \\
&\iff \Phi^{-1}(\mathrm{Cg}_{\mathsf{K}}(x, y)) \subseteq \sigma_{x,y}^{-1}[\sigma_{\varphi,\psi}^{-1}[\Gamma]] \\
&\iff \mathrm{Cg}_{\mathsf{K}}(x, y) \subseteq \Phi(\sigma_{x,y}^{-1}[\sigma_{\varphi,\psi}^{-1}[\Gamma]]) \\
&\iff \mathrm{Cg}_{\mathsf{K}}(x, y) \subseteq \sigma_{x,y}^{-1}[\sigma_{\varphi,\psi}^{-1}[\Phi(\Gamma)]] \\
&\iff \langle x, y \rangle \in \sigma_{x,y}^{-1}[\sigma_{\varphi,\psi}^{-1}[\Phi(\Gamma)]] \\
&\iff \langle \sigma_{\varphi,\psi}(\sigma_{x,y}(x)), \sigma_{\varphi,\psi}(\sigma_{x,y}(y)) \rangle \in \Phi(\Gamma) \\
&\iff \langle \varphi, \psi \rangle \in \Phi(\Gamma).
\end{aligned}$$

The equivalences above are justified as follows: the first and the last hold by the definition of $\sigma_{x,y}$ and $\sigma_{\varphi,\psi}$, the second by the definition of $\Delta(x, y)$, and the third and the seventh are obvious. To prove the fourth equivalence, recall that we assumed that Γ is a theory. Therefore, $\sigma_{x,y}^{-1}[\sigma_{\varphi,\psi}^{-1}[\Gamma]]$ is also a theory, by Corollary 1.11. Consequently, the assumption that the map $\Phi\colon \mathsf{Th}(\vdash) \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})$ is a bijection guarantees the validity of the fourth equivalence. The fifth equivalence holds because, by assumption, $\Gamma \in \mathsf{Th}(\vdash)$ and the map $\Phi\colon \mathsf{Th}(\vdash) \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})$ commutes with substitutions. It only remains to prove the sixth equivalence. To this end, observe that the assumption that $\Gamma \in \mathsf{Th}(\vdash)$ ensures that $\Phi(\Gamma) \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})$. By Proposition 5.9, this implies that $\sigma_{x,y}^{-1}[\sigma_{\varphi,\psi}^{-1}[\Phi(\Gamma)]]$ is also a K-congruence of $\boldsymbol{Fm}$. Since $\mathrm{Cg}_{\mathsf{K}}(x, y)$ is the least K-congruence of $\boldsymbol{Fm}$ containing the pair $\langle x, y \rangle$, this yields that $\sigma_{x,y}^{-1}[\sigma_{\varphi,\psi}^{-1}[\Phi(\Gamma)]]$ extends $\mathrm{Cg}_{\mathsf{K}}(x, y)$ iff it contains the pair $\langle x, y \rangle$. ⊠

Claim 5.23. *$\Delta(x, y)$ is a set of equivalence formulas for $\vdash$.*

Proof. It suffices to prove that $\Delta(x, y)$ satisfies the conditions in Theorem 3.2. First, observe that the relation $\Phi(\mathrm{Cn}_{\vdash}(\emptyset))$ is reflexive, because, by assumption, it is a congruence of $\boldsymbol{Fm}$. In particular, $\langle x, x \rangle \in \Phi(\mathrm{Cn}_{\vdash}(\emptyset))$. By Claim 5.22, this implies $\emptyset \vdash \Delta(x, x)$.

Then consider an n-ary connective f. By Claim 5.22, we have

$$\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle \in \Phi(\mathrm{Cn}_{\vdash}(\Delta(x_1, y_1) \cup \dots \cup \Delta(x_n, y_n))).$$

Since $\Phi(\mathrm{Cn}_{\vdash}(\Delta(x_1, y_1) \cup \dots \cup \Delta(x_n, y_n)))$ is a congruence of $\boldsymbol{Fm}$, this yields

$$\langle f(x_1, \dots, x_n), f(y_1, \dots, y_n) \rangle \in \Phi(\mathrm{Cn}_{\vdash}(\Delta(x_1, y_1) \cup \dots \cup \Delta(x_n, y_n)))$$

which, by Claim 5.22, entails $\Delta(x_1, y_1), \dots, \Delta(x_n, y_n) \vdash \Delta(f(x_1, \dots, x_n), f(y_1, \dots, y_n))$.

Therefore, it only remains to prove that $x, \Delta(x, y) \vdash y$. To this end, it suffices to show that

$$\Phi(\mathrm{Cn}_{\vdash}(\{y\})) \subseteq \Phi(\mathrm{Cn}_{\vdash}(\{x\} \cup \Delta(x, y))).$$

This is because, in this case, the assumption that $\Phi\colon \mathsf{Th}(\vdash) \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})$ is a lattice isomorphism yields $\mathrm{Cn}_{\vdash}(\{y\}) \subseteq \mathrm{Cn}_{\vdash}(\{x\} \cup \Delta(x, y))$, that is, $x, \Delta(x, y) \vdash y$.

To prove the above display, consider a pair $\varphi(x,y,\vec{z})$ and $\psi(x,y,\vec{z})$ of formulas such that $\langle \varphi(x,y,\vec{z}), \psi(x,y,\vec{z})\rangle \in \Phi(\mathrm{Cn}_{\vdash}(\{y\}))$. By Claim 5.22, we have $y \vdash \Delta(\varphi(x,y,\vec{z}), \psi(x,y,\vec{z}))$ which, by substitution invariance, yields $x \vdash \Delta(\varphi(x,x,\vec{z}), \psi(x,x,\vec{z}))$. From Claim 5.22 it follows

$$\langle \varphi(x,x,\vec{z}), \psi(x,x,\vec{z})\rangle, \langle x,y\rangle \in \Phi(\mathrm{Cn}_{\vdash}(\{x\} \cup \Delta(x,y))).$$

Since $\Phi(\mathrm{Cn}_{\vdash}(\{x\} \cup \Delta(x,y)))$ is a congruence of $\boldsymbol{Fm}$, this yields

$$\langle \varphi(x,y,\vec{z}), \psi(x,y,\vec{z})\rangle \in \Phi(\mathrm{Cn}_{\vdash}(\{x\} \cup \Delta(x,y))).$$

Thus, we conclude that $\Phi(\mathrm{Cn}_{\vdash}(\{y\})) \subseteq \Phi(\mathrm{Cn}_{\vdash}(\{x\} \cup \Delta(x,y)))$, as desired. ⊠

Now, we will prove that $\Phi(\Gamma) = \boldsymbol{\Omega}\Gamma$, for every $\Gamma \in \mathsf{Th}(\vdash)$. To this end, consider $\Gamma \in \mathsf{Th}(\vdash)$. By Proposition 1.7, the matrix $\langle \boldsymbol{Fm}, \Gamma\rangle$ is a model of $\vdash$. Therefore, Claim 5.23 ensures that

$$\boldsymbol{\Omega}\Gamma = \{\langle \varphi, \psi\rangle \in Fm \times Fm : \Delta(\varphi, \psi) \subseteq \Gamma\}.$$

Together with Claim 5.22, this yields $\Phi(\Gamma) = \boldsymbol{\Omega}\Gamma$, as desired.

Consequently, from the assumption that $\Phi\colon \mathsf{Th}(\vdash)^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})^{+}$ is an isomorphism it follows that so is $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash)^{+} \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})^{+}$. In particular, the Leibniz operator $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}(\boldsymbol{Fm})$ is completely order reflecting, order preserving, and commutes with inverse substitutions. By Theorems 4.15 and 3.4, this yields that $\vdash$ is both truth equational and equivalential. Therefore, we can apply Corollary 5.14, obtaining that $\vdash$ is algebraizable.

In order to conclude the proof, it only remains to show that K is the equivalent algebraic semantics of $\vdash$. Since $\vdash$ is algebraizable, its equivalent algebraic semantics is $\mathsf{Alg}^{*}(\vdash)$, by Theorem 5.13(i). Therefore, we will prove that $\mathsf{K} = \mathsf{Alg}^{*}(\vdash)$. To this end, observe that

$$\mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm}) = \{\boldsymbol{\Omega}\Gamma : \Gamma \in \mathsf{Th}(\vdash)\} = \mathsf{Con}_{\mathsf{Alg}^{*}(\vdash)}(\boldsymbol{Fm}), \tag{5.6}$$

where the first equality follows from the fact that the map $\boldsymbol{\Omega}\colon \mathsf{Th}(\vdash) \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})$ is surjective and the second from Lemma 5.18. As the sets of the countably generated members of K and $\mathsf{Alg}^{*}(\vdash)$ are, respectively,

$$\mathbb{I}\{\boldsymbol{Fm}/\theta : \theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{Fm})\} \text{ and } \mathbb{I}\{\boldsymbol{Fm}/\theta : \theta \in \mathsf{Con}_{\mathsf{Alg}^{*}(\vdash)}(\boldsymbol{Fm})\},$$

Condition (5.6) implies that K and $\mathsf{Alg}^{*}(\vdash)$ have the same countably generated members. Since K and $\mathsf{Alg}^{*}(\vdash)$ are both generalized quasivarieties (the first by assumption and the second because it is the equivalent algebraic semantics of $\vdash$), from Remark 5.3 it follows $\mathsf{K} = \mathsf{Alg}^{*}(\vdash)$, as desired. ⊠

The main conceptual difference between the isomorphism theorem for weakly algebraizable logics (i.e., Theorem 5.20) and that for algebraizable logics (i.e., Theorem 5.21) is that, in the latter, the isomorphisms are not explicitly required to be witnessed by the Leibniz operator. As a consequence, any robust correspondence between special sets and congruences indicates the presence of an algebraizable logic. Because of that, it should not come as a surprise that most of the correspondences between filters and congruences in the algebra of logic can be viewed as special instances of the isomorphism theorem of algebraizable logics.

Example 5.24. Given a modal algebra $\boldsymbol{A}$ and a Heyting algebra $\boldsymbol{B}$, let

$$\begin{aligned}\mathsf{Op}(\boldsymbol{A}) &:= \text{the lattice of open lattice filters of } \boldsymbol{A};\\ \mathsf{Fi}(\boldsymbol{B}) &:= \text{the lattice of lattice filters of } \boldsymbol{B}.\end{aligned}$$

We will use the isomorphism theorem (of algebraizable logics) to prove that

$$\mathsf{Op}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A}) \;\text{ and }\; \mathsf{Fi}(\boldsymbol{B}) \cong \mathsf{Con}(\boldsymbol{B}).$$

To this end, recall from Example 5.12 that K_g is algebraizable with equivalent algebraic semantics the class MA of modal algebras. Therefore, the isomorphism theorem guarantees that $\mathsf{Fi}_{\mathsf{K}_g}(\boldsymbol{A}) \cong \mathsf{Con}_{\mathsf{MA}}(\boldsymbol{A})$. This result can be improved as follows. On the one hand, as MA is closed under the formation of homomorphic images and $\boldsymbol{A} \in \mathsf{MA}$, we have $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{MA}}(\boldsymbol{A})$. On the other hand, the assumption that $\boldsymbol{A}$ is a modal algebra yields $\mathsf{Fi}_{\mathsf{K}_g}(\boldsymbol{A}) = \mathsf{Op}(\boldsymbol{A})$ (see Example 1.12). Therefore, we conclude that

$$\mathsf{Op}(\boldsymbol{A}) = \mathsf{Fi}_{\mathsf{K}_g}(\boldsymbol{A}) \cong \mathsf{Con}_{\mathsf{MA}}(\boldsymbol{A}) = \mathsf{Con}(\boldsymbol{A}),$$

as desired. An analogous argument shows that $\mathsf{Fi}(\boldsymbol{B}) \cong \mathsf{Con}(\boldsymbol{B})$. ⊠

The applicability of the isomorphism theorem, however, goes beyond the algebra of logic, as we proceed to illustrate.

Example 5.25. Given a group $\boldsymbol{A}$ [58], let

$$\mathsf{N}(\boldsymbol{A}) := \text{the lattice of normal subgroups of } \boldsymbol{A}.$$

We will use the isomorphism theorem to prove that

$$\mathsf{N}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A}).$$

To this end, we adopt the multiplicative notation for groups. Let $\vdash_{\mathsf{G}}$ be the assertional logic of the class G of groups. We begin by proving that

$$\mathsf{Fi}_{\vdash_{\mathsf{G}}}(\boldsymbol{A}) = \mathsf{N}(\boldsymbol{A}), \text{ for every group } \boldsymbol{A}. \tag{5.7}$$

Accordingly, consider a group $\boldsymbol{A}$. On the one hand, the definition of $\vdash_{\mathsf{G}}$ guarantees the validity of the rules

$$\emptyset \rhd 1 \qquad x, y \rhd x \cdot y \qquad x \rhd x^{-1} \qquad x \rhd y \cdot (x \cdot y^{-1}).$$

This, in turn, implies that the deductive filters of $\vdash_{\mathsf{G}}$ on $\boldsymbol{A}$ are normal subgroups. On the other hand, let N be a normal subgroup of $\boldsymbol{A}$ and consider the congruence of $\boldsymbol{A}$ associated with it, namely,

$$\theta_N := \{\langle a, b\rangle \in A \times A : a \cdot b^{-1} \in N\}.$$

It is well known that $p_{\theta_N}^{-1}[\{1/\theta_N\}] = N$, where $p_\theta \colon \boldsymbol{A} \to \boldsymbol{A}/\theta_N$ is the canonical homomorphism given by the rule $p_{\theta_N}(a) := a/\theta_N$. The definition of $\vdash_{\mathsf{G}}$ ensures that $\{1/\theta_N\}$ is a deductive filter of $\vdash_{\mathsf{G}}$ on the group $\boldsymbol{A}/\theta_N$. Therefore, from Proposition 1.10 it follows that N is a deductive filter of $\vdash_{\mathsf{G}}$ on $\boldsymbol{A}$. This establishes Condition (5.7).

Now, we will show that $\vdash_{\mathsf{G}}$ is algebraizable as witnessed by G and

$$\tau(x) := \{x \approx 1\} \;\text{ and }\; \Delta(x, y) := \{x \cdot y^{-1}\}.$$

First, the definition of $\vdash_{\mathsf{G}}$ guarantees the validity of Condition (Alg1). Furthermore, Condition (Alg4) amounts to

$$x \approx y \vDash_{\mathsf{G}} x \cdot y^{-1} \approx 1 \text{ and } x \cdot y^{-1} \approx 1 \vDash_{\mathsf{G}} x \approx y,$$

which is easily seen to be true. Therefore, from Proposition 5.11 it follows that τ, Δ, and G witness the algebraization of $\vdash_{\mathsf{G}}$.

As a consequence, we can apply the isomorphism theorem and Condition (5.7) obtaining that $\mathsf{N}(\boldsymbol{A}) \cong \mathsf{Con}_{\mathsf{G}}(\boldsymbol{A})$, for every group $\boldsymbol{A}$. Since the class of groups is closed under homomorphic images, we have $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{G}}(\boldsymbol{A})$, whence $\mathsf{N}(\boldsymbol{A}) \cong \mathsf{Con}(\boldsymbol{A})$, as desired.

The correspondence between ideals and congruences typical of ring theory can be derived from the isomorphism theorem in a similar manner. ⊠

CHAPTER 6

Farewell

The diagram in Figure 6.1 illustrates the dependence relations between the levels of the Leibniz hierarchy described so far. More precisely, given a class K of logics, we let

$$\mathsf{K}^{+} := \mathsf{K} \cup \text{the class of almost inconsistent logics;}$$
$$\mathsf{K}^{-} := \text{the class of logics with theorems in } \mathsf{K}.$$

Furthermore, we say that a logic is *regularly algebraizable* (resp. *regularly weakly algebraizable*) if it is assertional and algebraizable (resp. weakly algebraizable). Our aim is to prove the following:

Theorem 6.1. *The poset depicted in Figure 6.1 is obtained by ordering the classes of logics in the picture under the superset relation. Moreover, the joins in this poset coincide with intersections.*

Proof. We begin with the following observation:

Claim 6.2. *The joins in the poset depicted in Figure 6.1 are intersections.*

Proof of the Claim. Since this poset is finite, each of its elements is a join of join-irreducible ones. Consequently, it will be enough to prove that the join of any two join-irreducible elements is their intersection. To this end, observe that the join-irreducible elements of the poset are the classes of

(i) assertional logics;

(ii) logics with theorems;

(iii) logics $\vdash$ for which truth is implicitly definable in $\mathsf{Mod}^{*}(\vdash)$;

(iv) protoalgebraic logics;

(v) equivalential logics;

(vi) parametrically truth equational logics.

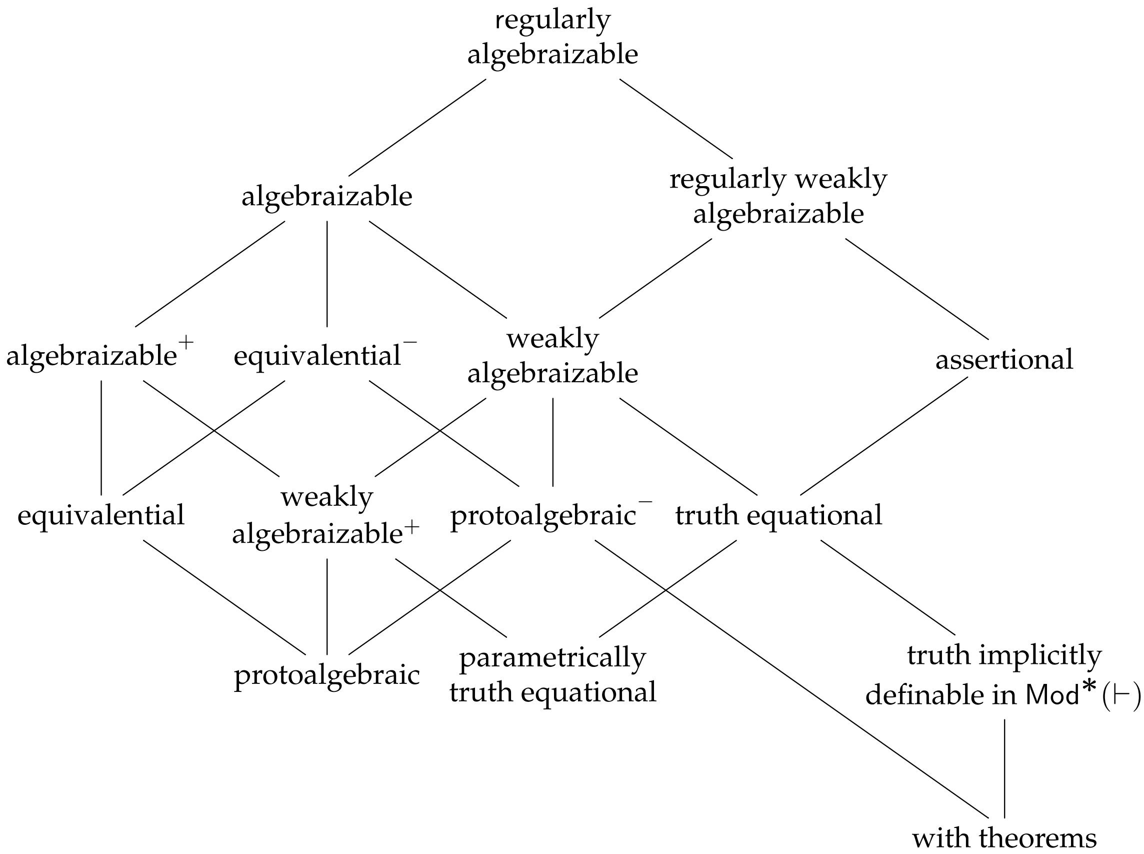


Figure 6.1: A portion of the Leibniz hierarchy.

We will first prove that, when restricted to the above classes, the order relation of Figure 6.1 is the superset relation. On the one hand, by definition, the class of equivalential logics is included in that of protoalgebraic logics. On the other hand, by Example 4.7, the class of assertional logics is included in that of truth equational logics. The latter is obviously included in the class of parametrically truth equational logics and, in view of Remark 4.17, in that of logics $\vdash$ for which truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$ as well. Lastly, Condition (4.5) and the fact that the matrix $\langle \mathbf{1}, \{1\} \rangle$, where $\mathbf{1}$ is the trivial algebra with unique element 1, is a reduced model of every logic imply that the class of logics $\vdash$ for which truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$ is included in that of logics with theorems.

Now, since the order relation between the join-irreducible elements of the poset is the superset relation, to conclude the proof, it suffices to show that the join $\mathsf{K} \lor \mathsf{W}$ of any pair K and W of *incomparable* join-irreducible elements is their intersection.

Accordingly, consider two classes K and W of logics in the above list that are incomparable according to the order of Figure 6.1. We begin by considering the case where one of K and W, say K, is the class of assertional logics. Since K and W are incomparable, the class W must be either that of protoalgebraic logics or that of equivalential logics. Suppose first that W is the class of protoalgebraic logics. In this case, $\mathsf{K} \lor \mathsf{W}$ is the class of regularly weakly algebraizable logics. By definition, this is the class of logics that are protoalgebraic, truth equational, and assertional. Since, in view of Example 4.7, every assertional logic is truth equational, we conclude that the class of regularly algebraizable logics is the intersection of the classes of protoalgebraic and assertional logics, that is, $\mathsf{K} \cap \mathsf{W}$. The case where W is the class of equivalential logics is handled similarly, using the fact that the class of algebraizable logics is the intersection of the classes of truth equational and equivalential logics (Corollary 5.14).

Then we may assume that neither K nor W is the class of assertional logics. We consider the case in which one of them, say K, is the class of logics with theorems. Then W is either the class of protoalgebraic logics or that of equivalential logics or that of parametrically truth equational logics. Since the first two cases are straightforward, we detail only the one in which W is the class of parametrically truth equational logics. Then $\mathsf{K} \vee \mathsf{W}$ is the class of truth equational logics which, by Corollary 4.12, is precisely the class of logics with theorems that are parametrically truth equational, that is, $\mathsf{K} \cap \mathsf{W}$.

Consequently, from now on we may assume that neither K nor W is the class of logics with theorems. We consider the case in which one of them, say K, is the class of logics $\vdash$ for which truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$. Then W is either the class of protoalgebraic logics or that of equivalential logics or that of parametrically truth equational logics. First suppose that W is the class of protoalgebraic logics. Then $\mathsf{K} \vee \mathsf{W}$ is the class of weakly algebraizable logics. Since Theorem 4.23 implies that the class of weakly algebraizable logics is $\mathsf{K} \cap \mathsf{W}$, we are done. The case where W is the class of equivalential logics is handled analogously. It only remains to consider the case where W is the class of parametrically truth equational logics. In this case $\mathsf{K} \vee \mathsf{W}$ is the class of truth equational logics. Since we already proved that this class is the intersection of those of parametrically truth equational logics and of logics with theorems, it suffices to prove that the class of logics $\vdash$ for which truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$ extends the class of truth equational logics and is contained into that of logics with theorems. But this holds, because the order relation between the join-irreducible elements of the poset is the superset relation.

Therefore, we may assume that neither K nor W is the class of logics $\vdash$ for which truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$. It follows that K and W are some of the classes of logics occurring in Conditions (iv), (v), or (vi). Since they are incomparable, we may assume, without loss of generality, that K is the class of parametrically truth equational logics and that W is either the class of protoalgebraic logics or that of equivalential logics. We detail only the case where W is the class of protoalgebraic logics, since the other is analogous. In this case, $\mathsf{K} \vee \mathsf{W}$ is the class of logics that are either weakly algebraizable or almost inconsistent. In view of Corollary 2.12 and the definition of an almost inconsistent logic, the almost inconsistent logics are precisely the protoalgebraic logics without theorems. Furthermore, the almost inconsistent logics are parametrically truth equational as witnessed by the set of equations $\tau(x, \vec{z}) := \emptyset$, because their reduced models are, up to isomorphism, only $\langle \mathbf{1}, \{1\} \rangle$ and $\langle \mathbf{1}, \emptyset \rangle$. Therefore, $\mathsf{K} \cap \mathsf{W}$ is the class of logics that are either almost inconsistent or protoalgebraic with theorems and parametrically truth equational. By Corollary 4.12, this is precisely the class of logics that are either almost inconsistent or weakly algebraizable. ⊠

We denote the order relation of the poset in Figure 6.1 by $\leqslant$. In view of the above Claim, it only remains to prove the following:

Claim 6.3. *For every pair* K *and* W *of classes of logics in Figure 6.1,*

$$\mathsf{K} \leqslant \mathsf{W} \iff \mathsf{W} \subseteq \mathsf{K}.$$

Proof of the Claim. To prove the left to right implication, suppose that $\mathsf{K} \leqslant \mathsf{W}$. Then let $\{k_1, \dots, k_n\}$ and $\{w_1, \dots, w_m\}$ be the sets of the join-irreducible elements below K and W, respectively. Clearly,

$$\mathsf{K} = k_1 \vee \dots \vee k_n \quad \text{and } \mathsf{W} = w_1 \vee \dots \vee w_m.$$

Together with Claim 6.2, this yields

$$\mathsf{K} = k_1 \cap \dots \cap k_n \quad \text{and } \mathsf{W} = w_1 \cap \dots \cap w_m.$$

Observe that from the assumption that $\mathsf{K} \leqslant \mathsf{W}$ it follows $\{k_1, \dots, k_n\} \subseteq \{w_1, \dots, w_m\}$. In view of the above display, this yields $\mathsf{W} \subseteq \mathsf{K}$.

To prove the implication from right to left, we reason by contraposition. It will be enough to establish that, for every join-irreducible K and every maximal element W in the subposet with universe $(\uparrow\mathsf{K})^c$, we have $\mathsf{W} \nsubseteq \mathsf{K}$. For suppose that this condition holds and consider a pair K and W of elements such that $\mathsf{K} \nleqslant \mathsf{W}$. Then there exists a join-irreducible $\mathsf{J} \leqslant \mathsf{K}$ such that $\mathsf{J} \nleqslant \mathsf{W}$. As $\mathsf{J} \nleqslant \mathsf{W}$, there exists a maximal element I in $(\uparrow\mathsf{J})^c$ such that $\mathsf{W} \leqslant \mathsf{I}$. Since J is join-irreducible and I maximal in $(\uparrow\mathsf{J})^c$, we can apply the assumption, obtaining $\mathsf{I} \nsubseteq \mathsf{J}$. Moreover, from $\mathsf{J} \leqslant \mathsf{K}$ and $\mathsf{W} \leqslant \mathsf{I}$ and the left to right implication in the statement it follows $\mathsf{K} \subseteq \mathsf{J}$ and $\mathsf{I} \subseteq \mathsf{W}$. Together with $\mathsf{I} \nsubseteq \mathsf{J}$, this yields $\mathsf{W} \nsubseteq \mathsf{K}$, as desired.

Then consider a join-irreducible element K and a maximal element $\mathsf{W} \in (\uparrow\mathsf{K})^c$. We need to prove that $\mathsf{W} \nsubseteq \mathsf{K}$. To this end, recall that, being join-irreducible, K is one of the classes of logics listed in Conditions (i)–(vi) in proof of Claim 6.2.

We begin with the case where K is the class of assertional logics. Then W must be the class of algebraizable logics. In view of Examples 4.2 and 5.12, the logic FL is algebraizable and fails to be assertional, whence $\mathsf{FL} \in \mathsf{W} \smallsetminus \mathsf{K}$ and $\mathsf{W} \nsubseteq \mathsf{K}$.

Then we consider the case where K is the class of logics with theorems. In this case, W is the class of logics that are either algebraizable or almost inconsistent. Since, by definition, almost inconsistent logics lack theorems, we obtain $\mathsf{W} \nsubseteq \mathsf{K}$.

Now, suppose that K is the class of logics $\vdash$ for which truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$. Then W is either the class of logics that are algebraizable or almost inconsistent or that of equivalential logics with theorems. Suppose first that W is the class of logics that are either algebraizable or almost inconsistent. Since $\langle \mathbf{1}, \{1\} \rangle$ and $\langle \mathbf{1}, \emptyset \rangle$ are reduced models of every almost inconsistent logic, truth is never implicitly definable in the class of the reduced models of an almost inconsistent logic. Consequently, $\mathsf{W} \nsubseteq \mathsf{K}$. Then we consider the case where W is the class of equivalential logics with theorems. Recall from Examples 2.6 and 4.18 that the logic K_ℓ is equivantial (and, obviously, it has theorems), but truth is not implicitly definable in $\mathsf{Mod}^*(\mathsf{K}_\ell)$. Consequently, $\mathsf{K}_\ell \in \mathsf{W} \smallsetminus \mathsf{K}$ and, therefore, $\mathsf{W} \nsubseteq \mathsf{K}$.

Then we consider the case where K is the class of protoalgebraic logics. In this case, W is the class of assertional logics. In view of Examples 2.17 and 4.7, we have $\mathsf{IPC}^- \in \mathsf{W} \smallsetminus \mathsf{K}$, whence $\mathsf{W} \nsubseteq \mathsf{K}$.

Consider then the case where K is the class of equivalential logics. In this case, W is the class of regularly weakly algebraizable logics. In view of Example 3.7, the logic MOL is assertional and protoalgebraic, but it fails to be equivalential. Since, by Example 4.7, assertional logics are truth equational, we conclude that $\mathsf{MOL} \in \mathsf{W} \smallsetminus \mathsf{K}$ and, therefore, that $\mathsf{W} \nsubseteq \mathsf{K}$.

Lastly, suppose that K is the class of parametrically truth equational logics. Then W is either the class of equivalential logics with theorems or that of logics $\vdash$ for which truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$. In view of Examples 2.6 and 4.18, the logic K_ℓ is equivalential (and, clearly, it has theorems), but it fails to be parametrically truth equational. Therefore, it only remains to exhibit a logic $\vdash$ for which truth is implicitly definable in $\mathsf{Mod}^*(\vdash)$ that is not parametrically truth equational. The only example we are aware of is an ad hoc one. Because of its artificial nature, we decided to omit it, but the interested reader can find it in [79, Example 2]. $\boxtimes$

The statement is an immediate consequence of Claims 6.2 and 6.3. $\boxtimes$

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