THE POSET OF ALL LOGICS III:
FINITELY PRESENTABLE LOGICS

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Abstract. A logic in a finite language is said to be finitely presentable if it is axiomatized by finitely many finite rules. It is proved that binary non-indexed products of logics that are both finitely presentable and finitely equivalential are essentially finitely presentable. This result does not extend to binary non-indexed products of arbitrary finitely presentable logics, as shown by a counterexample. Finitely presentable logics are then exploited to introduce finitely presentable Leibniz classes, and to draw a parallel between the Leibniz and the Maltsev hierarchies.

1. Introduction

The *interpretability* relation introduced in [15] is a preorder on the class of all (propositional) logics. Its associated partially ordered class Log, consisting of equivalence classes of equi-interpretable logics, was investigated in [17, 15, 16] under the name of the *poset of all logics*. We shall exploit this formalism to draw a precise relation between the Leibniz and Maltsev hierarchies, respectively of abstract algebraic logic and universal algebra.

More in detail, the *Maltsev hierarchy* is a taxonomy of varieties of algebras in terms of syntactic principles describing the structure of congruence lattices [14, 18, 25, 29, 31]. The *Leibniz hierarchy* [16] plays a similar role in algebraic logic, providing a classification of logics in terms of rule schemata that govern the interplay between lattices of logical theories and congruences lattices [2, 3, 8, 9, 19, 26].

Even though the apparent analogy between the Maltsev and Leibniz hierarchies was fairly well known [27] and inspired some investigations in algebraic logic [5, 6, 20, 21], the problem of understanding whether these hierarchies are two faces of the same coin remained essentially open. This should probably be attributed to the fact that, until recently, a framework in which a positive solution could be formulated was missing. In this paper we show that such a framework is provided by the study of the poset of all logics Log.

To this end, it is convenient to introduce some new concept. A logic in a finite language is said to be *finitely presentable* if it is axiomatized by finitely many finite rules. Moreover, *finitely presentable Leibniz classes* are the classes logics that can be faithfully identified with filters of Log generated by (equivalence classes of) logics that are both finitely presentable and finitely equivalential. Equivalently, they can be characterized in terms of closure properties as the classes of finitely equivalential logics closed under the formation of term-equivalent logics, compatible expansions, and binary non-indexed products that, moreover, satisfy the following requirement: each of their members compatibly extends some of their finitely presentable ones (see [15] for the relevant definitions). Finally, finitely presentable Leibniz classes can be viewed as classes of logics

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globally satisfying special rule schemata, here called *finitely presentable Leibniz conditions*. The equivalence between these definitions (Theorem 4.2) rests on the observation that the fact of "being both finitely presentable and finitely equivalential" is essentially preserved by the formation of binary non-indexed products (Theorem 2.4).

Finally, the poset of all finitely presentable Leibniz classes is called the *finite companion of the Leibniz hierarchy*. As a matter of fact, this concept can be exported straightforwardly from the setting of logics to that of two-deductive systems, i.e. substitution invariant consequence relations on the set of pairs of formulas [4]. Typical examples of two-deductive systems are equational consequences relative to varieties (once equations are identified with pairs of formulas in the natural way [15, Ex. 8.1]). Bearing this in mind, the relation between the Maltsev and Leibniz hierarchies can be phrased as the slogan "the Maltsev hierarchy is the restriction of the finite companion of the Leibniz hierarchy of two-deductive systems to equational consequences relative to varieties" (Theorem 5.1).

2. Finitely presentable logics

We use the same notation as in [15, 16].

Definition 2.1. A logic $\vdash$ is *finitely presentable* if $\mathscr{L}_{\vdash}$ is finite and $\vdash$ is axiomatized by a finite set of finite rules and is formulated in countably many variables.

Recall that a logic $\vdash$ is said to be *finitely equivalential* [2, 8, 10, 13] if it has a finite set of congruence formulas $\Delta(x,y)$, i.e. if there is a *finite* non-empty set $\Delta(x,y)$ of formulas such that for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $a, b \in A$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \Longleftrightarrow \Delta^{\boldsymbol{A}}(a,b) \subseteq F.$$

Our aim is to prove that binary non-indexed products of logics that are both finitely presentable and finitely equivalential are essentially finitely presentable and finitely equivalential (Theorem 2.4).

To this end, given two logics $\vdash_1$ and $\vdash_2$, and two basic operations $f \in \mathscr{L}_{\vdash_1}$ and $g \in \mathscr{L}_{\vdash_2}$, respectively n-ary and m-ary, we consider the basic operations of $\vdash_1 \otimes \vdash_2$

$$\begin{aligned} f_+(x_1,\dots,x_n) &\coloneqq \langle f(x_1,\dots,x_n), x_1\rangle \\ g_+(x_1,\dots,x_m) &\coloneqq \langle x_1, g(x_1,\dots,x_m)\rangle \\ x \cdot y &\coloneqq \langle \pi^2_1(x,y), \pi^2_2(x,y)\rangle, \end{aligned}$$

where π^2_i is the binary projection map on the i-th coordinate. Observe that for every $\mathscr{L}_{\vdash_1}$-algebra $\boldsymbol{A}$, $\mathscr{L}_{\vdash_2}$-algebra $\boldsymbol{B}$, and $\langle a_1, b_1\rangle, \dots, \langle a_{n+m}, b_{n+m}\rangle \in A \times B$,

$$\begin{aligned} f_+^{\boldsymbol{A}\otimes\boldsymbol{B}}(\langle a_1,b_1\rangle,\dots,\langle a_n,b_n\rangle) &= \langle f^{\boldsymbol{A}}(a_1,\dots,a_n), b_1\rangle \\ g_+^{\boldsymbol{A}\otimes\boldsymbol{B}}(\langle a_1,b_1\rangle,\dots,\langle a_m,b_m\rangle) &= \langle a_1, g^{\boldsymbol{B}}(b_1,\dots,b_m)\rangle \\ \langle a_1,b_1\rangle \cdot^{\boldsymbol{A}\otimes\boldsymbol{B}} \langle a_2,b_2\rangle &= \langle a_1,b_2\rangle. \end{aligned}$$

The importance of the operations f_+, g_+, and $\cdot$ is clarified by the following result:

Lemma 2.2. *Let $\vdash_1$ and $\vdash_2$ be two logics. Then $\vdash_1 \otimes \vdash_2$ is term-equivalent to its fragment $\vdash_+$ in the language $\mathscr{L}_+ = \{f_+ : f \in \mathscr{L}_{\vdash_1}\} \cup \{g_+ : g \in \mathscr{L}_{\vdash_2}\} \cup \{\cdot\}$. Moreover,* $\mathsf{Mod}^{\equiv}(\vdash_+)$ *is the class of $\mathscr{L}_+$-reducts of the matrices in* $\mathsf{Mod}^{\equiv}(\vdash_1 \otimes \vdash_2)$.

Proof. Let K be the class of $\mathscr{L}_+$-reducts of the matrices in $\mathsf{Mod}^{\equiv}(\vdash_1 \otimes \vdash_2)$. Clearly, $\vdash_+$ is the logic in the language $\mathscr{L}_+$ formulated in $\lambda := \kappa_{\vdash_1 \otimes \vdash_2}$ variables and induced by K.

We claim that $\mathsf{Mod}^{\equiv}(\vdash_+) = \mathsf{K}$. To prove this, recall from the definition of $\vdash_1 \otimes \vdash_2$ that $\lambda \geqslant \max\{|Fm(\vdash_+)|, |Fm(\vdash_1 \otimes \vdash_2)|\}$. By [15, Cor. 2.5] this implies

$$\mathsf{Mod}^{\equiv}(\vdash_+) = \mathbb{P}_{\text{SD}}\mathbb{R}\mathbb{S}\mathbb{P}_{\text{R}_{\lambda^+}}(\mathsf{K})$$
$$\mathsf{Mod}^{\equiv}(\vdash_1 \bigotimes \vdash_2) = \mathbb{P}_{\text{SD}}\mathbb{R}\mathbb{S}\mathbb{P}_{\text{R}_{\lambda^+}}(\mathsf{Mod}^{\equiv}(\vdash_1 \bigotimes \vdash_2)).$$

It is easy to see that K and $\mathsf{Mod}^{\equiv}(\vdash_1 \otimes \vdash_2)$ are term-equivalent classes. Together with the fact that K is the class of $\mathscr{L}_+$-reducts of $\mathsf{Mod}^{\equiv}(\vdash_1 \otimes \vdash_2)$, this guarantees that $\mathbb{P}_{\text{SD}}\mathbb{R}\mathbb{S}\mathbb{P}_{\text{R}_{\lambda^+}}(\mathsf{K})$ is the class of $\mathscr{L}_+$-reducts of $\mathbb{P}_{\text{SD}}\mathbb{R}\mathbb{S}\mathbb{P}_{\text{R}_{\lambda^+}}(\mathsf{Mod}^{\equiv}(\vdash_1 \otimes \vdash_2))$. In virtue of the above display this implies $\mathsf{Mod}^{\equiv}(\vdash_+) = \mathsf{K}$, establishing the claim.

Finally, the fact that K and $\mathsf{Mod}^{\equiv}(\vdash_1 \otimes \vdash_2)$ are term-equivalent classes and the claim imply that the logics $\vdash_+$ and $\vdash_1 \otimes \vdash_2$ are term-equivalent. ⊠

Moreover, we rely on the following observation:

Lemma 2.3. *Let $\vdash$ and $\vdash'$ be two logics.*

(i) *If $\vdash$ is finitely equivalential and $\vdash \leqslant \vdash'$, then $\vdash'$ is also finitely equivalential.*
(ii) *If $\vdash$ and $\vdash'$ are finitely equivalential, then $\vdash \otimes \vdash'$ is also finitely equivalential.*

Proof. (i): Let $\boldsymbol{\tau}$ be an interpretation of $\vdash$ into $\vdash'$, and $\Delta(x, y)$ a finite set of congruence formulas for $\vdash$. The proof of [15, Prop. 6.1(i)] shows that $\boldsymbol{\tau}[\Delta]$ is a set of congruence formulas for $\vdash'$. Since $\boldsymbol{\tau}[\Delta]$ is finite, we conclude that $\vdash'$ is finitely equivalential.

(ii): Let $\Delta(x, y)$ and $\Delta'(x, y)$ be finite sets of congruence formulas for $\vdash$ and $\vdash'$, respectively. The proof of [15, Prop. 6.1(ii)] shows that $\Delta \times \Delta'$ is a set of congruence formulas for $\vdash \otimes \vdash'$. As $\Delta \times \Delta'$ is finite, $\vdash \otimes \vdash'$ is finitely equivalential. ⊠

We are now ready to prove the main result of this part.

Theorem 2.4. *Binary non-indexed products of logics that are finitely presentable and finitely equivalential are term-equivalent to logics that are finitely presentable and finitely equivalential.*

Proof. Let $\vdash_1$ and $\vdash_2$ two finitely presentable and finitely equivalential logics. Let also $\vdash_+$ be the $\mathscr{L}_+$-fragment of $\vdash_1 \otimes \vdash_2$ identified in Lemma 2.2. We know that $\vdash_+$ and $\vdash_1 \otimes \vdash_2$ are term-equivalent. Therefore, to conclude the proof, it will be enough to show that $\vdash_+$ is term-equivalent to a logic that is finitely presentable and finitely equivalential.

To this end, recall that $\vdash_1$ is axiomatized by finitely many finite rules

$$\begin{array}{ccccc} \gamma_1^1 & \dots & \gamma_1^n & \rhd & \varphi_1 \\ \vdots & & \vdots & & \vdots \\ \gamma_m^1 & \dots & \gamma_m^n & \rhd & \varphi_m. \end{array}$$

Similarly, $\vdash_2$ is axiomatized by finitely many finite rules

$$\begin{array}{ccccc} \delta_1^1 & \dots & \delta_1^n & \rhd & \psi_1 \\ \vdots & & \vdots & & \vdots \\ \delta_m^1 & \dots & \delta_m^n & \rhd & \psi_m. \end{array}$$

Let us remark that the above notation contains a small abuse of notation, since some of the rules may have an empty antecedent.[1]

For every formula γ of $\vdash_1$ we define recursively a formula γ_+ of $\vdash_+$ as follows:

$$x_+ := x, \text{ for every variable } x$$
$$f(\psi_1, \ldots, \psi_n)_+ := f_+(\psi_{1+}, \ldots, \psi_{n+}), \text{ for every } f \in \mathscr{L}_{\vdash_1}.$$

In the same way, every formula δ of $\vdash_2$ is associated with a formula δ_+ of $\vdash_+$.

Observe that $\vdash_+$ is term-equivalent to a binary non-indexed product of two finitely equivalential logics, whence $\vdash_+$ is finitely equivalential by Lemma 2.3. Then let $\Delta(x, y)$ be a finite set of equivalence formulas for $\vdash_+$. Moreover, consider a variable x that does not appear in the rules axiomatizing $\vdash_1$ and $\vdash_2$. We consider the logic $\vdash$ in the language $\mathscr{L}_+$ formulated in countably many variables and axiomatized by the following rules:

$$\emptyset \rhd \Delta(x, x) \tag{1}$$
$$x, \Delta(x, y) \rhd y \tag{2}$$
$$\Delta(x_1, y_1), \ldots, \Delta(x_k, y_k) \rhd \Delta(*(x_1, \ldots, x_k), *(y_1, \ldots, y_k)) \tag{3}$$
$$\emptyset \rhd \Delta(x \cdot x, x) \tag{4}$$
$$\emptyset \rhd \Delta((x \cdot y) \cdot (u \cdot v), x \cdot v) \tag{5}$$
$$\emptyset \rhd \Delta(f_+(x_1 \cdot y_1, \ldots, x_k \cdot y_k), f_+(x_1, \ldots, x_k) \cdot f_+(y_1, \ldots, y_k)) \tag{6}$$
$$\emptyset \rhd \Delta(g_+(x_1 \cdot y_1, \ldots, x_k \cdot y_k), g_+(x_1, \ldots, x_k) \cdot g_+(y_1, \ldots, y_k)) \tag{7}$$
$$\emptyset \rhd \Delta(y \cdot f_+(x_1, \ldots, x_k), y \cdot x_1) \tag{8}$$
$$\emptyset \rhd \Delta(g_+(x_1, \ldots, x_k) \cdot y, x_1 \cdot y) \tag{9}$$
$$x, y \rhd x \cdot y \tag{10}$$
$$\gamma^1_{i+} \cdot x, \ldots, \gamma^n_{i+} \cdot x \rhd \varphi_{i+} \cdot x \tag{11}$$
$$x \cdot \delta^1_{i+}, \ldots, x \cdot \delta^n_{i+} \rhd x \cdot \psi_{i+} \tag{12}$$

for every $i \leqslant m$, every $f \in \mathscr{L}_{\vdash_1}$ and $g \in \mathscr{L}_{\vdash_2}$, and every $* \in \mathscr{L}_+$. It is clear that $\vdash$ is finitely presentable. Moreover, with an application of [15, Thm. 2.7] to the rules (1, 2, 3), we obtain that $\vdash$ is finitely equivalential with set of congruence formulas $\Delta(x, y)$.

Therefore, to conclude the proof, it will be enough to show that $\vdash_+$ and $\vdash$ are term-equivalent. More precisely, we shall see that $\mathsf{Mod}^{\equiv}(\vdash_+) = \mathsf{Mod}^{\equiv}(\vdash)$.

First observe that the rules axiomatizing $\vdash$ are valid in $\vdash_+$, whence $\vdash_+$ extends $\vdash$, and $\mathsf{Mod}^{\equiv}(\vdash_+) \subseteq \mathsf{Mod}^{\equiv}(\vdash)$. To prove the other inclusion, consider $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{\equiv}(\vdash)$. Observe that for every $a, b \in A$,

$$a = b \Longleftrightarrow \Delta^{\boldsymbol{A}}(a, b) \subseteq F. \tag{13}$$

The above display follows from the fact that Δ is a set of congruence formulas for $\vdash$ and $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{\equiv}(\vdash) = \mathbb{R}(\mathsf{Mod}(\vdash))$.

We claim that for every $i = 1, 2$ there is an $\mathscr{L}_{\vdash_i}$-algebra $\boldsymbol{A}_i$ such that $\boldsymbol{A}$ is isomorphic to the $\mathscr{L}_+$-reduct of $\boldsymbol{A}_1 \otimes \boldsymbol{A}_2$. To prove this, choose an element $a \in A$ and let

$$A_1 := \{b \cdot^{\boldsymbol{A}} a : b \in A\}.$$

[1] This innocuous abuse will occur repeatedly in the paper without further notice.

Then let $\boldsymbol{A}_1$ be the $\mathscr{L}_{\vdash_1}$-algebra with universe A_1 and with basic n-ary operations f interpreted as

$$f^{\boldsymbol{A}_1}(b_1 \cdot^{\boldsymbol{A}} a, \dots, b_n \cdot^{\boldsymbol{A}} a) := f_+^{\boldsymbol{A}}(b_1 \cdot^{\boldsymbol{A}} a, \dots, b_n \cdot^{\boldsymbol{A}} a)$$

for every $b_1 \cdot^{\boldsymbol{A}} a, \dots, b_n \cdot^{\boldsymbol{A}} a \in A_1$. To prove that $\boldsymbol{A}_1$ is well defined, we need to show that the range of the operation $f^{\boldsymbol{A}_1}$ is included in $\boldsymbol{A}_1$. Since $\langle \boldsymbol{A}, F\rangle$ is a model of the rules (6, 8), we obtain

$$\begin{aligned}&\Delta^{\boldsymbol{A}}(f^{\boldsymbol{A}_1}(b_1 \cdot^{\boldsymbol{A}} a, \dots, b_n \cdot^{\boldsymbol{A}} a), f_+^{\boldsymbol{A}}(b_1, \dots, b_n) \cdot^{\boldsymbol{A}} f_+^{\boldsymbol{A}}(a, \dots, a))\\ =\,&\Delta^{\boldsymbol{A}}(f_+^{\boldsymbol{A}}(b_1 \cdot^{\boldsymbol{A}} a, \dots, b_n \cdot^{\boldsymbol{A}} a), f_+^{\boldsymbol{A}}(b_1, \dots, b_n) \cdot^{\boldsymbol{A}} f_+^{\boldsymbol{A}}(a, \dots, a)) \subseteq F\end{aligned}$$

and

$$\Delta^{\boldsymbol{A}}(f_+^{\boldsymbol{A}}(b_1, \dots, b_n) \cdot^{\boldsymbol{A}} f_+^{\boldsymbol{A}}(a, \dots, a), f_+^{\boldsymbol{A}}(b_1, \dots, b_n) \cdot^{\boldsymbol{A}} a)) \subseteq F.$$

Together with (13), the above displays imply

$$f^{\boldsymbol{A}_1}(b_1 \cdot^{\boldsymbol{A}} a, \dots, b_n \cdot^{\boldsymbol{A}} a) = f_+^{\boldsymbol{A}}(b_1, \dots, b_n) \cdot^{\boldsymbol{A}} f_+^{\boldsymbol{A}}(a, \dots, a) = f_+^{\boldsymbol{A}}(b_1, \dots, b_n) \cdot^{\boldsymbol{A}} a \in A_1.$$

As a consequence, $\boldsymbol{A}_1$ is well defined.

Similarly, let $\boldsymbol{A}_2$ be the $\mathscr{L}_{\vdash_2}$-algebra with universe $A_2 := \{a \cdot^{\boldsymbol{A}} b \colon b \in A\}$ and with basic n-ary operations g interpreted as

$$g^{\boldsymbol{A}_2}(a \cdot^{\boldsymbol{A}} b_1, \dots, a \cdot^{\boldsymbol{A}} b_n) := g_+^{\boldsymbol{A}}(a \cdot^{\boldsymbol{A}} b_1, \dots, a \cdot^{\boldsymbol{A}} b_n)$$

for every $a \cdot^{\boldsymbol{A}} b_1, \dots, a \cdot^{\boldsymbol{A}} b_n \in A_2$. An argument similar to the one employed for $\boldsymbol{A}_1$ shows that $\boldsymbol{A}_2$ is well defined.

Consider the map $\kappa \colon A \to A_1 \times A_2$, defined for every $b \in A$ as

$$\kappa(b) := \langle b \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} b\rangle.$$

We shall see that κ is an isomorphism between $\boldsymbol{A}$ and the $\mathscr{L}_+$-reduct of $\boldsymbol{A}_1 \otimes \boldsymbol{A}_2$. To prove that κ is injective, consider $b, c \in A$ such that $\kappa(b) = \kappa(c)$, i.e.

$$b \cdot^{\boldsymbol{A}} a = c \cdot^{\boldsymbol{A}} a \text{ and } a \cdot^{\boldsymbol{A}} b = a \cdot^{\boldsymbol{A}} c.$$

By the above display and the fact that $\langle \boldsymbol{A}, F\rangle$ is a model of the rule (5),

$$\begin{aligned}\Delta^{\boldsymbol{A}}((b \cdot^{\boldsymbol{A}} a) \cdot (a \cdot^{\boldsymbol{A}} b), c \cdot^{\boldsymbol{A}} c) = \Delta^{\boldsymbol{A}}((c \cdot^{\boldsymbol{A}} a) \cdot (a \cdot^{\boldsymbol{A}} c), c \cdot^{\boldsymbol{A}} c) \subseteq F\\ \Delta^{\boldsymbol{A}}((b \cdot^{\boldsymbol{A}} a) \cdot (a \cdot^{\boldsymbol{A}} b), b \cdot^{\boldsymbol{A}} b) \subseteq F.\end{aligned}$$

By (13) we obtain

$$b \cdot^{\boldsymbol{A}} b = (b \cdot^{\boldsymbol{A}} a) \cdot (a \cdot^{\boldsymbol{A}} b) = c \cdot^{\boldsymbol{A}} c. \tag{14}$$

Now, since $\langle \boldsymbol{A}, F\rangle$ is a model of the rule (4), we have $\Delta^{\boldsymbol{A}}(b \cdot^{\boldsymbol{A}} b, b) \cup \Delta^{\boldsymbol{A}}(c \cdot^{\boldsymbol{A}} c, c) \subseteq F$. Consequently, by (13), $b = b \cdot^{\boldsymbol{A}} b$ and $c = c \cdot^{\boldsymbol{A}} c$. Together with the above display, this yields $b = c$. Hence κ is injective.

To prove that κ is surjective, consider a generic element $\langle b \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} c\rangle$ of $A_1 \times A_2$. As $\langle \boldsymbol{A}, F\rangle$ is a model of the rules (4, 5), we have $\Delta^{\boldsymbol{A}}(a \cdot^{\boldsymbol{A}} a, a) \cup \Delta^{\boldsymbol{A}}((b \cdot^{\boldsymbol{A}} c) \cdot^{\boldsymbol{A}} (a \cdot^{\boldsymbol{A}} a), b \cdot^{\boldsymbol{A}} a) \subseteq F$. Thus by (13),

$$(b \cdot^{\boldsymbol{A}} c) \cdot^{\boldsymbol{A}} a = (b \cdot^{\boldsymbol{A}} c) \cdot^{\boldsymbol{A}} (a \cdot^{\boldsymbol{A}} a) = b \cdot^{\boldsymbol{A}} a. \tag{15}$$

Similarly, one shows

$$a \cdot^{\boldsymbol{A}} (b \cdot^{\boldsymbol{A}} c) = (a \cdot^{\boldsymbol{A}} a) \cdot^{\boldsymbol{A}} (b \cdot^{\boldsymbol{A}} c) = a \cdot^{\boldsymbol{A}} c. \tag{16}$$

From (15, 16) it follows

$$\kappa(b \cdot^{\boldsymbol{A}} c) = \langle (b \cdot^{\boldsymbol{A}} c) \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} (b \cdot^{\boldsymbol{A}} c)\rangle = \langle b \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} c\rangle,$$

whence κ is surjective.

To prove that $\kappa \colon \boldsymbol{A} \to \boldsymbol{A}_1 \otimes \boldsymbol{A}_2$ preserves the operations of the language $\mathscr{L}_+$, consider a basic n-ary operation $f \in \mathscr{L}_{\vdash_1}$. For every $b_1, \dots, b_n \in A$,

$$\begin{aligned}
\kappa(f_+^{\boldsymbol{A}}(b_1, \dots, b_n)) &= \langle f_+^{\boldsymbol{A}}(b_1, \dots, b_n) \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} f_+^{\boldsymbol{A}}(b_1, \dots, b_n)\rangle \\
&= \langle f_+^{\boldsymbol{A}}(b_1, \dots, b_n) \cdot^{\boldsymbol{A}} f_+^{\boldsymbol{A}}(a, \dots, a), a \cdot^{\boldsymbol{A}} b_1\rangle \\
&= \langle f_+^{\boldsymbol{A}}(b_1 \cdot a, \dots, b_n \cdot a), a \cdot^{\boldsymbol{A}} b_1\rangle \\
&= \langle f^{\boldsymbol{A}_1}(b_1 \cdot^{\boldsymbol{A}} a, \dots, b_n \cdot^{\boldsymbol{A}} a), a \cdot^{\boldsymbol{A}} b_1\rangle \\
&= f_+^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2}(\langle b_1 \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} b_1\rangle, \dots, \langle b_n \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} b_n\rangle) \\
&= f_+^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2}(\kappa(b_1), \dots, \kappa(b_n)).
\end{aligned}$$

The non-obvious equalities above are justified as follows. The second equality is obtained by an application of the rule (8) and of (13) to both coordinates, while the third equality follows from an application of the rule (6) and of (13) to the first coordinate.

Similarly, one shows that κ preserves g_+, for every $g \in \mathscr{L}_{\vdash_2}$. To prove that κ preserves the operation $\cdot$, consider $b, c \in A$. We have

$$\begin{aligned}
\kappa(b \cdot^{\boldsymbol{A}} c) &= \langle (b \cdot^{\boldsymbol{A}} c) \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} (b \cdot^{\boldsymbol{A}} c)\rangle \\
&= \langle b \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} c\rangle \\
&= \langle b \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} b\rangle \cdot^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2} \langle c \cdot^{\boldsymbol{A}} a, a \cdot^{\boldsymbol{A}} c\rangle \\
&= \kappa(b) \cdot^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2} \kappa(c).
\end{aligned}$$

The only non-obvious equality above is the second, which follows from (15, 16). Hence κ preserves also $\cdot$. We conclude that it is an isomorphism between $\boldsymbol{A}$ and the $\mathscr{L}_+$-reduct of $\boldsymbol{A}_1 \otimes \boldsymbol{A}_2$, establishing the claim.

Now, let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be the algebras given by the claim. We can assume without loss of generality that $\boldsymbol{A}$ is the $\mathscr{L}_+$-reduct of $\boldsymbol{A}_1 \otimes \boldsymbol{A}_2$. In particular, under this identification, $F \subseteq A_1 \times A_2$. Then we define $F_1 := \pi_1[F] \subseteq A_1$ and $F_2 := \pi_2[F] \subseteq A_2$, where π_1 and π_2 are the natural projection maps. Clearly, $F \subseteq F_1 \times F_2$. To prove the other inclusion, consider $\langle b_1, b_2\rangle \in F_1 \times F_2$. There are $c_1 \in A_1$ and $c_2 \in A_2$ such that $\langle b_1, c_2\rangle, \langle c_1, b_2\rangle \in F$. Since $\langle \boldsymbol{A}, F\rangle$ is a model of the rule (10),

$$\langle b_1, b_2\rangle = \langle b_1, c_2\rangle \cdot^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2} \langle c_1, b_2\rangle = \langle b_1, c_2\rangle \cdot^{\boldsymbol{A}} \langle c_1, b_2\rangle \in F.$$

Thus $F = F_1 \times F_2$ and, therefore,

$$\langle \boldsymbol{A}, F\rangle \text{ is the } \mathscr{L}_+\text{-reduct of } \langle \boldsymbol{A}_1 \bigotimes \boldsymbol{A}_2, F_1 \times F_2\rangle. \tag{17}$$

We shall see that $\langle \boldsymbol{A}_1, F_1\rangle \in \mathsf{Mod}^{\equiv}(\vdash_1)$. To prove that $\langle \boldsymbol{A}_1, F_1\rangle$ is a model of $\vdash_1$, it suffices to show that it is a model of the rule $\gamma_i^1 \dots \gamma_i^n \rhd \varphi_i$, for every $i \leqslant m$. To this end, fix $i \leqslant m$ and let $z_1, \dots, z_k$ be the variables appearing in the rule $\gamma_i^1 \dots \gamma_i^n \rhd \varphi_i$. Consider also a tuple $\vec{c} = c_1, \dots, c_k \in A_1$ such that

$$\gamma_i^{1\boldsymbol{A}_1}(\vec{c}), \dots, \gamma_i^{n\boldsymbol{A}_1}(\vec{c}) \in F_1. \tag{18}$$

Recall that $\vdash$ is equivalential, whence it has theorems. Together with the fact that $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{\equiv}(\vdash)$ and $F = F_1 \times F_2$, this implies $F_1 \times F_2 \neq \emptyset$. Then we can choose an element $b \in F_2$. Recall also that the variables occurring in the rule (11) are among $\{x, z_1, \dots, z_k\}$ and that x does not appear in $z_1, \dots, z_k$. Then we consider the following assignment of the variables $\{z_1, \dots, z_k, x\}$ in $\boldsymbol{A}$:

$$z_j \longmapsto \langle c_j, b\rangle, \text{ for all } j \leqslant k \text{ and } x \longmapsto \langle c_1, b\rangle.$$

For every $t \leqslant n$,

$$\begin{aligned}\gamma_{i+}^{t\boldsymbol{A}}(\langle c_1, b\rangle, \dots, \langle c_k, b\rangle) \cdot^{\boldsymbol{A}} \langle c_1, b\rangle &= \gamma_{i+}^{t\boldsymbol{A}_1 \otimes \boldsymbol{A}_2}(\langle c_1, b\rangle, \dots, \langle c_k, b\rangle) \cdot^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2} \langle c_1, b\rangle \\ &= \langle \gamma_i^{t\boldsymbol{A}_1}(c_1, \dots, c_k), b\rangle \\ &\in F_1 \times F_2 = F.\end{aligned}$$

In the above display, the fact that $\langle \gamma_i^{t\boldsymbol{A}_1}(c_1, \dots, c_k), b\rangle \in F_1 \times F_2$ follows from (18) and $b \in F_2$.

Now, from the above display and the fact that $\langle \boldsymbol{A}, F\rangle$ is a model of (11) it follows

$$\begin{aligned}\langle \varphi_i^{\boldsymbol{A}_1}(c_1, \dots, c_k), b\rangle &= \varphi_{i+}^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2}(\langle c_1, b\rangle, \dots, \langle c_k, b\rangle) \cdot^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2} \langle c_1, b\rangle \\ &= \varphi_{i+}^{\boldsymbol{A}}(\langle c_1, b\rangle, \dots, \langle c_k, b\rangle) \cdot^{\boldsymbol{A}} \langle c_1, b\rangle \\ &\in F = F_1 \times F_2.\end{aligned}$$

As a consequence, we obtain $\varphi_i^{\boldsymbol{A}_1}(c_1, \dots, c_k) \in F_1$. This concludes the proof that $\langle \boldsymbol{A}_1, F_1\rangle$ is a model of $\gamma_i^1 \dots \gamma_i^n \rhd \varphi_i$ and, therefore, of $\vdash_1$.

Then we turn to prove that $\langle \boldsymbol{A}_1, F_1\rangle \in \mathbb{R}(\mathsf{Mod}(\vdash_1))$. Since $\langle \boldsymbol{A}_1, F_1\rangle$ is a model of $\vdash_1$, it will be enough to show that it is reduced. To this end, consider two distinct $b, c \in A_1$. Then choose an element $d \in A_2$. By (13) there is $\varphi(x, y) \in \Delta(x, y)$ such that

$$\varphi^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2}(\langle b, d\rangle, \langle c, d\rangle) = \varphi^{\boldsymbol{A}}(\langle b, d\rangle, \langle c, d\rangle) \notin F = F_1 \times F_2$$
$$\varphi^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2}(\langle b, d\rangle, \langle b, d\rangle) = \varphi^{\boldsymbol{A}}(\langle b, d\rangle, \langle b, d\rangle) \in F = F_1 \times F_2.$$

Now, the operation $\varphi^{\boldsymbol{A}_1 \otimes \boldsymbol{A}_2}$ can be viewed as a pair $\langle \varphi_1, \varphi_2\rangle$ of binary operations, respectively of $\vdash_1$ and $\vdash_2$. Bearing this in mind, the above display yields

$$\langle \varphi_1^{\boldsymbol{A}_1}(b, c), \varphi_2^{\boldsymbol{A}_2}(d, d)\rangle \notin F_1 \times F_2 \text{ and } \langle \varphi_1^{\boldsymbol{A}_1}(b, b), \varphi_2^{\boldsymbol{A}_2}(d, d)\rangle \in F_1 \times F_2.$$

As a consequence,

$$\varphi_1^{\boldsymbol{A}_1}(b, c) \notin F_1 \text{ and } \varphi_1^{\boldsymbol{A}_1}(b, b) \in F_1.$$

By [15, Prop. 2.2(i)] we conclude that $\langle b, c\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{A}_1} F_1$. Hence the matrix $\langle \boldsymbol{A}_1, F_1\rangle$ is reduced, whence $\langle \boldsymbol{A}_1, F_1\rangle \in \mathbb{R}(\mathsf{Mod}(\vdash_1)) \subseteq \langle \mathsf{Mod}^{\equiv}(\vdash_1)$.

Similarly, one proves that $\langle \boldsymbol{A}_2, F_2\rangle \in \mathsf{Mod}^{\equiv}(\vdash_2)$. By [15, Cor. 4.14], the fact that $\langle \boldsymbol{A}_i, F_i\rangle \in \mathsf{Mod}^{\equiv}(\vdash_i)$ for every $i = 1, 2$ implies $\langle \boldsymbol{A}_1 \otimes \boldsymbol{A}_2, F_1 \times F_2\rangle \in \mathsf{Mod}^{\equiv}(\vdash_1 \otimes \vdash_2)$. But, together with (17), this yields that $\langle \boldsymbol{A}, F\rangle$ is the $\mathscr{L}_+$-reduct of a matrix in $\mathsf{Mod}^{\equiv}(\vdash_1 \otimes \vdash_2)$. By Lemma 2.2 this guarantees that $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{\equiv}(\vdash_+)$, concluding the proof that $\mathsf{Mod}^{\equiv}(\vdash_+) = \mathsf{Mod}^{\equiv}(\vdash)$. ⊠

Theorem 2.4 was first discovered by Neumann in the setting of varieties of algebras [23, Appendix 4] (see also [22]). More precisely, a variety K is said to be *finitely presentable* if its language is finite and K is axiomatized by finitely many equations.

Corollary 2.5 (Neumann)**.** *Binary non-indexed products of finitely presentable varieties are finitely presentable.*

Proof sketch. Observe that a variety K is finitely presentable exactly when its relative equational consequence $\vDash_{\mathsf{K}}$ is finitely presentable as a 2-deductive system [4]. Moreover, the 2-deductive system $\vDash_{\mathsf{K}}$ is well known to be finitely equivalential.

Now, consider two finitely presentable varieties K and V. In virtue of the above remarks, the 2-deductive systems $\vDash_{\mathsf{K}}$ and $\vDash_{\mathsf{V}}$ are both finitely presentable and finitely equivalential. A straightforward generalization of Theorem 2.4 to 2-deductive systems shows that the non-indexed product $\vDash_{\mathsf{K}} \otimes \vDash_{\mathsf{V}}$ is also finitely presentable. By a simple variant of [15, Cor. 4.14], $\vDash_{\mathsf{K}} \otimes \vDash_{\mathsf{V}}$ is the equational consequence relative to the variety $\mathsf{K} \otimes \mathsf{V}$, whence $\mathsf{K} \otimes \mathsf{V}$ is finitely presentable. ⊠

3. A combinatorial counterexample

Recall that *binary* non-indexed products of logics that are finitely presentable and finitely equivalential are term-equivalent to finitely presentable logics (Theorem 2.4). This section is entirely devoted to prove that the restriction to finitely equivalential logics in this result cannot be dropped. Remarkably, this shows that the study of logics (as opposed to varieties) allows us spot implicit assumptions on deductive systems that are not immediately visible from the perspective of universal algebra (cf. Corollary 2.5). More precisely, we prove the following strong negative result:[2]

Theorem 3.1. *Binary non-indexed products of finitely presentable logics need not be equi-interpretable with any finitely presentable logic.*

The above result is proved by exhibiting an involved combinatorial counterexample. The uninterested reader may safely move on to the next section.

Let $\vdash_{\neg}$ be the logic in the language consisting of a single unary connective $\neg$ and formulated in countably many variables, axiomatized by the rule

$$x, \neg x \rhd y.$$

In other words, $\vdash_{\neg}$ is the logic of the *ex falso sequitur quodlibet* rule. Similarly, let $\vdash_{\Box}$ be the logic in the language with a single unary connective $\Box$ and formulated in countably many variables, axiomatized by the rule

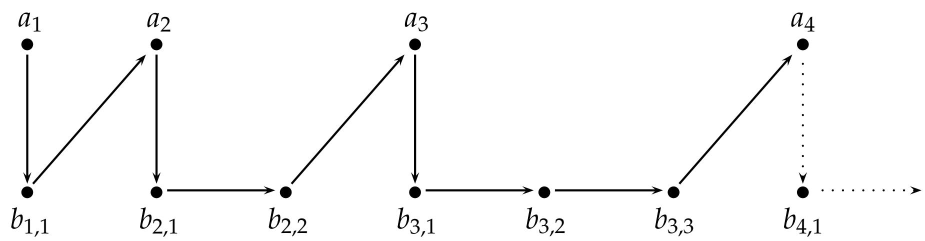
$$\Box x \rhd x.$$

Clearly $\vdash_{\neg}$ and $\vdash_{\Box}$ are finitely presentable. Therefore, to establish Theorem 3.1, it will be enough to show that the equivalence class $[\![\vdash_{\neg} \otimes \vdash_{\Box}]\!]$ does not contain any finitely presentable logic.

To this end, it is convenient to recall that algebras $\boldsymbol{A} = \langle A; f^{\boldsymbol{A}} \rangle$ such that f is a unary operation are is one-to-one correspondence with directed graphs G in which each vertex is the origin of exactly one edge. More precisely, given such an algebra $\boldsymbol{A}$, we consider the direct graph with set of vertexes A and with set of edges $\{\langle a, c \rangle \in A^2 \colon f^{\boldsymbol{A}}(a) = c\}$. Conversely, every directed graph G as above is associated with with the algebra whose universe is the set of vertexes of G and in which $f^{\boldsymbol{A}}(a)$ is the unique vertex b such that there is an edge connecting a to b in G. Bearing this in mind, we can define algebras by displaying suitable directed graphs.

[2]Here "strong" refers to the fact that Theorem 3.1 involves *equi-interpretability*, as opposed to simple *term-equivalence*.

3.1. **The algebra $\boldsymbol{A}$.** Let $\boldsymbol{A} = \langle A; \neg^{\boldsymbol{A}} \rangle$ be the algebra depicted below:



We have

$$A = \{a_n : 0 < n \in \omega\} \bigcup_{0<n\in\omega} \{b_{n,1}, b_{n,2}, \ldots, b_{n,n}\}.$$

Moreover, for every $0 < n \in \omega$ and $0 < m < n$,

$$\neg^{\boldsymbol{A}} a_n = b_{n,1} \quad \neg^{\boldsymbol{A}} b_{n,m} = b_{n,m+1} \quad \neg^{\boldsymbol{A}} b_{n,n} = a_{n+1}.$$

Finally, let $F := \{a_n : 0 < n \in \omega\} \subseteq A$.

Fact 1. $\langle \boldsymbol{A}, F \rangle \in \mathbb{R}(\mathsf{Mod}(\vdash_\neg))$.

Proof. It is straightforward to see $\langle \boldsymbol{A}, F \rangle$ is a model of the rule $x, \neg x \rhd y$, whence $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}(\vdash_\neg)$. Hence it only remains to prove that $\langle \boldsymbol{A}, F \rangle$ is reduced.

We make extensive use of [15, Prop. 2.2(i)] without notice. Given $0 < n \in \omega$, we consider the unary polynomial function

$$p_n(x) := \underbrace{\neg^{\boldsymbol{A}} \ldots \neg^{\boldsymbol{A}}}_{n+1\text{-times}} x.$$

We have $p_n(a_n) = a_{n+1} \in F$ and $p_n(a_m) \notin F$ for every $m > n$. As a consequence, the singleton $\{a_n\}$ is a block of $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ for every $0 < n \in \omega$.

Similarly, given $0 < m \leqslant n \in \omega$, we consider the unary polynomial functions

$$q^+_{n,m}(x) := \underbrace{\neg^{\boldsymbol{A}} \ldots \neg^{\boldsymbol{A}}}_{(n+1-m)\text{-times}} x \text{ and } q^-_{n,m}(x) := \underbrace{\neg^{\boldsymbol{A}} \ldots \neg^{\boldsymbol{A}}}_{(2n+3-m)\text{-times}} x.$$

Observe that $b_{n,m}$ is the unique element c in the set $\{b_{s,t} \in A : \text{either } n < s \text{ or } (n = s \text{ and } m \leqslant t)\}$ such that

$$q^+_{n,m}(c), q^-_{n,m}(c) \in F.$$

This easily implies that the singleton $\{b_{n,m}\}$ is a block of $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ for every $0 < m \leqslant n \in \omega$. Hence we conclude that $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is the identity relation. ⊠

Now, consider the following unary and binary connectives of $\vdash_\neg \otimes \vdash_\Box$, where π^2_i is the binary projection map on the i-th coordinate:

$$\neg_+ x := \langle \neg x, x \rangle \quad \Box_+ x := \langle x, \Box x \rangle \quad x \cdot y := \langle \pi^2_1(x,y), \pi^2_2(x,y) \rangle.$$

By Lemma 2.2 $\vdash_\neg \otimes \vdash_\Box$ is term-equivalent to its $\langle \neg_+, \Box_+, \cdot \rangle$-fragment. Bearing this in mind, from now on we identify $\vdash_\neg \otimes \vdash_\Box$ with its $\langle \neg_+, \Box_+, \cdot \rangle$-fragment. In particular, let $\boldsymbol{A}^-$ be the algebra in the language of $\vdash_\neg \otimes \vdash_\Box$ with universe A and basic operations defined as follows for every $a, c \in A$,

$$\neg_+^{\boldsymbol{A}^-} a := \neg^{\boldsymbol{A}} a \quad \Box_+^{\boldsymbol{A}^-} a := a \quad a \cdot^{\boldsymbol{A}^-} c := a. \tag{19}$$

Fact 2. $\langle \boldsymbol{A}^-, F\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg \otimes \vdash_\Box)$.

Proof. It is clear that $\langle \boldsymbol{A}^-, F\rangle \cong \langle \boldsymbol{A} \otimes \boldsymbol{1}, F \times \{1\}\rangle$ where $\boldsymbol{1}$ is the trivial algebra in the language of $\vdash_\Box$. By Fact 1 and [15, Cor. 4.14] we are done. ⊠

Fact 3. The following rules hold in $\vdash_\neg \otimes \vdash_\Box$:

(i) $\Box_+ x \rhd x$.
(ii) $x, \neg_+ x \rhd \underbrace{\neg_+ \ldots \neg_+}_{n\text{-times}} x$ for every $n \in \omega$.

Proof. (i): From the definition of $\vdash_\Box$ it follows that $\Box x \vdash_\Box x$. Together with the definition of $\vdash_\neg \otimes \vdash_\Box$, this yields $\Box_+ x \rhd x$. Condition (ii) is proved similarly. ⊠

For the present purpose, it is convenient to recall that the *subformula tree* of a formula φ in some language $\mathscr{L}$ is defined by recursion on the construction of φ as follows. The subformula tree of a variable x is the one-element tree, whose unique node is labelled by x. The subformula tree of a complex formula φ is obtained as follows. Suppose that $\varphi = g(\psi_1, \ldots, \psi_n)$ for some basic n-ary symbol g and formulas $\psi_1, \ldots, \psi_n$. First we pick the disjoint union of the subformula trees of $\psi_1, \ldots, \psi_n$, and we relabel the root of the subformula tree of each ψ_i by $\langle \psi_i, i\rangle$. Second we add to these trees a common root labelled by φ. The root of the subformula tree is the bottom element and the leaves are the top ones, and we use the expression *successors* and *predecessors* accordingly.

Definition 3.2. Let φ be a formula of the logic $\vdash_\neg \otimes \vdash_\Box$.

(i) An occurrence of the symbol $\neg_+$ in φ is said to be *faithful* if, in this occurrence, $\neg_+$ is the principal connective of a subformula ψ of φ that is not preceded in the subformula tree of φ by any formula γ labelled as $\langle \gamma, 2\rangle$ and whose immediate predecessor is labelled by $\delta \cdot \gamma$ or $\langle \delta \cdot \gamma, n\rangle$ for some $n \in \omega$.
(ii) An occurrence of the symbol $\Box_+$ in φ is said to be *faithful* if, in this occurrence, $\Box_+$ is the principal connective of a subformula ψ of φ that is not preceded in the subformula tree of φ by any formula γ labelled as $\langle \gamma, 1\rangle$ and whose immediate predecessor is labelled by $\gamma \cdot \delta$ or $\langle \gamma \cdot \delta, n\rangle$ for some $n \in \omega$.

Given $n \in \omega$, we say that φ contains n faithful occurrences of $\neg_+$ (resp. of $\Box_+$), if there are at least n different faithful occurrences of $\neg_+$ (resp. $\Box_+$) in φ.

As an exemplification of the above definition, observe that the only occurrence of $\neg_+$ in the formula $\Box_+(\neg_+ x \cdot y) \cdot \Box_+ x$ is faithful, while its occurrence in $\Box_+ x \cdot \Box_+(\neg_+ x \cdot y)$ is not.

Consider a logic $\vdash$ equi-interpretable with $\vdash_\neg \otimes \vdash_\Box$. To establish Theorem 3.1, it suffices to show that $\vdash$ is not finitely presentable. To this end, observe that there are interpretations $\boldsymbol{\tau}$ of $\vdash$ into $\vdash_\neg \otimes \vdash_\Box$, and $\boldsymbol{\rho}$ of the latter logic into the first.

Fact 4. If $\varphi(x)$ is a formula of $\vdash_\neg \otimes \vdash_\Box$, and n the exact number of faithful occurrences of $\neg_+$ in φ, then for every $a \in A$,

$$\varphi^{\boldsymbol{A}^-}(a) = \underbrace{\neg^{\boldsymbol{A}} \ldots \neg^{\boldsymbol{A}}}_{n\text{-times}} a.$$

Proof. By induction on the construction of the formula φ. ⊠

Fact 5. The following conditions hold:

(i) there is no faithful occurrence of $\neg_+$ in $\boldsymbol{\tau\rho}(\Box_+)$; and
(ii) there is exactly one faithful occurrence of $\neg_+$ in $\boldsymbol{\tau\rho}(\neg_+)$.

Proof. (i): Suppose the contrary, with a view to contradiction. By Fact 4 there is $n \geqslant 1$ such that for every $a \in A$,

$$\Box_+^{\boldsymbol{A}^{-\tau\rho}} a = \boldsymbol{\tau\rho}(\Box_+)^{\boldsymbol{A}^-} a = \underbrace{\neg^{\boldsymbol{A}} \dots \neg^{\boldsymbol{A}}}_{n\text{-times}} a. \tag{20}$$

Since $n \geqslant 1$, we can interrogate the definition $\boldsymbol{A}$, obtaining an element $c \in A$ such that

$$\underbrace{\neg^{\boldsymbol{A}} \dots \neg^{\boldsymbol{A}}}_{n\text{-times}} c \in F \text{ and } c \notin F.$$

Together with (20) this yields

$$\Box_+^{\boldsymbol{A}^{-\tau\rho}} c \in F \text{ and } c \notin F,$$

whence the matrix $\langle \boldsymbol{A}^{-\tau\rho}, F \rangle$ is not a model of the rule $\Box_+ x \rhd x$. By condition (i) of Fact 3 we conclude that $\langle \boldsymbol{A}^{-\tau\rho}, F \rangle$ is not a model of $\vdash_\neg \otimes \vdash_\Box$.

On the other hand, the opposite conclusion can be drawn from Fact 2, since $\boldsymbol{\tau\rho}$ is an interpretation of $\vdash_\neg \otimes \vdash_\Box$ into itself. Hence we reached a contradiction, as desired.

(ii): Let n be the number of faithful occurrences of $\neg_+$ in $\boldsymbol{\tau\rho}(\neg_+)$. From Fact 4 it follows that for every $a \in A$,

$$\neg_+^{\boldsymbol{A}^{-\tau\rho}} a = \underbrace{\neg^{\boldsymbol{A}} \dots \neg^{\boldsymbol{A}}}_{n\text{-times}} a. \tag{21}$$

Moreover, from Fact 2 it follows $\langle \boldsymbol{A}^{-\tau\rho}, F \rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg \otimes \vdash_\Box)$. By [15, Cor. 4.14] there are $\langle \boldsymbol{A}_1, F_1 \rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg)$ and $\langle \boldsymbol{A}_2, F_2 \rangle \in \mathsf{Mod}^{\equiv}(\vdash_\Box)$ such that

$$\langle \boldsymbol{A}^{-\tau\rho}, F \rangle = \langle \boldsymbol{A}_1 \bigotimes \boldsymbol{A}_2, F_1 \times F_2 \rangle.$$

Suppose, with a view to contradiction, that $n = 0$. Together with (21), this implies that $\neg_+^{\boldsymbol{A}^{-\tau\rho}}$ is the identity map. As $\boldsymbol{A}^{-\tau\rho} = \boldsymbol{A}_1 \otimes \boldsymbol{A}_2$, we conclude that $\neg^{\boldsymbol{A}_1}$ is also the identity map. Now, since $\emptyset \neq F = F_1 \times F_2$, we can choose $a \in F_1$, obtaining $\{\neg^{\boldsymbol{A}_1} a, a\} = \{a\} \subseteq F_1$. Since $x, \neg x \vdash_\neg y$ and $\langle \boldsymbol{A}_1, F_1 \rangle$ is a model of $\vdash_\neg$, we get $F_1 = A_1$. By [16, Lem. 5.5] we conclude that $\langle \boldsymbol{A}_1, F_1 \rangle = \langle \mathbf{1}, \{1\} \rangle$. In particular, this implies

$$\omega = |F| = |F_1 \times F_2| = |\{1\} \times F_2| = |F_2|.$$

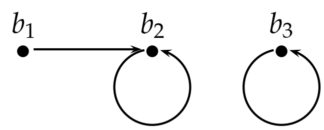
Then we can choose two different $a, c \in F_2$. Since $\langle \boldsymbol{A}_2, F_2 \rangle \in \mathsf{Mod}^{\equiv}(\vdash_\Box)$, by [16, Prop. 2.2(ii)] there is a unary polynomial function $p(x)$ of $\boldsymbol{A}_2$ such that $\mathrm{Fg}_{\vdash_\Box}^{\boldsymbol{A}_2}(F_2, p(a)) \neq \mathrm{Fg}_{\vdash_\Box}^{\boldsymbol{A}_2}(F_2, p(c))$. Hence, $\boldsymbol{A}_2$ possesses a unary polynomial function (namely p) that is neither the identity map nor a constant map. As a consequence, $\Box^{\boldsymbol{A}_2}$ is not the identity map. On the other hand, condition (i) and Fact 4 guarantees that $\Box_+^{\boldsymbol{A}^{-\tau\rho}}$ is the identity map. Since $\boldsymbol{A}^{-\tau\rho} = \boldsymbol{A}_1 \otimes \boldsymbol{A}_2$, this yields that so is $\Box^{\boldsymbol{A}_2}$. Hence we reached the desired contradiction.

Now, we know that $n > 0$. Suppose, with a view to contradiction, that $n \geqslant 2$. Looking at the definition of $\boldsymbol{A}$, we can find an element $a \in F$ such that

$$\underbrace{\neg^{\boldsymbol{A}} \dots \neg^{\boldsymbol{A}}}_{n\text{-times}} a \in F \text{ and } \underbrace{\neg^{\boldsymbol{A}} \dots \neg^{\boldsymbol{A}}}_{n\text{-times}} \underbrace{\neg^{\boldsymbol{A}} \dots \neg^{\boldsymbol{A}}}_{n\text{-times}} a \notin F.$$

By (21) we have
$$a, \neg_+^{A^{-\tau\rho}} a \in F \text{ and } \neg_+^{A^{-\tau\rho}} \neg_+^{A^{-\tau\rho}} a \notin F.$$
Thus $\langle \boldsymbol{A}^{-\tau\rho}, F\rangle$ is not a model of the rule $x, \neg_+ x \rhd \neg_+\neg_+ x$. But, in the light of condition (ii) of Fact 3, this contradicts the fact that $\langle \boldsymbol{A}^{-\tau\rho}, F\rangle$ is a model of $\vdash_\neg \otimes \vdash_\Box$. Hence we reached a contradiction, as desired. ⊠

3.2. **The algebra $\boldsymbol{B}$.** Now, consider the algebra $\boldsymbol{B} = \langle B; \Box^{\boldsymbol{B}}\rangle$ depicted below.



Let $\boldsymbol{B}^-$ be the algebra with universe B and basic operations defined for every $a, c \in B$ as follows:
$$\neg_+^{\boldsymbol{B}^-} a := a \quad \Box_+^{\boldsymbol{B}^-} a := \Box^{\boldsymbol{B}} a \quad a \cdot^{\boldsymbol{B}^-} c := c.$$
Setting $G := \{b_1, b_3\} \subseteq B$, we have that following:

Fact 6. $\langle \boldsymbol{B}^-, G\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg \otimes \vdash_\Box)$.

Proof. First observe that $\langle \boldsymbol{B}, G\rangle$ is a reduced model of $\vdash_\Box$, whence $\langle \boldsymbol{B}, G\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\Box)$. Moreover, $\langle \boldsymbol{B}^-, G\rangle \cong \langle \boldsymbol{1} \otimes \boldsymbol{B}, \{1\} \times G\rangle$ where $\boldsymbol{1}$ is the trivial algebra in the language of $\vdash_\neg$. By [15, Cor. 4.14] we are done. ⊠

Fact 7. If $\varphi(x)$ is a formula of $\vdash_\neg \otimes \vdash_\Box$, and n the exact number of faithful occurrences of $\Box_+$ in the formula φ, then for every $a \in B$,
$$\varphi^{\boldsymbol{B}^-}(a) = \underbrace{\Box^{\boldsymbol{B}} \dots \Box^{\boldsymbol{B}}}_{n\text{-times}} a.$$

Proof. By induction on the construction of φ. ⊠

Fact 8. There is no faithful occurrence of $\Box_+$ in $\boldsymbol{\tau\rho}(\neg_+)$.

Proof. Suppose the contrary, with a view to contradiction. Then let $n \geqslant 1$ be the exact number of faithful occurrences of $\Box_+$ in $\boldsymbol{\tau\rho}(\neg_+)$. By Fact 7 and $\boldsymbol{B} \vDash \Box x \approx \Box\Box x$, for every $a \in B$,
$$\neg_+^{\boldsymbol{B}^{-\tau\rho}} a = \underbrace{\Box^{\boldsymbol{B}} \dots \Box^{\boldsymbol{B}}}_{n\text{-times}} a = \Box^{\boldsymbol{B}} a. \tag{22}$$
Moreover, by Fact 6 and the fact that $\boldsymbol{\tau\rho}$ is an interpretation of $\vdash_\neg \otimes \vdash_\Box$ into itself, we have $\langle \boldsymbol{B}^{-\tau\rho}, G\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg \otimes \vdash_\Box)$. Thus we can assume without loss of generality that
$$\langle \boldsymbol{B}^{-\tau\rho}, G\rangle = \langle \boldsymbol{B}_1 \bigotimes \boldsymbol{B}_2, G_1 \times G_2\rangle.$$
for some $\langle \boldsymbol{B}_1, G_1\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg)$ and $\langle \boldsymbol{B}_2, G_2\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\Box)$ [15, Cor. 4.14].

Now, the definition of $\boldsymbol{B}$ and (22) imply that the map $\neg_+^{\boldsymbol{B}^{-\tau\rho}}$ is not the identity. As a consequence, also $\neg^{\boldsymbol{B}_1}$ is not the identity map, whence $\boldsymbol{B}_1$ is non-trivial. Together with the fact that $B = B_1 \times B_2$ is a three-element set, this implies that B_2 is a singleton and, therefore, $\langle \boldsymbol{B}_2, G_2\rangle = \langle \boldsymbol{1}, \{1\}\rangle$, where $\boldsymbol{1}$ is the trivial algebra in the language of $\vdash_\Box$. Thus
$$\langle \boldsymbol{B}^{-\tau\rho}, G\rangle = \langle \boldsymbol{B}_1 \bigotimes \boldsymbol{1}, G_1 \times \{1\}\rangle. \tag{23}$$

Then consider $a \in B_1$ such that $b_3 = \langle a, 1\rangle$. By (22) we obtain

$$\langle \neg^{\boldsymbol{B}_1} a, 1\rangle = \neg_+^{\boldsymbol{B}_1 \otimes \boldsymbol{1}} \langle a, 1\rangle = \neg_+^{\boldsymbol{B}^{-\tau\rho}} b_3 = \Box^{\boldsymbol{B}} b_3 = b_3 \in G = G_1 \times G_2$$
$$\langle a, 1\rangle = b_3 \in G = G_1 \times G_2.$$

As a consequence, we get $a, \neg^{\boldsymbol{B}_1} a \in G_1$. Since $\langle \boldsymbol{B}_1, G_1\rangle$ is a model of the rule $x, \neg x \rhd y$, we obtain $G_1 = B_1$. By [16, Lem. 5.5] this yields $\langle \boldsymbol{B}_1, G_1\rangle = \langle \boldsymbol{1}, \{1\}\rangle$, where $\boldsymbol{1}$ is the trivial algebra in the language of $\vdash_\neg$. By (23) we conclude that $B = \{1\} \times \{1\}$. But this contradicts the fact that B is a three-element set, as desired. $\boxtimes$

Fact 9. $\boldsymbol{\tau\rho}(\Box_+ x) \neq x$.

Proof. Suppose the contrary, with a view to contradiction. By Fact 6 we get $\langle \boldsymbol{B}^{-\tau\rho}, G\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg \otimes \vdash_\Box)$. Moreover, by [15, Cor. 4.14] there are $\langle \boldsymbol{B}_1, G_1\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg)$ and $\langle \boldsymbol{B}_2, G_2\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\Box)$ such that

$$\langle \boldsymbol{B}^{-\tau\rho}, G\rangle = \langle \boldsymbol{B}_1 \bigotimes \boldsymbol{B}_2, G_1 \times G_2\rangle.$$

Since B is a three-element set, there is $i = 1, 2$ such that $\langle \boldsymbol{B}_i, G_i\rangle$ is a trivial matrix. In particular, this implies that $\cdot^{\boldsymbol{B}^{-\tau\rho}}$ is either the projection on the first argument or the projection on the second one. Moreover, by the assumption $\Box_+^{\boldsymbol{B}^{-\tau\rho}}$ is the identity map. The same holds for $\neg_+^{\boldsymbol{B}^{-\tau\rho}}$ by Facts 7 and 8. Hence all the basic operations of $\boldsymbol{B}^{-\tau\rho}$ are projection maps.

In particular, this yields that the unary polynomial functions of $\boldsymbol{B}^{-\tau\rho}$ are just the constant maps and the identity map. Together with [15, Prop. 2.2(ii)] and $G = \{b_1, b_3\}$, this implies $\langle b_1, b_3\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}^{-\tau\rho}}_{\vdash_\neg \otimes \vdash_\Box} G$. But, as desired, this contradicts the fact that $\langle \boldsymbol{B}^{-\tau\rho}, G\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg \otimes \vdash_\Box)$. $\boxtimes$

3.3. **The algebra $\boldsymbol{C}$.** Suppose, with a view to contradiction, that the logic $\vdash$ is finitely presentable. From now on, our goal is to reach a contradiction.

To this end, observe that $\vdash$ is axiomatized by a finite set of finite rules

$$\begin{array}{ccccc} \gamma_1^1 & \dots & \gamma_1^k & \rhd & \varphi_1 \\ \vdots & & \vdots & & \vdots \\ \gamma_m^1 & \dots & \gamma_m^k & \rhd & \varphi_m. \end{array}$$

Consider a natural number n strictly greater than the number of faithful occurrences of $\neg_+$ in the following set of formulas

$$\{\boldsymbol{\tau}(\gamma_i^j) : i \leqslant m \text{ and } j \leqslant k\} \cup \{\boldsymbol{\tau}(\varphi_i) : i \leqslant m\}.$$

Moreover, with every formula φ in the language $\langle \neg_+, \Box_+, \cdot\rangle$ we associate a formula φ^* in the language $\langle \neg_+, \Box_+\rangle$ defined by recursion as follows:

$$\begin{aligned} x^* &:= x, \text{ for every variable } x \\ (\neg_+ \psi)^* &:= \neg_+ \psi^* \\ (\Box_+ \psi)^* &:= \Box_+ \psi^* \\ (\psi_1 \cdot \psi_2)^* &:= \Box_+ \psi_1^*. \end{aligned}$$

Fact 10. There are $s, t \in \omega$ such that

$$\boldsymbol{\tau\rho}(\neg_+)^* = \underbrace{\Box_+ \ldots \Box_+}_{s\text{-times}} \neg_+ \underbrace{\Box_+ \ldots \Box_+}_{t\text{-times}} x_1.$$

Proof. Immediate from condition (ii) of Fact 5. ⊠

Let t be the natural number given by Fact 10. We define an algebra $\boldsymbol{C}$ in the language $\langle \neg_+, \Box_+, \cdot \rangle$ with universe

$$C := \{b_1, \ldots, b_{t+1}\} \cup \{a_1, \ldots, a_{t+n+2}\}$$

and operations defined for every $p, q \in C$, and every $1 \leqslant i \leqslant t+1$ and $1 \leqslant j \leqslant t+n+2$ as follows:

$$p \cdot^{\boldsymbol{C}} q := \begin{cases} p & \text{if } p \in \{a_1, \ldots, a_{t+n+2}\} \\ a_1 & \text{if } p = b_{t+1} \\ b_{r+1} & \text{if } p = b_r \text{ for some } r \leqslant t \end{cases} \qquad \neg_+^{\boldsymbol{C}} b_i = \Box_+^{\boldsymbol{C}} b_i := \begin{cases} a_1 & \text{if } i = t+1 \\ b_{i+1} & \text{if } i \leqslant t \end{cases}$$

$$\neg_+^{\boldsymbol{C}} a_j := \begin{cases} a_{t+n+2} & \text{if } j = t+n+2 \\ a_{j+1} & \text{if } j \leqslant t+n+1 \end{cases} \qquad \Box_+^{\boldsymbol{C}} a_j := a_j.$$

Fact 11. If $\varphi(x)$ is a formula in the language $\langle \neg_+, \Box_+, \cdot \rangle$, then $\varphi^{\boldsymbol{C}}(c) = \varphi^{*\boldsymbol{C}}(c)$ for every $c \in C$.

Proof. We reason by induction on the construction of φ. The case where φ is a variable, and the cases where $\varphi = \neg_+\psi$ or $\varphi = \Box_+\psi$ (for some formula ψ) are straightforward. Thus we only detail the case where $\varphi = \psi_1 \cdot \psi_2$. Consider $c \in C$. Looking at the definition of $\boldsymbol{C}$, we obtain

$$\varphi^{\boldsymbol{C}}(c) = \psi_1^{\boldsymbol{C}}(c) \cdot^{\boldsymbol{C}} \psi_2^{\boldsymbol{C}}(c) = \begin{cases} \psi_1^{\boldsymbol{C}}(c) & \text{if } \psi_1^{\boldsymbol{C}}(c) \in \{a_1, \ldots, a_{t+n+2}\} \\ \\ \Box_+^{\boldsymbol{C}} \psi_1^{\boldsymbol{C}}(c) & \text{if } \psi_1^{\boldsymbol{C}}(c) \in \{b_1, \ldots, b_{t+1}\} \end{cases}$$

and

$$\varphi^{*\boldsymbol{C}}(c) = \Box_+ \psi_1^{*\boldsymbol{C}}(c) = \begin{cases} \psi_1^{*\boldsymbol{C}}(c) & \text{if } \psi_1^{*\boldsymbol{C}}(c) \in \{a_1, \ldots, a_{t+n+2}\} \\ \\ \Box_+^{\boldsymbol{C}} \psi_1^{*\boldsymbol{C}}(c) & \text{if } \psi_1^{*\boldsymbol{C}}(c) \in \{b_1, \ldots, b_{t+1}\} \end{cases}$$

Now, from the inductive hypothesis it follows that $\psi_1^{\boldsymbol{C}}(c) = \psi_1^{*\boldsymbol{C}}(c)$. Together with the above displays, this implies $\varphi^{\boldsymbol{C}}(c) = \varphi^{*\boldsymbol{C}}(c)$. ⊠

Fact 12. For every $1 \leqslant j \leqslant t+n+2$ and $1 < i \leqslant t+1$ we have:

(i) $\boldsymbol{\tau\rho}(\neg_+)^{\boldsymbol{C}}(a_j) = \neg_+^{\boldsymbol{C}} a_j$;
(ii) $\boldsymbol{\tau\rho}(\Box_+)^{\boldsymbol{C}}(a_j) = a_j$;
(iii) $\boldsymbol{\tau\rho}(\neg_+)^{\boldsymbol{C}}(b_1) = a_1$; and
(iv) $\boldsymbol{\tau\rho}(\neg_+)^{\boldsymbol{C}}(b_i) = a_2$.

Proof. Recall from Fact 10 that

$$\boldsymbol{\tau\rho}(\neg_+)^* = \underbrace{\Box_+ \ldots \Box_+}_{s\text{-times}} \neg_+ \underbrace{\Box_+ \ldots \Box_+}_{t\text{-times}} x_1.$$

Together with the definition of $\boldsymbol{C}$, this implies

$$\boldsymbol{\tau\rho}(\neg_+)^{*\boldsymbol{C}}(a_j) = \neg_+^{\boldsymbol{C}} a_j, \quad \boldsymbol{\tau\rho}(\neg_+)^{*\boldsymbol{C}}(b_1) = a_1, \text{ and } \boldsymbol{\tau\rho}(\neg_+)^{*\boldsymbol{C}}(b_i) = a_2.$$

In turn, with an application of Fact 11, this establishes conditions (i), (iii), and (iv).

(ii): From condition (i) of Fact 5 it follows that

$$\boldsymbol{\tau\rho}(\Box_+)^* = \underbrace{\Box_+ \dots \Box_+}_{p\text{-times}} x_1$$

for some $p \in \omega$. Together with the definition of $\boldsymbol{C}$, this implies $\boldsymbol{\tau\rho}(\Box_+)^{*\boldsymbol{C}}(a_j) = a_j$. By Fact 11 we conclude $\boldsymbol{\tau\rho}(\Box_+)^{\boldsymbol{C}}(a_j) = a_j$. ⊠

Fact 13. Let $\varphi(y_1, \dots, y_{t+1}, x_1, \dots, x_{t+n+2})$ be a formula of $\vdash_\neg \otimes \vdash_\Box$ with exactly p faithful occurrences of $\neg_+$. Then

(i) either $\varphi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2}) = \varphi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2})$,

(ii) or both $\varphi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2})$ and $\varphi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2})$ belong to $\{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}$, where q is the minimum of $\{p+1, t+n+2\}$.

Proof. We reason by induction on the complexity of φ. In the base case, φ is a variable among $y_1, \dots, y_{t+1}, x_1, \dots, x_{t+n+2}$. If $\varphi = y_i$, then condition (ii) holds, while if $\varphi = x_i$, then (i) holds. Then we move to the induction step.

First suppose that $\varphi = \Box_+\psi$ for some formula ψ. By the induction hypothesis, ψ satisfies either (i) or (ii). If ψ satisfies (i), then the same holds for φ. Then suppose that ψ satisfies (ii), let p be the number of faithful occurrences of $\neg_+$ in ψ, and set $q := \min\{p+1, t+n+2\}$. Then

$$\psi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}$$
$$\psi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}.$$

Inspecting the definition of $\boldsymbol{C}$, one sees that the set $\{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}$ is closed under the operation $\Box_+^{\boldsymbol{C}}$. Thus

$$\varphi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}$$
$$\varphi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}.$$

Since p is also the number of faithful occurrences of $\neg_+$ in φ, we conclude that φ satisfies condition (ii).

Then suppose that $\varphi = \neg_+\psi$ for some formula ψ. Again, ψ satisfies either (i) or (ii). If ψ satisfies (i), so does φ. Then suppose that ψ satisfies (ii), let p be the number of faithful occurrences of $\neg_+$ in ψ, and set $q := \min\{p+1, t+n+2\}$. Then

$$\psi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}$$
$$\psi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}.$$

Define also $q' := \min\{p+1, t+n+2\}$. Looking at the definition of $\boldsymbol{C}$, one sees that if $c \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}$, then $\neg_+^{\boldsymbol{C}} c \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_{q'}\}$. In particular,

$$\varphi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_{q'}\}$$
$$\varphi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_{q'}\}.$$

Together with the fact that $p+1$ is the number of faithful occurrences of $\neg_+$ in φ, this implies that φ satisfies condition (ii).

Finally, suppose that $\varphi = \psi_1 \cdot \psi_2$ for some formulas ψ_1 and ψ_2. By the induction hypothesis, ψ_1 satisfies either (i) or (ii). Interrogating the definition of $\cdot^{\boldsymbol{C}}$ one sees that if ψ_1 satisfies (i), so does φ. Then suppose that ψ_1 satisfies condition (ii). Moreover, let p be the number of faithful occurrences of $\neg_+$ in ψ_1, and set $q := \min\{p+1, t+n+2\}$. We have

$$\psi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}$$
$$\psi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}.$$

Thus, looking at the definition of $\cdot^{\boldsymbol{C}}$, we obtain

$$\begin{aligned}\varphi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2}) &\in \{b_2, \dots, b_{t+1}, a_1, \psi_1^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2})\} \\ &\subseteq \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}\end{aligned}$$

and

$$\begin{aligned}\varphi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2}) &\in \{b_2, \dots, b_{t+1}, a_1, \psi_1^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2})\} \\ &\subseteq \{b_1, \dots, b_{t+1}, a_1, \dots, a_q\}.\end{aligned}$$

Since p is also the number of faithful occurrences of $\neg_+$ in φ, we conclude that φ satisfies condition (ii). ⊠

Fact 14. Let $\varphi(y_1, \dots, y_{t+1}, x_1, \dots, x_{t+n+2})$ be a formula of $\vdash_\neg \bigotimes \vdash_\Box$ with exactly $< n$ faithful occurrences of $\neg_+$. Then

$$\varphi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2}) = a_{t+n+1} \Longleftrightarrow \varphi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2}) = a_{t+n+1}.$$

Proof. We know that φ satisfies either condition (i) or (ii) of Fact 13. If φ satisfies condition (i), we are done. Then suppose that φ satisfies condition (ii). Moreover, let $p < n$ be the number of faithful occurrences of $\neg_+$ in φ. By (ii) we get

$$\varphi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_{p+1}\}$$
$$\varphi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2}) \in \{b_1, \dots, b_{t+1}, a_1, \dots, a_{p+1}\}.$$

Since $p < n$, we have $p+1 < t+n+1$. Thus neither $\varphi^{\boldsymbol{C}}(b_1, \dots, b_{t+1}, a_1, a_2, \dots, a_{t+n+2})$ nor $\varphi^{\boldsymbol{C}}(a_1, \dots, a_1, a_1, a_2, \dots, a_{t+n+2})$ is equal to a_{t+n+1}. ⊠

Fact 15. $\langle \boldsymbol{C}^{\boldsymbol{\tau}}, \{a_{t+n+1}\}\rangle \in \mathsf{Mod}(\vdash)$.

Proof. Recall that $\vdash$ it axiomatized by the rules $\gamma_i^1, \dots, \gamma_i^k \rhd \varphi_i$ with $i \leqslant m$. Therefore, it will be enough to show that the matrix $\langle \boldsymbol{C}, \{a_{t+n+1}\}\rangle$ is a model of the rule $\boldsymbol{\tau}(\gamma_i^1), \dots, \boldsymbol{\tau}(\gamma_i^k) \rhd \boldsymbol{\tau}(\varphi_i)$ for every $i \leqslant m$. Let $\boldsymbol{D}$ be the subalgebra of $\boldsymbol{C}$ with universe $\{a_1, \dots, a_{t+n+2}\}$. Recall that each formula in the rules $\boldsymbol{\tau}(\gamma_i^1), \dots, \boldsymbol{\tau}(\gamma_i^k) \rhd \boldsymbol{\tau}(\varphi_i)$ have $< n$ faithful occurrences of $\neg_+$. In the light of Fact 14, this implies that if $\langle \boldsymbol{D}, \{a_{t+n+1}\}\rangle$ is a model of the rules $\boldsymbol{\tau}(\gamma_i^1), \dots, \boldsymbol{\tau}(\gamma_i^k) \rhd \boldsymbol{\tau}(\varphi_i)$, then so is $\langle \boldsymbol{C}, \{a_{t+n+1}\}\rangle$. Therefore, to conclude the proof, it suffices to show that $\langle \boldsymbol{D}, \{a_{t+n+1}\}\rangle$ is a model of the rules $\boldsymbol{\tau}(\gamma_i^1), \dots, \boldsymbol{\tau}(\gamma_i^k) \rhd \boldsymbol{\tau}(\varphi_i)$ or, equivalently, that $\langle \boldsymbol{D}^{\boldsymbol{\tau}}, \{a_{t+n+1}\}\rangle$ is a model of $\vdash$.

This is what we do now. Let $\boldsymbol{E} = \langle D; \neg^{\boldsymbol{E}}\rangle$ be the algebra where $\neg^{\boldsymbol{E}}$ is defined as $\neg_+^{\boldsymbol{D}}$. It is not hard to see that $\langle \boldsymbol{E}, \{a_{t+n+1}\}\rangle \in \mathbb{R}(\mathsf{Mod}(\vdash_\neg))$. As a consequence, we obtain $\langle \boldsymbol{E} \bigotimes \boldsymbol{1}, \{\langle a_{t+n+1}, 1\rangle\}\rangle \in \mathsf{Mod}^{\equiv}(\vdash_\neg \bigotimes \vdash_\Box)$ where $\boldsymbol{1}$ is the trivial algebra in the language of $\vdash_\Box$ [15, Cor. 4.14]. Since $\boldsymbol{\tau}$ is an interpretation of $\vdash$ into $\vdash_\neg \bigotimes \vdash_\Box$,

$$\langle (\boldsymbol{E} \bigotimes \boldsymbol{1})^{\boldsymbol{\tau}}, \{\langle a_{t+n+1}, 1\rangle\}\rangle \in \mathsf{Mod}^{\equiv}(\vdash). \tag{24}$$

Moreover, since $\Box_+^{\boldsymbol{D}}$ is the identity map and $\cdot^{\boldsymbol{D}}$ is the projection on the first coordinate, the function

$$\pi\colon \langle \boldsymbol{E} \bigotimes \boldsymbol{1}, \{\langle a_{t+n+1}, 1\rangle\}\rangle \to \langle \boldsymbol{D}, \{a_{t+n+1}\}\rangle,$$

defined as $\pi(\langle c, 1\rangle) := c$ for every $c \in D$, is an isomorphism. Together with (24) this implies that $\langle \boldsymbol{D}^\tau, \{a_{t+n+1}\}\rangle$ is a model of $\vdash$, as desired. ⊠

Fact 16. $\langle \boldsymbol{C}^{\tau\rho}, \{a_{t+n+1}\}\rangle^* \in \mathsf{Mod}^{\equiv}(\vdash_\neg \otimes \vdash_\Box)$.

Proof. By Fact 15 and [15, Prop. 3.3] we obtain $\langle \boldsymbol{C}^{\tau\rho}, \{a_{t+n+1}\}\rangle \in \mathsf{Mod}(\vdash_\neg \otimes \vdash_\Box)$. Hence $\langle \boldsymbol{C}^{\tau\rho}, \{a_{t+n+1}\}\rangle^* \in \mathbb{R}(\mathsf{Mod}(\vdash_\neg \otimes \vdash_\Box)) \subseteq \mathsf{Mod}^{\equiv}(\vdash_\neg \otimes \vdash_\Box)$. ⊠

Fact 17. The operation $\Box_+$ is the identity map on $\boldsymbol{C}^{\tau\rho}/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}$.

Proof. Recall from condition (i) of Fact 12 that $\neg_+^{\boldsymbol{C}^{\tau\rho}} a_j = \neg_+^{\boldsymbol{C}} a_j$ for every $1 \leqslant j \leqslant t+n+2$. We will make systematic use of this fact without further notice.

Given $1 \leqslant i \leqslant t+n+1$, we consider the following unary polynomial function of $\boldsymbol{C}^{\tau\rho}$:

$$p_i(x) := \underbrace{\neg_+^{\boldsymbol{C}^{\tau\rho}} \dots \neg_+^{\boldsymbol{C}^{\tau\rho}}}_{t+n+1-i\text{-times}} x.$$

For every $1 \leqslant i \leqslant t+n+1$ and $1 \leqslant j \leqslant t+n+2$,

$$p_i(a_j) = a_{t+n+1} \iff i = j.$$

By [15, Prop. 2.2(i)] we get

$$\langle a_i, a_j\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}, \text{ for every } 1 \leqslant i < j \leqslant t+n+2. \tag{25}$$

Now, by Fact 16 and [15, Cor. 4.14] there are algebras $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ in the languages of $\vdash_\neg$ and $\vdash_\Box$, respectively, such that $\boldsymbol{C}^{\tau\rho}/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\} = \boldsymbol{C}_1 \otimes \boldsymbol{C}_2$. Then there are $e \in C_1$ and $g \in C_2$ such that $a_1/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\} = \langle e, g\rangle$. In particular, for every $1 \leqslant j \leqslant t+n+2$,

$$\begin{aligned}
a_j/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\} &= (\underbrace{\neg_+^{\boldsymbol{C}} \dots \neg_+^{\boldsymbol{C}}}_{j-1\text{-times}} a_1)/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\} \\
&= (\underbrace{\neg_+^{\boldsymbol{C}^{\tau\rho}} \dots \neg_+^{\boldsymbol{C}^{\tau\rho}}}_{j-1\text{-times}} a_1)/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\} \\
&= \underbrace{\neg_+^{\boldsymbol{C}^{\tau\rho}/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}} \dots \neg_+^{\boldsymbol{C}^{\tau\rho}/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}}}_{j-1\text{-times}} (a_1/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}) \\
&= \underbrace{\neg_+^{\boldsymbol{C}_1 \otimes \boldsymbol{C}_2} \dots \neg_+^{\boldsymbol{C}_1 \otimes \boldsymbol{C}_2}}_{j-1\text{-times}} \langle e, g\rangle \\
&= \langle \underbrace{\neg^{\boldsymbol{C}_1} \dots \neg^{\boldsymbol{C}_1}}_{j-1\text{-times}} e, g\rangle.
\end{aligned}$$

Together with (25), the above display implies that the following elements are different from one another:

$$\{\underbrace{\neg^{\boldsymbol{C}_1} \dots \neg^{\boldsymbol{C}_1}}_{j-1\text{-times}} e \colon 1 \leqslant j \leqslant t+n+2\}$$

Thus C_1 has at least $t+n+2$ elements, call them $e_1, \dots, e_{t+n+2}$.

Then suppose, with a view to contradiction, that $\Box_+$ is not the identity map on $\boldsymbol{C}^{\tau\rho}/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}$. There is $\langle c, d\rangle \in C_1 \times C_2 = C/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}$ such that

$$\langle c, d\rangle \neq \Box_+^{\boldsymbol{C}^{\tau\rho}/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}}\langle c, d\rangle = \Box_+^{\boldsymbol{C}_1 \otimes \boldsymbol{C}_2}\langle c, d\rangle = \langle c, \Box^{\boldsymbol{C}_2} d\rangle.$$

As a consequence, $d \neq \Box^{\boldsymbol{C}_2} d$. Now, consider the distinct elements

$$\langle e_1, d\rangle, \dots, \langle e_{t+n+2}, d\rangle \in C_1 \times C_2 = C/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}.$$

For every $1 \leqslant j \leqslant t+n+2$,

$$\langle e_j, d\rangle \neq \langle e_j, \Box^{\boldsymbol{C}_2}, d\rangle = \Box_+^{\boldsymbol{C}_1 \otimes \boldsymbol{C}_2}\langle e_j, d\rangle = \Box_+^{\boldsymbol{C}^{\tau\rho}/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}}\langle e_j, d\rangle.$$

Thus the algebra $\boldsymbol{C}^{\tau\rho}/\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}$ has at least $t+n+2$ elements that are not fixed points of $\Box_+$. Consequently, also $\boldsymbol{C}^{\tau\rho}$ has $t+n+2$ elements that are not fixed points of $\Box_+$, call them $g_1, \dots, g_{t+n+2}$. By condition (ii) of Fact 12, $\{g_1, \dots, g_{t+n+2}\} \subseteq \{b_1, \dots, t_{t+1}\}$. But this contradicts the fact that $t+1 < t+n+2$. Hence we reached a contradiction, as desired. ⊠

Fact 18. The singleton $\{b_1\}$ is a block of $\boldsymbol{\Omega}^{\boldsymbol{C}^{\tau\rho}}\{a_{t+n+1}\}$.

Proof. By [15, Prop. 2.2(i)] it will be enough to show that for every $c \in C \smallsetminus \{b_1\}$ there exists a unary polynomial function $p(x)$ of $\boldsymbol{C}^{\tau\rho}$ such that $p(b_1) = a_{t+n+1}$ if and only if $p(c) \neq a_{t+n+1}$. To this end, consider the unary polynomial function

$$p(x) \coloneqq \underbrace{\neg_+^{\boldsymbol{C}^{\tau\rho}} \dots \neg_+^{\boldsymbol{C}^{\tau\rho}}}_{t+n+1\text{-times}} x.$$

Conditions (i), (iii), and (iv) of Fact 12 guarantee that $p(b_1) = a_{t+n+1}$ and $p(c) \neq a_{t+n+1}$ for every $c \in C \smallsetminus \{b_1\}$. ⊠

Fact 19. $\Box_+^{\boldsymbol{C}^{\tau\rho}} b_1 = b_1$.

Proof. Immediate from Facts 17 and 18. ⊠

Fact 20. $\boldsymbol{\tau\rho}(\Box_+ x) = x$.

Proof. From the definition of $\boldsymbol{C}$ it follows that if $\varphi(x)$ is a formula in the language $\langle \neg_+, \Box_+, \cdot\rangle$ and $b_1 \in \varphi^{\boldsymbol{C}}[C]$, then φ is a variable. On the other hand, by Fact 19 $b_1 = \boldsymbol{\tau\rho}(\neg_+ x)^{\boldsymbol{C}}(b_1)$. Since $\boldsymbol{\tau\rho}(\neg_+ x)$ is a formula of $\langle \neg_+, \Box_+, \cdot\rangle$, we conclude that $\boldsymbol{\tau\rho}(\neg_+ x)$ is a variable, whence it is the variable x. ⊠

We are now ready to conclude the proof of Theorem 3.1. To establish the theorem, it is enough to show $\vdash_\neg \otimes \vdash_\Box$ is not equi-interpretable with a finitely presentable logic. To prove this, we supposed, with a view to contradiction, that $\vdash_\neg \otimes \vdash_\Box$ is equi-interpretable with any finitely presentable logic $\vdash$. This allowed us to establish Facts 9 and 20, thus reaching the desired contradiction.

4. Finitely presentable Leibniz classes

Definition 4.1. A *finitely presentable Leibniz condition* Φ is a family $\{\Phi_n : n \in \omega\}$ of logics that are finitely presentable and finitely equivalential such that if $n \leqslant m$, then $\vdash_{\Phi_m} \leqslant \vdash_{\Phi_n}$. A logic $\vdash$ is said to *satisfy* Φ if $\vdash_{\Phi_n} \leqslant \vdash$ for some $n \in \omega$, and the class of logics satisfying Φ is denoted by $\mathrm{Log}(\Phi)$.

Accordingly, a class $\mathbb{K}$ of logics is a *finitely presentable Leibniz class* if it is of the form $\mathsf{Log}(\Phi)$ for some finitely presentable Leibniz condition Φ. ⊠

Finitely presentable Leibniz classes can be characterized in a way similar to the one for Leibniz classes in [16, Thm. 2.2]. To explain how, recall that the poset of all logics Log is a meet-semilattice [15, Thm. 4.6]. A subcollection F of Log is a *filter* if it is a non-empty upset closed under binary infima. Moreover, given a class $\mathbb{K}$ of logics, we set

$$\mathbb{K}^{\dagger} := \{[\![\vdash]\!] \colon \vdash \in \mathbb{K}\}.$$

Theorem 4.2. *Let $\mathbb{K}$ be a class of logics. The following conditions are equivalent:*

(i) *$\mathbb{K}$ is a finitely presentable Leibniz class.*
(ii) *$\mathbb{K}$ is a non-empty class of finitely equivalential logics that is closed under under term-equivalence, compatible expansions, and finite (equiv. binary) non-indexed products of logics. Moreover, every $\vdash \in \mathbb{K}$ is a compatible expansion of a finitely presentable $\vdash' \in \mathbb{K}$.*
(iii) *$\mathbb{K}^{\dagger}$ is a filter of Log generated by elements of the form $[\![\vdash]\!]$ where $\vdash$ is finitely presentable and finitely equivalential, and $\mathbb{K} = \{\vdash \colon [\![\vdash]\!] \in \mathbb{K}^{\dagger}\}$.*

Proof. Part (i)⇒(iii) is proved as its alter ego in [16, Thm. 2.2].

(iii)⇒(ii): The proof of this part is also analogous to that of [16, Thm. 2.2]. The only substantial difference is that here we need to show that every logic $\vdash \in \mathbb{K}$ is a compatible expansion of a finitely presentable $\vdash' \in \mathbb{K}$. To this end, consider a logic $\vdash \in \mathbb{K}$. By the assumption there is a finitely presentable and finitely equivalential logic $\vdash_{fin}$ such that $\vdash_{fin} \leqslant \vdash$. The logic $\vdash_{fin}$ has a finite set of congruence formulas $\Delta(x,y)$ and a finite axiomatization

$$\begin{array}{ccccc} \gamma_1^1 & \dots & \gamma_1^k & \rhd & \varphi_1 \\ \vdots & & \vdots & & \vdots \\ \gamma_m^1 & \dots & \gamma_m^k & \rhd & \varphi_m. \end{array}$$

Now, since $\vdash_{fin} \leqslant \vdash$, the logic $\vdash$ is term-equivalent to a compatible expansion $\vdash'$ of $\vdash_{fin}$ [15, Prop. 3.8]. Then let $\boldsymbol{\tau}$ be the interpretation of $\vdash'$ into $\vdash$ witnessing the term-equivalence of these logics. Moreover, let $\mathscr{L}^{\dagger}$ be the set of basic operations of $\vdash$ occurring in the formulas $\{\boldsymbol{\tau}(*) \colon * \in \mathscr{L}_{\vdash_{fin}}\}$. We define a logic $\vdash_{\dagger}$ in the language $\mathscr{L}^{\dagger}$ by means of the following calculus:

$$\boldsymbol{\tau}(\gamma_i^1), \dots, \boldsymbol{\tau}(\gamma_i^k) \rhd \boldsymbol{\tau}(\varphi_i) \tag{26}$$

$$\emptyset \rhd \boldsymbol{\tau}[\Delta](x,x) \tag{27}$$

$$x, \boldsymbol{\tau}[\Delta(x,y)] \rhd y \tag{28}$$

$$\boldsymbol{\tau}[\Delta](x_1,y_1) \cup \dots \cup \boldsymbol{\tau}[\Delta](x_n,y_n) \rhd \boldsymbol{\tau}[\Delta](*(x_1,\dots,x_n), *(y_1,\dots,y_n)) \tag{29}$$

for every $i \leqslant m$ and $* \in \mathscr{L}^{\dagger}$. Clearly, $\vdash_{\dagger}$ is finitely presentable. Moreover, it is finitely equivalential with set of congruence formulas $\boldsymbol{\tau}[\Delta(x,y)]$ by [15, Thm. 2.7].

We shall see that $\vdash$ is a compatible expansion of $\vdash_{\dagger}$. To this end, consider a matrix $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{\equiv}(\vdash) = \mathbb{R}(\mathsf{Mod}(\vdash))$. The restriction $\boldsymbol{\tau}_{\restriction}$ of $\boldsymbol{\tau}$ to $\mathscr{L}_{\vdash_{fin}}$ is an interpretation of $\vdash_{fin}$ into $\vdash$. As a consequence, the proof of [15, Prop. 6.1(i)] guarantees that $\boldsymbol{\tau}[\Delta]$ is a set of congruence formulas for $\vdash$. In particular, this implies that for every pair of different $a, c \in A$ there is a formula $\varphi(x,y) \in \boldsymbol{\tau}[\Delta]$ such that $\varphi^{\boldsymbol{A}}(a,a) \in F$ and $\varphi^{\boldsymbol{A}}(a,c) \notin F$. By [15, Prop. 2.2(i)] the $\mathscr{L}^{\dagger}$-reduct of $\langle \boldsymbol{A}, F \rangle$ is reduced.

Since $\boldsymbol{\tau}[\Delta]$ is a set of congruence formulas for $\vdash$, the rules (27, 28, 29) are valid in $\vdash$ [15, Thm. 2.7]. Similarly, since the rules $\gamma_i^1, \dots, \gamma_i^k \rhd \varphi_i$ are valid in $\vdash_{fin}$ and $\boldsymbol{\tau}_{\upharpoonright}$ is an interpretation of $\vdash_{fin}$ into $\vdash$, we can apply [15, Prop. 3.3] obtaining the rules in (26) are valid in $\vdash$. Thus the $\mathscr{L}^{\dagger}$-reduct of $\langle \boldsymbol{A}, F \rangle$ is a (reduced) model of $\vdash_{+}$. Hence $\vdash$ is a compatible expansion of $\vdash_{+}$.

To conclude the proof, it will be enough to show that $\vdash_{+} \in \mathbb{K}$. To this end, observe that $\boldsymbol{\tau}_{\upharpoonright}$ can be regarded as a translation of $\vdash_{fin}$ into $\vdash_{+}$. We shall see that $\boldsymbol{\tau}_{\upharpoonright}$ is also an interpretation. To this end, consider $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{\equiv}(\vdash_{+}) = \mathbb{R}(\mathsf{Mod}(\vdash_{+}))$. The fact that $\langle \boldsymbol{A}, F \rangle$ is a model of the rules (26) guarantees that $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F \rangle \in \mathsf{Mod}(\vdash_{fin})$. Then consider two distinct elements $a, c \in A$. Since $\boldsymbol{\tau}[\Delta]$ is a set of congruence formulas for $\vdash_{+}$, there is $\varphi(x, y) \in \Delta$ such that

$$\varphi^{\boldsymbol{A}^{\boldsymbol{\tau}}}(a, a) = \boldsymbol{\tau}(\varphi)^{\boldsymbol{A}}(a, a) \in F \text{ and } \varphi^{\boldsymbol{A}^{\boldsymbol{\tau}}}(a, c) = \boldsymbol{\tau}(\varphi)^{\boldsymbol{A}}(a, c) \notin F.$$

Hence we conclude that $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F \rangle$ is reduced. As a consequence, $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F \rangle \in \mathsf{Mod}^{\equiv}(\vdash_{fin})$. Thus $\boldsymbol{\tau}_{\upharpoonright}$ is a translation of $\vdash_{fin}$ into $\vdash_{+}$. Since $\vdash_{fin} \in \mathbb{K}$ and $\mathbb{K}^{\dagger}$ is an upset of Log, this implies $[\![\vdash_{+}]\!] \in \mathbb{K}^{\dagger}$. Together with the fact that $\mathbb{K} = \{\vdash : [\![\vdash]\!] \in \mathbb{K}^{\dagger}\}$, this yields $\vdash_{+} \in \mathbb{K}$.

(ii)⇒(i): First we identify algebraic languages with their types. Under this identification there are only countably many finitely presentable logics. Then let $\{\vdash^{n} : n \in \omega\}$ be an enumeration of all finitely presentable logics in $\mathbb{K}$. By assumption each $\vdash^{n}$ is also finitely equivalential. Then we can apply Theorem 2.4, obtaining that for every $n \in \omega$ there is a finitely presentable and finitely equivalential logic $\vdash_{n}$ that is term-equivalent to $\vdash^{0} \otimes \cdots \otimes \vdash^{n}$. We set $\Phi := \{\vdash_{n} : n \in \omega\}$. Observe that Φ is a finitely presentable Leibniz condition. It is clear that if $n \leqslant m$, then $\vdash_{m} \leqslant \vdash_{n}$ (see [15, Thm. 4.6] if necessary). Moreover, from the assumption we know that the logics $\vdash^{0}, \dots, \vdash^{n} \dots$ are finitely presentable and finitely equivalential. By Theorem 2.4 we conclude that $\vdash_{n}$ is finitely presentable and finitely equivalential for every $n \in \omega$. As a consequence, Φ is a finitely presentable Leibniz condition.

It only remains to prove that $\mathrm{Log}(\Phi) = \mathbb{K}$. If $\vdash \in \mathrm{Log}(\Phi)$, then there is $n \in \omega$ such that $\vdash_{n} \leqslant \vdash$. By [15, Prop. 3.8] $\vdash$ is term-equivalent to a compatible expansion of a logic term-equivalent to a finite non-indexed product of elements of $\mathbb{K}$. Thus, by the assumption we conclude that $\vdash \in \mathbb{K}$. Conversely, consider $\vdash \in \mathbb{K}$. By the assumption there is $n \in \omega$ such that $\vdash$ is a compatible expansion of $\vdash^{n}$, whence $\vdash^{n} \leqslant \vdash$. Since $\vdash_{n} \leqslant \vdash^{n}$, we conclude that $\vdash_{n} \leqslant \vdash$, whence $\vdash \in \mathrm{Log}(\Phi)$. ⊠

In order to review some examples of finitely presentable Leibniz classes, recall that a logic $\vdash$ is said to be *finitely algebraizable* [10] if it is finitely equivalential and there is a finite set of equations $E(x)$ for such that every $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{\equiv}(\vdash)$ and $a \in A$,

$$a \in F \Longleftrightarrow \boldsymbol{A} \vDash E(a).$$

In this case, E is called a set of *defining equations* for $\vdash$. Similarly, $\vdash$ is said to be *finitely regularly algebraizable* if it is assertional and finitely equivalential [3, 8, 10]. It is easy to see that finitely regularly algebraizable logics are finitely algebraizable.

Theorem 4.3. *A logic $\vdash$ is finitely algebraizable if and only if there are a finite non-empty set of formulas $\Delta(x,y)$ and a finite set of equations $E(x)$ such that*

$$\begin{aligned}\emptyset &\vdash \Delta(x,x)\\ x, \Delta(x,y) &\vdash y\\ \Delta(x_1,y_1),\dots,\Delta(x_n,y_n) &\vdash \Delta(*(x_1,\dots,x_n), *(y_1,\dots,y_n))\\ x &\vdash \Delta(\varepsilon,\delta)\\ \bigcup\{\Delta(\varphi,\psi)\colon \varphi\approx\psi\in E(x)\} &\vdash x\end{aligned}$$

for every n-ary connective $*$, and $\varepsilon \approx \delta \in E(x)$. In this case, Δ and E are respectively a set of congruence formulas and a set of defining equations for $\vdash$. Moreover, $\vdash$ is finitely regularly algebraizable if and only if, in addition to the above requirements, $x, y \vdash \Delta(x,y)$.

Proof. See for instance [10, Thm. 3.21] and (essentially) [10, Thm. 3.52]. $\boxtimes$

Proposition 4.4. *Finitely equivalential, finitely algebraizable, and finitely regularly algebraizable logics form finitely presentable Leibniz classes.*

Proof. We detail the proof for finitely algebraizable logics only, since the other cases are analogous. For every $n \in \omega$, let $\mathscr{L}_n$ be the language consisting of the binary connectives $\{\multimap_k\colon k \leqslant n\}$ and the unary connectives $\{\Box_k\colon k\leqslant n\}\cup\{\Diamond_k\colon k\leqslant n\}$. Moreover, set

$$\Delta_n(x,y) \coloneqq \{x \multimap_k y\colon k\leqslant n\} \text{ and } E_n(x) \coloneqq \{\Box_k x\approx \Diamond_k x\colon k\leqslant n\}.$$

Let $\vdash^n_{\mathsf{A}}$ be the logic in the language $\mathscr{L}_n$ formulated in countably many variables and axiomatized by the following rules

$$\begin{aligned}\emptyset &\rhd \Delta_n(x,x)\\ x,\Delta_n(x,y) &\rhd y\\ \Delta_n(x_1,y_1),\dots,\Delta_n(x_m,y_m) &\rhd \Delta_n(*(x_1,\dots,x_m), *(y_1,\dots,y_m))\\ x &\rhd \Delta_n(\Box_k x, \Diamond_k x)\\ \Delta_n(\Box_0 x,\Diamond_0 x)\cup\dots\cup\Delta_n(\Box_n x,\Diamond_n x) &\rhd x\end{aligned}$$

for every $* \in \mathscr{L}_n$ and $k\leqslant n$. By Theorem 4.3 the logic $\vdash^n_{\mathsf{A}}$ is finitely algebraizable with set of congruence formulas $\Delta_n(x,y)$ and set of defining equations $E_n(x)$.

The sequence $\Phi \coloneqq \{\vdash^n_{\mathsf{A}}\colon n\in\omega\}$ is a finitely presentable Leibniz class. To prove this, observe that the members of Φ are clearly finitely presentable and finitely equivalential. Then consider two naturals $n \leqslant m$. Let $\boldsymbol{\tau}$ be the translation of $\mathscr{L}_m$ into $\mathscr{L}_n$ defined by the following rule: for every $k\leqslant m$,

$$\Diamond_k \longmapsto \Diamond_{\min\{k,n\}} x \text{ and } \multimap_k \longmapsto x \multimap_{\min\{k,n\}} y.$$

We shall see that $\boldsymbol{\tau}$ is an interpretation of $\vdash^m_{\mathsf{A}}$ into $\vdash^n_{\mathsf{A}}$. To this end, consider $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{\equiv}(\vdash^n_{\mathsf{A}}) = \mathbb{R}(\mathsf{Mod}^{\equiv}(\vdash^n_{\mathsf{A}}))$. Set

$$\boldsymbol{\tau}[E_m] \coloneqq \{\boldsymbol{\tau}(\varepsilon)\approx\boldsymbol{\tau}(\delta)\colon \varepsilon\approx\delta\in E_m\}.$$

Since $\boldsymbol{\tau}[\Delta_m] = \Delta_n$ and $\boldsymbol{\tau}[E_m] = E_n$, it is easy to see that $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F\rangle \in \mathsf{Mod}(\vdash^m_{\mathsf{A}})$. Then consider two distinct elements $a, c \in A$. Since Δ_n is a set of congruence formulas for $\vdash^n_{\mathsf{A}}$, there is $\varphi(x,y)\in\Delta_n$ such that $\varphi^{\boldsymbol{A}}(a,a)\in F$ and $\varphi^{\boldsymbol{A}}(a,c)\notin F$, whence

$$\varphi^{\boldsymbol{A}^{\boldsymbol{\tau}}}(a,a) = \boldsymbol{\tau}(\varphi)^{\boldsymbol{A}}(a,a) = \varphi^{\boldsymbol{A}}(a,a)\in F \text{ and } \varphi^{\boldsymbol{A}^{\boldsymbol{\tau}}}(a,c) = \boldsymbol{\tau}(\varphi)^{\boldsymbol{A}}(a,c) = \varphi^{\boldsymbol{A}}(a,c)\notin F.$$

By [15, Prop. 2.2(i)] we conclude that $\langle a, c\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{A}^{\boldsymbol{\tau}}} F$, whence the matrix $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F\rangle$ is reduced. As a consequence, $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F\rangle \in \mathsf{Mod}^{\equiv}(\vdash^m_{\mathsf{A}})$ and, therefore, $\boldsymbol{\tau}$ is an interpretation. Thus Φ is a finitely presentable Leibniz class.

It only remains to prove that $\mathrm{Log}(\Phi)$ is the class of finitely algebraizable logics. First consider $\vdash\, \in \mathrm{Log}(\Phi)$. There is $n \in \omega$ such that $\vdash^n_{\mathsf{A}} \leqslant\, \vdash$. Since $\vdash^n_{\mathsf{A}}$ is finitely equivalential, we can apply condition (i) of Lemma 2.3 obtaining that so is $\vdash$. Moreover, let $\boldsymbol{\tau}$ be an interpretation of $\vdash^n_{\mathsf{A}}$ into $\vdash$. Then consider $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{\equiv}(\vdash)$ and $a \in F$. Since $E_n(x)$ is a set of defining equations for $\vdash^n_{\mathsf{A}}$ and $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F\rangle \in \mathsf{Mod}^{\equiv}(\vdash^n_{\mathsf{A}})$,

$$a \in F \iff \boldsymbol{A}^{\boldsymbol{\tau}} \vDash E_n(a) \iff \boldsymbol{A} \vDash \boldsymbol{\tau}[E_n](a).$$

As a consequence, $\boldsymbol{\tau}[E_n]$ is a set of defining equations for $\vdash$. We conclude that $\vdash$ is finitely algebraizable.

Conversely, consider a finitely algebraizable logic $\vdash$. By Theorem 4.3 there are finite sets $\Delta(x, y)$ and $E(x)$, respectively, of congruence formulas and of defining equations for $\vdash$, satisfying the rules in the theorem. Consider a natural $n \geqslant \max\{|\Delta|, |E|\}$ and let $\boldsymbol{\tau}$ be a translation of $\mathscr{L}_n$ into $\mathscr{L}_\vdash$ such that $\boldsymbol{\tau}[\Delta_n] = \Delta$ and $\boldsymbol{\tau}[E_n] = E$. We shall see that $\boldsymbol{\tau}$ is an interpretation of $\vdash^n_{\mathsf{A}}$ into $\vdash$. To this end, consider $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{\equiv}(\vdash)$. The fact that $\Delta(x, y)$ and $E(x)$ satisfy the rules in Theorem 4.3 implies $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F\rangle \in \mathsf{Mod}(\vdash^n_{\mathsf{A}})$. Moreover, the matrix $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F\rangle$ is shown to be reduced with an argument similar to the one adopted in the proof that Φ is a finitely presentable Leibniz condition. Hence we conclude that $\langle \boldsymbol{A}^{\boldsymbol{\tau}}, F\rangle \in \mathbb{R}(\mathsf{Mod}(\vdash^n_{\mathsf{A}})) \subseteq \mathsf{Mod}^{\equiv}(\vdash^n_{\mathsf{A}})$. This shows that $\boldsymbol{\tau}$ is an interpretation of $\vdash^n_{\mathsf{A}}$ into $\vdash$, whence $\vdash\, \in \mathrm{Log}(\Phi)$. ⊠

Remark 4.5. Interestingly enough, it can be shown that the finitely presentable Leibniz classes individuated in Proposition 4.4 are not Leibniz classes in the sense of [16]. As an exemplification, we detail a proof of the fact that neither finitely equivalential nor finitely algebraizable logics form a Leibniz class, leaving the case of finitely regularly algebraizable logics to the reader.

Consider the finitely algebraizable logics $\vdash^n_{\mathsf{A}}$ defined in the proof of Proposition 4.4. We shall see that the non-indexed product $\bigotimes_{n\in\omega} \vdash^n_{\mathsf{A}}$ is not finitely equivalential. Suppose the contrary, with a view to contradiction. Then the logic $\bigotimes_{n\in\omega} \vdash^n_{\mathsf{A}}$ has a finite set of congruence formulas

$$\Delta(x, y) = \{\varphi_0, \dots, \varphi_m\}.$$

From the definition of $\bigotimes_{n\in\omega} \vdash^n_{\mathsf{A}}$ it follows that

$$\nabla_k(x, y) \coloneqq \{\varphi_0(k), \dots, \varphi_m(k)\}$$

is a set of congruence formulas with parameters for $\vdash^k_{\mathsf{A}}$, for every $k \in \omega$. In particular, ∇_{m+1} is a set of congruence formulas for $\vdash^{m+1}_{\mathsf{A}}$.

On cardinality grounds, there are $i, j \leqslant m+1$ such that $\multimap_i$ and $\diamond_j$ are not the main connective of any formula in ∇_{m+1}. Moreover, the elements of ∇_{m+1} are complex formulas, i.e. not variables. This is justified as follows. The fact that $\nabla_{m+1}(x, y)$ is a set of congruence formulas for $\vdash^{m+1}_{\mathsf{A}}$ implies that $\nabla_{m+1}(x, x)$ is a set of theorems of $\vdash^{m+1}_{\mathsf{A}}$. As $\vdash^{m+1}_{\mathsf{A}}$ is not inconsistent, the variable x is not one of its theorems, whence $x \notin \nabla_{m+1}(x, x)$. Therefore $\nabla_{m+1}(x, y)$ does not contain any variable.

Then consider the $\mathscr{L}_{m+1}$-algebra $\boldsymbol{A}$ with universe $\{0,1\}$ and operations defined as follows for every $a, c \in A$, and $k \leqslant m+1$:

$$a \multimap_k c := \begin{cases} 1 & \text{if } p = q \text{ or } k \neq i \\ 0 & \text{if } p \neq q \text{ and } k = i. \end{cases}$$

and

$$\Box_k a := 1 \quad \Diamond_k a := \begin{cases} 1 & \text{if } k \neq j \text{ or } a = 1 \\ 0 & \text{if } k = j \text{ and } a = 0. \end{cases}$$

Clearly, for every $a, c \in A$,

$$a = c \iff \Delta^{\boldsymbol{A}}_{m+1}(a, c) \subseteq \{1\}$$
$$a = 1 \iff \boldsymbol{A} \vDash E_{m+1}(a).$$

Together with the definition of $\vdash^{m+1}_{\mathsf{A}}$, this yields $\langle \boldsymbol{A}, \{1\} \rangle \in \mathsf{Mod}(\vdash^{m+1}_{\mathsf{A}})$. Moreover, the first of the conditions in the display guarantees that $\langle \boldsymbol{A}, \{1\} \rangle$ is reduced, whence $\langle \boldsymbol{A}, \{1\} \rangle \in \mathbb{R}(\mathsf{Mod}(\vdash^{m+1}_{\mathsf{A}}))$. Since ∇_{m+1} is a set of congruence formulas for $\vdash^{m+1}_{\mathsf{A}}$, this implies that for every $a, c \in A$,

$$a = c \iff \nabla^{\boldsymbol{A}}_{m+1}(a, c) \subseteq \{1\}.$$

But the definition of $\boldsymbol{A}$ and the fact that ∇_{m+1} is a set of complex formulas, whose principal connectives are neither $\multimap_i$ nor $\Diamond_j$, imply $\nabla^{\boldsymbol{A}}_{m+1}(a, c) \subseteq \{1\}$ for every $a, c \in A$, whence $\boldsymbol{A}$ should be the trivial algebra, which is false. Hence we reached a contradiction, as desired. We conclude that $\bigotimes_{n \in \omega} \vdash^n_{\mathsf{A}}$ is not finitely equivalential.

This shows that the classes of finitely equivalential logics and of finitely algebraizable logics are not closed under the formation of infinite non-indexed products. Thus by [16, Thm. 2.2] they are not Leibniz classes. ⊠

Remark 4.6. Similarly, not all Leibniz classes are finitely presentable. For instance, equivalential logics were shown to form a Leibniz class in [16, Rmk. 2.4]. On the other hand, it is well known that there are equivalential logics that are not finitely equivalential. By condition (ii) of Theorem 4.2, this implies that the Leibniz class of equivalential logics is not finitely presentable. ⊠

We refer to the poset of all finitely presentable Leibniz classes ordered under inclusion as to the *finite companion of the Leibniz hierarchy*. In the light of the above remarks, the Leibniz hierarchy as defined in [16], and its finite companion are incomparable. As detailed in [16, Sec. 4,5, and 6], the order-theoretic study of the finite companion of the Leibniz hierarchy may shed light on the importance of certain finitely presentable Leibniz classes in abstract algebraic logic (cf. also [1, 11, 24, 30]). Here we sketch only some very basic ideas in this direction.

A non-empty class of finitely equivalential logics is said to be a *filter class* if it is closed under term-equivalence, compatible expansions, and finite non-indexed products. Equivalently, filter classes are the classes $\mathbb{K}$ of finitely equivalential logics such that $\mathbb{K}^{\dagger}$ is a filter of Log and $\mathbb{K} = \{\vdash : [\![\vdash]\!] \in \mathbb{K}^{\dagger}\}$. Recall from Theorem 4.2 that finitely presentable Leibniz classes are special filter classes. Moreover, filter classes are easily seen to form a lattice, whose meets are intersections. Accordingly, a filter class $\mathbb{K}$ is said to be *meet-prime* among filter classes if for every pair of filters classes $\mathbb{K}_1$ and $\mathbb{K}_2$,

$$\text{if } \mathbb{K}_1 \cap \mathbb{K}_2 \subseteq \mathbb{K}, \text{ then either } \mathbb{K}_1 \subseteq \mathbb{K} \text{ or } \mathbb{K}_2 \subseteq \mathbb{K}.$$

Equivalently, $\mathbb{K}$ is meet-prime among filter classes if there is no pair of finitely equivalential logics $\vdash_1, \vdash_2 \notin \mathbb{K}$ such that $\vdash \in \mathbb{K}$, for every logic $\vdash$ such that $\vdash_1, \vdash_2 \leqslant \vdash$.

Theorem 4.7. *The finitely presentable Leibniz classes of finitely equivalential, and finitely regularly algebraizable logics are meet-prime among filter classes.*

Proof. The statement for the class of finitely equivalential logics is straightforward. Then we consider the case of finitely regularly algebraizable logics. Consider two finitely equivalential logics $\vdash_1$ and $\vdash_2$ that are not assertional. It will be enough to construct a logic $\vdash$ that is not assertional and in which $\vdash_1$ and $\vdash_2$ are interpretable. The fact that such a logic exists is an immediate consequence of [16, Thm. 5.11]. ⊠

Problem 1. Is the finitely presentable Leibniz class of finitely algebraizable logics meet-prime among filter classes?

5. Relations with Maltsev classes

We close this work by discussing the relations between the Maltsev hierarchy and the finite companion of the Leibniz hierarchy. To this end, recall that a variety K is *interpretable* [28] into another variety V, when V is term-equivalent to some variety V^* whose reducts (in a smaller signature) belong to K, in which case we write $\mathsf{K} \leqslant \mathsf{V}$. Then a *Maltsev condition* [12] is a sequence $\Phi = \{\mathsf{K}_n \colon n \in \omega\}$ of finitely presentable varieties such that

$$\text{if } n \leqslant m \text{, then } \mathsf{K}_m \leqslant \mathsf{K}_n.$$

In this case, we set

$$\mathrm{Var}(\Phi) := \{\mathsf{K} \colon \mathsf{K} \text{ is a variety and } \mathsf{K}_n \leqslant \mathsf{K} \text{ for some } n \in \omega\}.$$

A class $\mathbb{M}$ of varieties is a *Maltsev class* if $\mathbb{M} = \mathrm{Var}(\Phi)$ for some Maltsev condition Φ. The *Maltsev hierarchy* is the poset of Maltsev classes ordered under inclusion.

Recall from [15, Sec. 8] that the equational consequence $\vDash_{\mathsf{K}}$ relative to a variety K can be viewed as a 2-deductive system in the sense of [4]. Moreover, a variety K is finitely presentable exactly when $\vDash_{\mathsf{K}}$ is finitely presentable as a two deductive system, i.e. expressed in a finite language and axiomatized by finitely many finite rules. Similarly, a variety K is interpretable into another one V exactly when the 2-deductive system $\vDash_{\mathsf{K}}$ is interpretable into $\vDash_{\mathsf{V}}$.

The notion of a finitely presentable Leibniz class can be naturally extended to arbitrary two-deductive systems. More precisely, a *finitely presentable Leibniz class* of 2-deductive systems is a class of 2-deductive system satisfying condition (ii) of Theorem 4.2 once "logic" is replaced by "2-deductive system" (see [15, Sec. 8] if necessary). The following result explains the relation between Maltsev classes and finitely presentable Leibniz classes of 2-deductive systems.

Theorem 5.1. *A class $\mathbb{M}$ of varieties is a Maltsev class if and only if there is a finitely presentable Leibniz class $\mathbb{K}$ of 2-deductive systems such that*

$$\mathbb{M} = \{\mathsf{K} \colon \mathsf{K} \textit{ is a variety and } \vDash_{\mathsf{K}} \in \mathbb{K}\}.$$

Proof sketch. If $\vdash$ and $\vdash'$ are 2-deductive system, we denote by $\vdash \leqslant \vdash'$ the fact that $\vdash$ is interpretable into $\vdash'$.

First suppose that $\mathbb{M}$ is a Maltsev class. Then set

$$\mathbb{K} := \{\vdash \colon \vdash \text{ is a 2-deductive system and } \vDash_{\mathsf{K}} \leqslant \vdash \text{ for some } \mathsf{K} \in \mathbb{K}\}.$$

Clearly, $\mathbb{M} = \{\mathsf{K} : \mathsf{K} \text{ is a variety and } \vDash_{\mathsf{K}} \in \mathbb{K}\}$. Moreover, $\mathbb{K}$ is easily seen to be a finitely presentable Leibniz class of 2-deductive systems.

Conversely, suppose that $\mathbb{M}$ is a class of varieties and that there is a finitely presentable Leibniz class $\mathbb{K}$ of 2-deductive systems satisfying the display in the statement. As in the proof of Theorem 4.2, we identify the algebraic languages with their types. Under this identification there are only countably many finitely presentable varieties. We consider an enumeration $\{\mathsf{V}_n : n \in \omega\}$ of the finitely presentable varieties V_n such that $\vDash_{\mathsf{V}_n} \in \mathbb{K}$. For every $n \in \omega$, we set

$$\mathsf{W}_n := \mathsf{V}_0 \bigotimes \cdots \bigotimes \mathsf{V}_n.$$

By Corollary 2.5 the variety W_n is term-equivalent to a finitely presentable variety K_n. Then

$$\Phi := \{\mathsf{K}_n : n \in \omega\}$$

is a Maltsev condition. To conclude the proof, it suffices to show $\mathrm{Var}(\Phi) = \mathbb{M}$.

The inclusion from left to right is left as an exercise. To prove the other inclusion, consider a variety $\mathsf{K} \in \mathbb{M}$. By the assumption, $\vDash_{\mathsf{K}} \in \mathbb{K}$. Since $\mathbb{K}$ is a finitely presentable Leibniz class, $\vDash_{\mathsf{K}}$ is a compatible expansion of a finitely presentable 2-deductive system $\vdash$ in $\mathbb{K}$. Since $\vdash$ is finitely presentable, it is axiomatized by a finite set of finite rules

$$\begin{array}{ccccc}
\langle \varepsilon_1^1, \delta_1^1 \rangle & \dots & \langle \varepsilon_1^k, \delta_1^k \rangle & \rhd & \langle \varphi_1, \psi_1 \rangle \\
\vdots & & \vdots & & \vdots \\
\langle \varepsilon_m^1, \delta_m^1 \rangle & \dots & \langle \varepsilon_m^k, \delta_m^k \rangle & \rhd & \langle \varphi_m, \psi_m \rangle.
\end{array}$$

Moreover, since K is a variety, the 2-deductive system $\vDash_{\mathsf{K}}$ is axiomatized by the rules

$$\emptyset \rhd \langle x, x \rangle \tag{30}$$
$$\langle x, y \rangle \rhd \langle y, x \rangle \tag{31}$$
$$\langle x, y \rangle, \langle y, z \rangle \rhd \langle x, z \rangle \tag{32}$$
$$\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle \rhd \langle *(x_1, \dots, x_n), *(y_1, \dots, y_n) \rangle \tag{33}$$
$$\emptyset \rhd \langle \varepsilon, \delta \rangle \tag{34}$$

for every basic n-ary operation $*$ of K, and every equation $\varepsilon \approx \delta$ valid in K (see for instance [7, Sec. 1.2]).

Now, since $\vDash_{\mathsf{K}}$ is a compatible expansion of $\vdash$, the rules axiomatizing $\vdash$ are also valid in $\vDash_{\mathsf{K}}$. Since they are finitely many, they can also be derived from a finite set Σ of the rules axiomatizing $\vDash_{\mathsf{K}}$. Then consider the finite language

$$\mathscr{L} := \{* : * \text{ is a basic symbol occurring in } \Sigma\}.$$

Moreover, let $\vdash'$ be the 2-deductive system in the language $\mathscr{L}$ formulated in countably many variables and axiomatized by the rules in Σ plus the rules (30, 31, 32) and (33) restricted to the connectives $* \in \mathscr{L}$. Clearly, $\vdash'$ is an extension of $\vdash$, whence $\vdash' \in \mathbb{K}$ [16, Rmk. 2.5]. Moreover, $\vdash'$ is finitely presentable.

Observe that $\vdash'$ is of the form $\vDash_{\mathsf{V}}$ for some variety V (see [7, Sec. 1.2] if necessary). Thus V is finitely presentable, since so is $\vDash_{\mathsf{V}}$. Together with the fact that $\vDash_{\mathsf{V}} = \vdash' \in \mathbb{K}$, this implies $\mathsf{V} = \mathsf{V}_n$ for some $n \in \omega$. As a consequence, $\mathsf{K}_n \leqslant \mathsf{V}$. Now, the fact that the rules axiomatizing $\vDash_{\mathsf{V}}$ hold in $\vDash_{\mathsf{K}}$ guarantees that the $\mathscr{L}$-reduct of K belong to V, whence

$\mathsf{V} \leqslant \mathsf{K}$. Hence we conclude that $\mathsf{K}_n \leqslant \mathsf{K}$ and, therefore, $\mathsf{K} \in \mathrm{Var}(\Phi)$. This establishes that $\mathbb{M} = \mathrm{Var}(\Phi)$, whence $\mathbb{M}$ is a Maltsev class. $\boxtimes$

Theorem 5.1 states that Maltsev classes can be though as finitely presentable Leibniz classes of 2-deductive systems, suitably restricted to equational consequences relative to varieties. This shows that the Leibniz and Maltsev hierarchies of abstract algebraic logic and universal algebra, respectively, are indeed two faces of the same coin. More precisely, the Maltsev hierarchy (i.e. the poset of Maltsev classes) can be embedded into the finite companion of the Leibniz hierarchy of 2-deductive systems.

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