A COMPLETION OF REDUCED COMMUTATIVE RINGS

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Abstract. A commutative ring is *reduced* when it can be embedded into a direct product of fields. While the category of reduced commutative rings plays a fundamental role in affine geometry, it exhibits several structural deficiencies: it admits nonregular monomorphisms and epimorphisms, lacks amalgamation, and is not equationally axiomatizable. In this paper, we simultaneously repair these defects via a canonical completion in which all monomorphisms become regular. This completion is obtained by adjoining weak inverses and weak prime roots, turning the class of reduced commutative rings into a discriminator variety. As a consequence, we obtain an explicit description of dominions in every class of reduced commutative rings containing all fields. This description is strikingly simple compared to that of dominions in the category of all commutative rings, as reflected in the Isbell–Mazet–Silver Zigzag Theorem.

1. Introduction

Reduced commutative rings play a central role in algebraic geometry, where they arise as coordinate rings of affine algebraic sets (see, e.g., [28, p. 48]). They are characterized by the absence of nonzero nilpotent elements or, equivalently, by their embeddability into direct products of fields (see, e.g., [31, 3.14 p. 23]). Despite its ubiquity, the class of reduced commutative rings exhibits significant structural deficiencies: it admits nonregular monomorphisms and epimorphisms, lacks amalgamation, and is not even equationally axiomatizable.[1] In this paper, we show that these defects can be simultaneously repaired by a completion in which all monomorphisms become regular (see [12, Sec. 11]). Remarkably, this completion is obtained by adjoining two simple kinds of unary operations, namely weak inverses and weak prime roots, and turns the class of reduced commutative rings into a discriminator variety. Furthermore, this completion is canonical, in the sense that it coincides with the unique (up to term equivalence) equational class in which monomorphisms are regular that, moreover, is isomorphic via the forgetful functor to a mono-reflective subcategory of the category of reduced commutative rings (see Corollary 6.2).

As a further consequence, we obtain a novel description of dominions in every class of reduced commutative rings containing all fields, such as the class of all integral domains, in terms of weak inverses and weak prime roots. The simplicity of this description contrasts sharply with the one of dominions in the ambient category of all commutative rings given by the Isbell–Mazet–Silver Zigzag Theorem (see [20, Thm. 1.1]), which combines results from [26, p. 2-07] and [29, Prop. 1.1].

Dominions and the regularity of monos and epis. By an *algebra* $\boldsymbol{A}$ we understand a set A endowed with a family of finitary operations on A. Familiar examples of algebras include monoids,

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[1] In the category of reduced commutative rings, the existence of nonregular epimorphisms is equivalent to the existence of nonsurjective epimorphisms.

groups, rings, and Boolean algebras. Two algebras are said to be *similar* when they share the same language (see, e.g., [4, Chap. 1.1] and [15, Chap. 1.3]). Every class K of similar algebras can be viewed as a category whose objects are the members of K and whose arrows are the homomorphisms between them. We recall that an arrow in K is said to be a *monomorphism* (resp. *epimorphism*) in K when it is left (resp. right) cancellable. A monomorphism (resp. epimorphism) in K is *regular* when it is the equalizer (resp. coequalizer) of a parallel pair of arrows in K (see, e.g., [1, Defs. 7.56 & 7.71]).

While reduced commutative rings cannot be axiomatized equationally, they can still be axiomatized by implications between finite sets of equations. Classes of algebras of this kind are known as *quasivarieties* and were introduced by Maltsev (see [25]). We recall that quasivarieties form bicomplete categories (see, e.g., [5, Prop. 9.4.8 & Thm. 9.4.14]) in which injective arrows coincide with monomorphisms and surjective arrows with regular epimorphisms (see, e.g., [27, p. 222] and [2, Rem. 5.13(2)]).

In quasivarieties, the demand that all monomorphisms (resp. epimorphisms) be regular can be formulated in terms of Isbell's dominions (see [19]), as we proceed to recall. Let K be a class of similar algebras. Given a subalgebra $\boldsymbol{A}$ of a member $\boldsymbol{B}$ of K (in symbols $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$), the *dominion* of $\boldsymbol{A}$ in $\boldsymbol{B}$ relative to K is the set

$$\begin{aligned}\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \{b \in B : g(b) = h(b) \text{ for every pair of homomorphisms}\\ g, h \colon \boldsymbol{B} \to \boldsymbol{C} \text{ with } \boldsymbol{C} \in \mathsf{K} \text{ such that } g{\restriction}_A = h{\restriction}_A\}.\end{aligned}$$

When K is a quasivariety, the following holds (see, e.g., [22, Prop. 6.1] and [19, p. 1]):

$$\begin{aligned}\text{every monomorphism in } \mathsf{K} \text{ is regular } &\iff \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = A \text{ for all } \boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K};\\ \text{every epimorphism in } \mathsf{K} \text{ is regular } &\iff \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) \neq B \text{ for all } \boldsymbol{A} < \boldsymbol{B} \in \mathsf{K},\end{aligned}$$

where $\boldsymbol{A} < \boldsymbol{B}$ means that $\boldsymbol{A}$ is a proper subalgebra of $\boldsymbol{B}$. Consequently, in the context of quasivarieties, the regularity of monomorphisms implies that of epimorphisms.

Regularizing monomorphisms. As we mentioned, our aim is to complete the category of reduced commutative rings so that every monomorphism becomes regular. Since the requirement that all monomorphisms be regular in a quasivariety K can be expressed in terms of dominions, solving this problem necessitates a detailed analysis of dominions, which is facilitated by the following concept (see [12, Sec. 3]).

An n-ary *operation* of K is a family $f = \langle f^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K}\rangle$, where each $f^{\boldsymbol{A}} \colon \mathsf{dom}(f^{\boldsymbol{A}}) \to A$ is a partial n-ary function on A with domain $\mathsf{dom}(f^{\boldsymbol{A}}) \subseteq A^n$ that is globally preserved by the homomorphisms between members of K. The latter means that for every homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $\langle a_1, \dots, a_n\rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ we have

$$\langle h(a_1), \dots, h(a_n)\rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \text{ and } h(f^{\boldsymbol{A}}(a_1, \dots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \dots, h(a_n)).$$

An n-ary operation f of K is said to be *implicit* when it is defined by some first order formula $\varphi(x_1, \dots, x_n, y)$, in the sense that for all $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \dots, a_n, b \in A$,

$$\boldsymbol{A} \vDash \varphi(a_1, \dots, a_n, b) \iff \langle a_1, \dots, a_n\rangle \in \mathsf{dom}(f^{\boldsymbol{A}}) \text{ and } f^{\boldsymbol{A}}(a_1, \dots, a_n) = b.$$

For instance, "taking multiplicative inverses" is an implicit operation of commutative rings because it is defined by the equation $xy \approx 1$ and ring homomorphisms preserve multiplicative inverses, when they exist. Implicit operations and dominions are connected as follows (see [3, Thm. 1] and [9, Thm. 3.2]). When K is a quasivariety and $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$, the dominion $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$ is the result of closing A in B under all the implicit operations of K. Formally, $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$ consists of the elements

$b \in B$ for which there exist an implicit operation f of K and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \cap A^n$ such that $b = f^{\boldsymbol{B}}(a_1, \ldots, a_n)$.

This suggests a canonical method for completing a quasivariety K so that every monomorphism becomes regular. The overall strategy consists of adding enough implicit operations to K in such a way that the resulting expansion C is still a quasivariety and, moreover, satisfies the following closure property: if $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{C}$, then A is closed in B under the implicit operations of C. Consequently, for all $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{C}$ we have $\mathsf{d}_{\mathsf{C}}(\boldsymbol{A}, \boldsymbol{B}) = A$, which ensures that all monomorphisms in C are regular.

The result of such an expansion of K has been called a *Beth companion* of K in [12, Sec. 11]. For instance, the class of Abelian groups is a Beth companion of the quasivariety of cancellative commutative monoids (see [12, Thm. 11.9(i)]), obtained by adding the implicit operation of "taking inverses". While a quasivariety may lack a Beth companion (see, e.g., [12, Thm. 14.17] and [14, Thm. 6.1]), Beth companions are essentially unique when they exist because all Beth companions of a quasivariety K are term equivalent (see [12, Thm. 11.7]). In a slogan, Beth companions are the natural completions that regularize monomorphisms for quasivarieties.

The main result. Let RCR be the quasivariety of reduced commutative rings. In this paper, we show that RCR admits a Beth companion, obtained by adjoining the implicit operations of "taking weak inverses" and "taking weak prime roots" (see Theorem 6.1). In addition, this completion of RCR is canonical, in the sense that it coincides with the unique (up to term equivalence) quasivariety in which monomorphisms are regular that, moreover, is isomorphic via the forgetful functor to a mono-reflective subcategory of RCR (see Corollary 6.2). Beyond ensuring the regularity of monomorphisms (and hence of epimorphisms), it also repairs the failure of the amalgamation property and yields a discriminator variety (see Theorem 5.2 and Remark 6.11). Thus, a minimal expansion transforms RCR into a remarkably structured class.

In order to describe concretely the Beth companion of RCR, we recall that the characteristic of a field is either zero or prime (see, e.g., [24, p. 30]), and that every element a of a field of prime characteristic p has at most one p-th root that, when existing, will be denoted by $\sqrt[p]{a}$ (see, e.g., [24, F14 p. 71]). We say that a field $\boldsymbol{A}$ is *weakly rooted* when one of the following conditions holds:

(i) $\boldsymbol{A}$ has characteristic 0;
(ii) $\boldsymbol{A}$ has prime characteristic p and contains $\sqrt[p]{a}$ for every $a \in A$.

Besides the fields of characteristic zero, examples of weakly rooted fields include all finite fields and all algebraically closed fields (see Proposition 3.2). Given a prime p, the *weak p-root* of an element a of a weakly rooted field $\boldsymbol{A}$ is

$$r_p(a) = \begin{cases} \sqrt[p]{a} & \text{if } \boldsymbol{A} \text{ has characteristic } p; \\ 0 & \text{otherwise.} \end{cases}$$

In addition, the *weak inverse* of an element a of a field $\boldsymbol{A}$ is

$$a^* = \begin{cases} a^{-1} & \text{if } a \neq 0; \\ 0 & \text{if } a = 0, \end{cases}$$

where a^{-1} is the multiplicative inverse of a. Lastly, an *implicitly closed field* is an algebra $\langle A; +, \cdot, -, 0, 1, (\,)^*, \{r_p : p \text{ is prime}\}\rangle$, where $\langle A; +, \cdot, -, 0, 1\rangle$ is a weakly rooted field and $(\,)^*$ and r_p are the unary operations of taking weak inverses and weak p-roots, respectively.

Our main result states that the Beth companion of RCR is the class of all the algebras that can be embedded into a direct product of implicitly closed fields. We term these algebras *implicitly closed* *meadows* and show that they can be axiomatized by the axioms of commutative rings together with the equations

$$x \approx x^2 x^*, \quad x \approx x^{**}, \quad (r_p(x))^p \approx (1 - p^* p)x$$

for every prime p, where we denote by p the result of summing p-times the unit 1 (see Theorem 3.14).

Dominions. As a consequence, we obtain a novel characterization of dominions in every class $\mathsf{K} \subseteq \mathsf{RCR}$ containing all fields (see Theorem 7.3). In addition to RCR and the class of all fields, these comprise, for example, the class of all integral domains. Besides its wide applicability, an attractive feature of our characterization lies in its simplicity, which is in stark contrast with the more involved description of dominions in the category of all commutative rings given by the Isbell-Mazet-Silver Zigzag Theorem (see [19, Thm. 2.9], [20, Thm. 1.1], [26, p. 2-07], and [29, Prop. 1.1]).[2]

More precisely, for all $\boldsymbol{A} \in \mathsf{RCR}$ and prime ideals I of $\boldsymbol{A}$, let $\mathsf{acl}(\mathsf{frac}(\boldsymbol{A}/I))$ be the algebraic closure of the fraction field of $\boldsymbol{A}/I$, and $\mathsf{icf}_I(\boldsymbol{A})$ the unique implicitly closed field whose field reduct is $\mathsf{acl}(\mathsf{frac}(\boldsymbol{A}/I))$. We show that for every class $\mathsf{K} \subseteq \mathsf{RCR}$ that contains all fields and for all $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$,

$$\begin{aligned} \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \{b \in B : {} & \text{for every pair } I \text{ and } J \text{ of prime ideals of } \boldsymbol{B}, \\ & \langle b + I, b + J \rangle \text{ belongs to the subalgebra of } \mathsf{icf}_I(\boldsymbol{B}) \times \mathsf{icf}_J(\boldsymbol{B}) \\ & \text{generated by } \{\langle a + I, a + J \rangle : a \in A\}\}. \end{aligned}$$

This description becomes particularly simple when K is the class F of all fields, as for all $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{F}$ the dominion $\mathsf{d}_{\mathsf{F}}(\boldsymbol{A}, \boldsymbol{B})$ is the least subfield of the algebraic closure of $\boldsymbol{B}$ that contains A and, when $\boldsymbol{B}$ has characteristic p, is also closed under p-roots (see Corollary 7.4).

2. Field theory

2.1. **Algebras.** We recall that an *algebraic language* is a family $\mathscr{L}$ of function symbols together with a map $\mathsf{ar} \colon \mathscr{L} \to \mathbb{N}$ that associates an arity with every member of $\mathscr{L}$. Then, an *$\mathscr{L}$-algebra* is a structure $\boldsymbol{A} = \langle A; \{f^{\boldsymbol{A}} : f \in \mathscr{L}\}\rangle$, where A is set and $f^{\boldsymbol{A}}$ is a function on A of arity $\mathsf{ar}(f)$ for every $f \in \mathscr{L}$ (see, e.g., [4, Sec. 1.1] and [15, Sec. 1.3]). Two algebras are said to be *similar* when they have the same language. A class of similar algebras is called *elementary* when it can be axiomatized by first order formulas.

A *term* of a class of similar algebras K is a formal expression obtained by applying the function symbols of K to the set of variables. For instance, $x + (y \cdot -z)$ is a ring term. With every term $t(x_1, \dots, x_n)$ of K and $\boldsymbol{A} \in \mathsf{K}$ we associate a map $t^{\boldsymbol{A}} \colon A^n \to A$ that sends a tuple $\langle a_1, \dots, a_n\rangle \in A^n$ to the result of applying the interpretation of t in $\boldsymbol{A}$ to the elements $a_1, \dots, a_n$. For instance, if $t(x, y, z) = x + (y \cdot -z)$ and $\boldsymbol{A}$ is a ring, then $t^{\boldsymbol{A}} \colon A^3 \to A$ is the map defined as $t^{\boldsymbol{A}}(a, b, c) = a + (b \cdot -c)$ for all $a, b, c \in A$ (see, e.g., [4, Sec. 4.3]).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be a pair of similar algebras. We write $\boldsymbol{A} \leqslant \boldsymbol{B}$ to indicate that $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$. Moreover, when $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we denote the subalgebra of $\boldsymbol{B}$ with universe $h[A]$ by $h[\boldsymbol{A}]$. The subalgebra of $\boldsymbol{A}$ generated by some $X \subseteq A$ will be denoted by $\mathsf{Sg}^{\boldsymbol{A}}(X)$. We recall that

$$\mathsf{Sg}^{\boldsymbol{A}}(X) = \{t^{\boldsymbol{A}}(a_1, \dots, a_n) : t(x_1, \dots, x_n) \text{ is a term of } \boldsymbol{A} \text{ and } a_1, \dots, a_n \in A\}$$

(see, e.g., [4, Thm. 1.14]). Lastly, let $\mathbb{I}, \mathbb{H}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{u}}$ be the class operators of closure under isomorphisms, homomorphic images, subalgebras, direct products, and ultraproducts, respectively.

[2]We remark that our result does not follow from the Isbell-Mazet-Silver Zigzag Theorem because, when $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K} \subseteq \mathsf{K}'$ for a class of similar algebras K', the equality $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \mathsf{d}_{\mathsf{K}'}(\boldsymbol{A}, \boldsymbol{B})$ need not hold.

2.2. **Fields and integral domains.** Throughout this work, rings will be considered as algebras in the language $\mathscr{L} = \{+, \cdot, -, 0, 1\}$. Consequently, all rings will be assumed to be unital. When it exists, we denote the multiplicative inverse of an element a of a ring by a^{-1}.

Definition 2.1. A nontrivial commutative ring $\boldsymbol{A}$ is said to be:

(i) a *field* when every nonzero element of A has a multiplicative inverse;
(ii) an *integral domain* when for all $a, b \in A$ such that $ab = 0$ we have $0 \in \{a, b\}$ or, equivalently, when $\boldsymbol{A} \leqslant \boldsymbol{B}$ for some field $\boldsymbol{B}$ (see, e.g., [24, F7 p. 28]).

The classes of all fields and of all integral domains will be denoted, respectively, by F and ID.

Definition 2.2. A commutative ring $\boldsymbol{A}$ is said to be *reduced* when $a^2 = 0$ implies $a = 0$ for every $a \in A$ or, equivalently, when $\boldsymbol{A} \in \mathbb{ISP}(\mathsf{F})$ (see, e.g., [31, 3.14 p. 23]). We denote the class of reduced commutative rings by RCR.

By definition we have

$$\mathsf{ID} = \mathbb{S}(\mathsf{F}) \quad \text{and} \quad \mathsf{RCR} = \mathbb{ISP}(\mathsf{F}).$$

We recall that every ideal I of a commutative ring $\boldsymbol{A}$ induces a quotient ring $\boldsymbol{A}/I$.

Proposition 2.3 ([24, Def. 9, p. 41])**.** *Let I be an ideal of a commutative ring $\boldsymbol{A}$. Then I is prime if and only if $\boldsymbol{A}/I$ is an integral domain.*

We also recall that every integral domain $\boldsymbol{A}$ can be embedded into its field of fractions $\mathsf{frac}(\boldsymbol{A})$. To simplify the notation, from now on we will assume that $\boldsymbol{A} \leqslant \mathsf{frac}(\boldsymbol{A})$.

Proposition 2.4 ([24, F7 p. 28])**.** *Let $\boldsymbol{A} \in \mathsf{ID}$ and $\boldsymbol{B} \in \mathsf{F}$. For every embedding $h\colon \boldsymbol{A} \to \boldsymbol{B}$ there exists an embedding $g\colon \mathsf{frac}(\boldsymbol{A}) \to \boldsymbol{B}$ such that $g{\restriction}_A = h$.*

For every $n \in \mathbb{N}$ we define recursively a term nx in the language of rings by setting

$$0x = 0 \quad \text{and} \quad (n+1)x = nx + x.$$

Definition 2.5. The *characteristic* of a field $\boldsymbol{A}$ is the smallest $n \in \mathbb{Z}^+$ such that in $\boldsymbol{A}$ we have $n1 = 0$ if such an n exists and 0 otherwise.

Proposition 2.6 ([24, p. 30])**.** *The characteristic of a field is either prime or zero.*

2.3. **Field extensions.** We denote the ring of polynomials in variables $x_1, \dots, x_n$ with coefficients in a field $\boldsymbol{A}$ by $\boldsymbol{A}[x_1, \dots, x_n]$ and recall that $\boldsymbol{A}[x_1, \dots, x_n]$ is always an integral domain (see, e.g., [23, Remark p. 149]).

Definition 2.7. A *field extension* consists of a pair of fields $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$. Given a field extension $\boldsymbol{A} \leqslant \boldsymbol{B}$, an element $b \in B$ is said to be:

(i) *algebraic* over $\boldsymbol{A}$ when there exists a nonzero polynomial $p(x) \in \boldsymbol{A}[x]$ with coefficients in $\boldsymbol{A}$ such that $p(b) = 0$;
(ii) *transcendental* over $\boldsymbol{A}$ when it is not algebraic over $\boldsymbol{A}$.

Let $\boldsymbol{A} \leqslant \boldsymbol{B}$ be a field extension and $b \in B$ algebraic over $\boldsymbol{A}$. Then there exists a unique monic polynomial $\mu_b(x) \in \boldsymbol{A}[x]$, known as the *minimal polynomial* of b over $\boldsymbol{A}$, with $\mu_b(b) = 0$ that is of minimal degree among the members of $\boldsymbol{A}[x]$ with root b (see, e.g., [24, p.17]). It is well known that μ_b is irreducible (see, e.g., [24, F6 p. 26]). Lastly, given a field extension $\boldsymbol{A} \leqslant \boldsymbol{B}$ and $b_1, \dots, b_n \in B$, we denote the smallest subfield of $\boldsymbol{B}$ containing $A \cup \{b_1, \dots, b_n\}$ by $\boldsymbol{A}(b_1, \dots, b_n)$ (see, e.g., [24, p. 6]). Moreover, we write $\boldsymbol{A}(x_1, \dots, x_n)$ as a shorthand for $\mathsf{frac}(\boldsymbol{A}[x_1, \dots, x_n])$.

Proposition 2.8. *The following conditions hold for all field extensions $\boldsymbol{A} \leqslant \boldsymbol{B}$ and $b \in B$:*

(i) *if b is algebraic over $\boldsymbol{A}$ and c is a root of μ_b in some field $\boldsymbol{C}$ with $\boldsymbol{A} \leqslant \boldsymbol{C}$, there exists an isomorphism $h \colon \boldsymbol{A}(b) \to \boldsymbol{A}(c)$ such that*

$$h(b) = c \text{ and } h(a) = a \text{ for every } a \in A;$$

(ii) *if b is transcendental over $\boldsymbol{A}$, there exists an isomorphism $h \colon \boldsymbol{A}(b) \to \boldsymbol{A}(x)$ such that*

$$h(b) = x \text{ and } h(a) = a \text{ for every } a \in A.$$

Proof. See, e.g., [24, F1 p. 55] and [24, F9 p. 30]. ⊠

We will make use of the following concept.

Definition 2.9. A field $\boldsymbol{A}$ is said to be *algebraically closed* when every nonconstant polynomial in $\boldsymbol{A}[x]$ has a root in $\boldsymbol{A}$. Given a field extension $\boldsymbol{A} \leqslant \boldsymbol{B}$, we say that

(i) $\boldsymbol{A} \leqslant \boldsymbol{B}$ is *algebraic* when every member of $\boldsymbol{B}$ is algebraic over $\boldsymbol{A}$;
(ii) $\boldsymbol{B}$ is an *algebraic closure* of $\boldsymbol{A}$ when it is algebraically closed and $\boldsymbol{A} \leqslant \boldsymbol{B}$ is algebraic.

Theorem 2.10 ([24, Thm. 2 p. 57])**.** *Every field $\boldsymbol{A}$ has an algebraic closure that, moreover, is unique up to isomorphism, in the sense that if $\boldsymbol{B}$ and $\boldsymbol{C}$ are algebraic closures of $\boldsymbol{A}$, there exists an isomorphism $h \colon \boldsymbol{B} \to \boldsymbol{C}$ that fixes every member of $\boldsymbol{A}$.*

Consequently, we may talk about *the* algebraic closure of a field $\boldsymbol{A}$, which we denote by $\mathsf{acl}(\boldsymbol{A})$. We will rely on the next property of algebraic closures (see, e.g., [23, Thm. 9.23 p. 463]), for which we recall that every homomorphism between fields is an embedding (see, e.g., [24, F15 p. 42]).

Proposition 2.11. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{F}$. For every embedding $h \colon \boldsymbol{A} \to \boldsymbol{B}$ there exists an embedding $g \colon \mathsf{acl}(\boldsymbol{A}) \to \mathsf{acl}(\boldsymbol{B})$ such that $h = g{\restriction}_A$.*

Given a ring $\boldsymbol{A}$, the degree of a polynomial $p \in \boldsymbol{A}[x]$ will be denoted by $\mathsf{deg}(p)$. We will make use of separable extensions and perfect fields, which are usually defined in terms of splitting fields (see, e.g., [24, Chap. 7]). However, for the present purpose, it is convenient to adopt the following definition, which is equivalent because every splitting field of a field $\boldsymbol{A}$ embeds into the algebraic closure of $\boldsymbol{A}$ (see, e.g., [24, proof of F4 p. 61]).

Definition 2.12. Given a field $\boldsymbol{A}$, we say that

(i) an algebraic field extension $\boldsymbol{A} \leqslant \boldsymbol{B}$ is *separable* when the minimal polynomial μ_b of every $b \in B$ over $\boldsymbol{A}$ has $\mathsf{deg}(\mu_b)$ distinct roots in $\mathsf{acl}(\boldsymbol{A})$;
(ii) $\boldsymbol{A}$ is *perfect* when every algebraic extension of $\boldsymbol{A}$ is separable.

3. Implicitly closed meadows

3.1. **Field expansions.** Recall from Proposition 2.6 that the characteristic of a field is either zero or prime. An element a of a field of prime characteristic p has at most one p-th root that, when it exists, will be denoted by $\sqrt[p]{a}$ (see, e.g., [24, F14 p. 71]).

Definition 3.1. A field $\boldsymbol{A}$ is *weakly rooted* when one of the following conditions holds:

(i) $\boldsymbol{A}$ has characteristic 0;
(ii) $\boldsymbol{A}$ has prime characteristic p and contains $\sqrt[p]{a}$ for every $a \in A$.

The class of weakly rooted fields will be denoted by WRF.

Proposition 3.2. *Every finite or algebraically closed field is weakly rooted.*

Proof. From [24, Remark, p. 71] it follows that every finite field is weakly rooted. Then let $\boldsymbol{A}$ be an algebraically closed field. If $\boldsymbol{A}$ has characteristic zero, we are done. Then we consider the case where $\boldsymbol{A}$ has prime characteristic p. We need to prove that $\sqrt[p]{a}$ exists for every $a \in A$. To this end, consider $a \in A$. As $\boldsymbol{A}$ is algebraically closed, the polynomial $x^p - a$ has a root b in $\boldsymbol{A}$, i.e., $b^p = a$. Hence, we conclude that $b = \sqrt[p]{a}$. ⊠

From Theorem 2.10 and Proposition 3.2 we deduce the following.

Corollary 3.3. *Every field can be extended to a weakly rooted one.*

We will also make use of the following observation (see, e.g., [24, F11 p. 70 and F19 p. 73]).

Theorem 3.4. *Every weakly rooted field is perfect.*

We say that an algebra $\boldsymbol{A}$ is a *reduct* of an algebra $\boldsymbol{B}$ when there exists a sublanguage $\mathscr{L}$ of the language of $\boldsymbol{B}$ such that $\boldsymbol{A} = \langle B; \{f^{\boldsymbol{B}} : f \in \mathscr{L}\}\rangle$. In this case, we also say that $\boldsymbol{B}$ is an *expansion* of $\boldsymbol{A}$. A *field expansion* is an expansion of a field. The *characteristic* of a field expansion $\boldsymbol{A}$ is the characteristic of the field reduct of $\boldsymbol{A}$.

We will focus on field expansions obtained adding the following operations. Given a field $\boldsymbol{A}$, let $(\,)^*\colon A \to A$ be the map defined for every $a \in A$ as

$$a^* = \begin{cases} a^{-1} & \text{if } a \neq 0; \\ 0 & \text{if } a = 0. \end{cases} \tag{1}$$

The element a^* is said to be a *weak inverse* of a. Moreover, if $\boldsymbol{A}$ is weakly rooted field and p prime, let $r_p\colon A \to A$ be the map defined for every $a \in A$ as

$$r_p(a) = \begin{cases} \sqrt[p]{a} & \text{if } \boldsymbol{A} \text{ has characteristic } p; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We call $r_p(a)$ the *weak p-root* of a. When convenient, we write $a^{*\boldsymbol{A}}$ and $r_p^{\boldsymbol{A}}(a)$ (as opposed to a^* and $r_p(a)$ only) to stress that these elements are computed in $\boldsymbol{A}$.

Definition 3.5. An algebra $\langle A; +, \cdot, -, 0, 1, (\,)^*\rangle$ is said to be a *zero-totalized field* when $\langle A; +, \cdot, -, 0, 1\rangle$ is a field and $(\,)^*$ the unary operation in (1). The class of zero-totalized fields will be denoted by ZTF. A zero-totalized field is *weakly rooted* when its field reduct is weakly rooted.

In the context of zero-totalized weakly rooted fields, the operations r_p can be defined equationally, as we proceed to illustrate. With every prime p we associate the equation

$$\mathsf{root}_p(x, y) = y^p \approx (1 - p^* p)x$$

in the language of ZTF. Observe that the expression $1 - p^* p$ equals 1 in zero-totalized fields of characteristic p, and equals 0 otherwise.

Proposition 3.6. *Let $\boldsymbol{A} \in \mathsf{ZTF}$ be weakly rooted. For every prime p and $a, b \in A$,*

$$\boldsymbol{A} \vDash \mathsf{root}_p(a, b) \iff b = r_p(a).$$

Proof. We have two cases depending on whether $\boldsymbol{A}$ has characteristic p or not (see Proposition 2.6). First, suppose that $\boldsymbol{A}$ has characteristic p. We have

$$\boldsymbol{A} \vDash \mathsf{root}_p(a, b) \iff b^p = (1 - p^* p)a \iff b^p = a \iff b = \sqrt[p]{a} \iff b = r_p(a),$$

where the first equivalence above holds by the definition of the equation root_p, the second by the assumption that $\boldsymbol{A}$ has characteristic p, the third is straightforward, and the fourth holds by the definition of a weak p-root.

Lastly, we consider the case where $\boldsymbol{A}$ does not have characteristic p. We have

$$\boldsymbol{A} \vDash \mathsf{root}_p(a,b) \iff b^p = (1-p^*p)a \iff b^p = 0 \iff b = 0,$$

where the first equivalence above holds by the definition of the equation root_p, the second because $\boldsymbol{A}$ does not have characteristic p, and the third because fields are integral domains. ⊠

For the present purpose, the most important field expansions are the following.

Definition 3.7. An *implicitly closed field* is an algebra $\langle A; +, \cdot, -, 0, 1, (\,)^*, \{r_p : p \text{ is prime}\}\rangle$, where $\langle A; +, \cdot, -, 0, 1, (\,)^*\rangle$ is a zero-totalized weakly rooted field and the r_p's are the unary operations in (2). The class of implicitly closed fields will be denoted by ICF.

Definition 3.8. We denote by $\boldsymbol{A}^+$ the field expansion of a weakly rooted field $\boldsymbol{A}$ obtained by adding the operation $(\,)^*$ as well as all the r_p's for p prime.

From Proposition 3.2 we deduce the following.

Proposition 3.9. *Let $\boldsymbol{A}$ be a field. Then $\mathsf{acl}(\boldsymbol{A})^+$ is an implicitly closed field.*

The next observation will be needed later on.

Proposition 3.10. *We have that WRF and ICF are elementary classes. Moreover, $\mathsf{WRF} = \mathbb{P}_{\mathrm{u}}(\mathsf{WRF})$ and $\mathsf{ICF} = \mathbb{IS}\mathbb{P}_{\mathrm{u}}(\mathsf{ICF})$.*

Proof. The class WRF can be axiomatized by adding the following first order formulas to the equational axioms of commutative rings:

$$\forall x(x \not\approx 0 \to \exists y(xy \approx 1)) \text{ and } p \approx 0 \to \forall x \exists y(x \approx y^p) \text{ for all primes } p.$$

Therefore, WRF is an elementary class. Since every elementary class is closed under $\mathbb{P}_{\mathrm{u}}$ (see, e.g., [8, Thm. V.2.16]), it follows that so is WRF. As the inclusion $\mathsf{WRF} \subseteq \mathbb{P}_{\mathrm{u}}(\mathsf{WRF})$ is straightforward, we conclude that $\mathsf{WRF} = \mathbb{P}_{\mathrm{u}}(\mathsf{WRF})$.

The class ICF can be axiomatized by adding the following formulas to the equational axioms of commutative rings: for all prime p,

$$\forall x(x \approx 0 \to x^* \approx 0), \qquad \forall x(x \not\approx 0 \to x^*x \approx 1),$$
$$\forall x(p \approx 0 \to x \approx (r_p(x))^p), \qquad \forall x(p \not\approx 0 \to r_p(x) \approx 0).$$

Thus, ICF is an elementary class. Since every class of algebras that can be axiomatized by first order formulas of the form $\forall x_1, \dots, x_n \varphi$ with φ quantifier-free is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_{\mathrm{u}}$ (see, e.g., [8, Thm. V.2.20]), so is ICF. As the inclusion $\mathsf{ICF} \subseteq \mathbb{IS}\mathbb{P}_{\mathrm{u}}(\mathsf{ICF})$ is straightforward, we conclude that $\mathsf{ICF} = \mathbb{IS}\mathbb{P}_{\mathrm{u}}(\mathsf{ICF})$. ⊠

3.2. **Implicitly closed meadows.** We begin by recalling the following concepts.

Definition 3.11. A class of similar algebras is

(i) a *quasivariety* when it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{u}}$ or, equivalently, when it can be axiomatized by a set of *quasiequations*, that is, formulas of the form $\bigsqcap \Phi \to \varphi$, where $\Phi \cup \{\varphi\}$ is a finite set of equations and $\bigsqcap$ the conjunction symbol (see, e.g., [8, Thm. V.2.25] due to Maltsev);

(ii) a *variety* when it is closed under $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$ or, equivalently, when it can be axiomatized by a set of equations (see, e.g., [8, Thm. II.11.9] due to Birkhoff).

Observe that every variety is a quasivariety because every equation φ is equivalent to the quasiequation $\bigsqcap \emptyset \to \varphi$. However, the converse does not need to hold in general: for example, RCR is a quasivariety axiomatized by the equations of commutative rings together with the quasiequation $x^2 \approx 0 \to x \approx 0$, but it is not a variety because it is not closed under $\mathbb{H}$. To prove the latter, observe that the ring of integers $\mathbb{Z}$ is reduced and $\mathbb{Z}/4\mathbb{Z} \in \mathbb{H}(\mathbb{Z})$ is not, for $2^2 = 0$ in $\mathbb{Z}/4\mathbb{Z}$.

For every class of similar algebras K there exist the least quasivariety and the least variety containing K, which we denote by $\mathbb{Q}(\mathsf{K})$ and $\mathbb{V}(\mathsf{K})$, respectively. By theorems of Maltsev and Tarski, respectively, the latter can be described as follows (see, e.g., [8, Thms. V.2.25 and II.9.5]).

Theorem 3.12. *Let* K *be a class of similar algebras. Then*

$$\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{u}}(\mathsf{K}) \quad \textit{and} \quad \mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K}).$$

The following structures will play a fundamental role in this paper.

Definition 3.13. We write ICM as a shorthand for $\mathbb{ISP}(\mathsf{ICF})$. An *implicitly closed meadow* is a member of ICM.

With every prime p we associate the equation

$$\mathsf{root}_p(x, r_p(x)) = (r_p(x))^p \approx (1 - p^* p)x$$

in the language of ICM. Our first goal will be to axiomatize ICM. For this, we recall that the class of commutative rings can be axiomatized by equations.

Theorem 3.14. ICM *is a variety axiomatized by the equations*

$$x \approx x^2 x^*, \quad x \approx x^{**}, \quad \mathsf{root}_p(x, r_p(x))$$

for each prime p*, together with the axioms of commutative rings.*

The next concept will be instrumental for this purpose.

Definition 3.15. We write M as a shorthand for $\mathbb{ISP}(\mathsf{ZTF})$. A *meadow* is a member of M (see [6, Sec. 3.2]).

Theorem 3.16 ([6])**.** M *is a variety axiomatized by the equations*

$$x \approx x^2 x^* \quad \textit{and} \quad x \approx x^{**},$$

together with the axioms of commutative rings.

We are now ready to prove Theorem 3.14.

Proof. We begin by proving that ICM satisfies the axioms in the statement. Recall that $\mathsf{ICM} = \mathbb{ISP}(\mathsf{ICF})$ by definition and observe that ICF satisfies the (equational) axioms of commutative rings as well as the equations $x \approx x^2 x^*$ and $x \approx x^{**}$. Therefore, so does ICM because the validity of equations is preserved by $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}$. Similarly, to prove that ICM satisfies the equation $\mathsf{root}_p(x, r_p(x))$ for a prime p, it will be enough to show that so does ICF. Then, consider $\boldsymbol{A} \in \mathsf{ICF}$ and $a \in A$. As $\boldsymbol{A}$ has a zero-totalized field reduct that is weakly rooted, from Proposition 3.6 it follows that $\boldsymbol{A} \vDash \mathsf{root}_p(a, r_p(a))$.

Next, let Σ be the set of axioms in the statement. We will prove that every algebra $\boldsymbol{A} = \langle A; +, \cdot, -, 0, 1, ()^*, \{r_p : p \text{ is prime}\}\rangle$ satisfying Σ belongs to ICM. To this end, consider an algebra

$\boldsymbol{A}$ as above satisfying Σ. In view of Theorem 3.16, $\boldsymbol{A}$ has a meadow reduct $\boldsymbol{A}_{\mathsf{m}}$. Therefore, $\boldsymbol{A}_{\mathsf{m}} \in \mathsf{M} = \mathbb{ISP}(\mathsf{ZTF})$. Then, there exist $\{\boldsymbol{B}_i : i \in I\} \subseteq \mathsf{ZTF}$ and an embedding $h \colon \boldsymbol{A}_{\mathsf{m}} \to \prod_{i \in I} \boldsymbol{B}_i$. By Corollary 3.3 each $\boldsymbol{B}_i$ extends to a weakly rooted $\boldsymbol{C}_i \in \mathsf{ZTF}$. Since $\boldsymbol{B}_i \leqslant \boldsymbol{C}_i$ for each $i \in I$, we can view h as an embedding $h \colon \boldsymbol{A}_{\mathsf{m}} \to \prod_{i \in I} \boldsymbol{C}_i$. Furthermore, as each $\boldsymbol{C}_i$ is weakly rooted, it is the meadow reduct of some $\boldsymbol{D}_i \in \mathsf{ICF}$. Therefore, in order to conclude the proof, it only remains to show that $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{D}_i$ is also an embedding, for in this case we would have $\boldsymbol{A} \in \mathbb{ISP}(\{\boldsymbol{D}_i : i \in I\}) \subseteq \mathbb{ISP}(\mathsf{ICF}) = \mathsf{ICM}$, as desired.

First, observe that $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{D}_i$ is a well-defined injective map that preserves all the basic operations of meadows because $h \colon \boldsymbol{A}_{\mathsf{m}} \to \prod_{i \in I} \boldsymbol{C}_i$ is an embedding from the meadow reduct of $\boldsymbol{A}$ to the meadow reduct of $\prod_{i \in I} \boldsymbol{D}_i$. Therefore, to show that h is an embedding of implicitly closed meadows, it suffices to show that h preserves the operations of the form r_p. To this end, consider a prime p and $a \in A$. We need to prove that $h(r_p^{\boldsymbol{A}}(a)) = r_p^{\prod_{i \in I} \boldsymbol{D}_i}(h(a))$, that is,

$$(h(r_p^{\boldsymbol{A}}(a)))(i) = r_p^{\boldsymbol{D}_i}((h(a))(i)) \text{ for every } i \in I. \tag{3}$$

In order to prove the above display, consider $j \in I$. First, recall that $\boldsymbol{A}$ satisfies the equation $\mathsf{root}_p(x, r_p(x))$ by assumption. Therefore,

$$(r_p^{\boldsymbol{A}}(a))^p = (1 - p^* p) a.$$

Let $\pi_j \colon \prod_{i \in I} \boldsymbol{C}_i \to \boldsymbol{C}_j$ be the projection onto the j-th coordinate. As $\pi_j \circ h \colon \boldsymbol{A}_{\mathsf{m}} \to \boldsymbol{C}_j$ is a homomorphism of meadows, from the above display it follows that the following holds in $\boldsymbol{C}_j$:

$$((h(r_p^{\boldsymbol{A}}(a)))(j))^p = (1 - p^* p)(h(a)(j)).$$

Together with the definition of $\mathsf{root}_p(x, y)$, this yields

$$\boldsymbol{C}_j \vDash \mathsf{root}_p((h(a))(j), (h(r_p^{\boldsymbol{A}}(a)))(j)).$$

Recall that $\boldsymbol{C}_i$ is weakly rooted because it is the meadow reduct of $\boldsymbol{D}_i \in \mathsf{ICF}$. Consequently, we can apply Proposition 3.6 to the above display, obtaining that $(h(r_p^{\boldsymbol{A}}(a)))(i)$ is $r_p((h(a))(i))$ in $\boldsymbol{C}_i$. As $\boldsymbol{C}_i$ is the meadow reduct of $\boldsymbol{D}_i$, we have $(h(r_p^{\boldsymbol{A}}(a)))(i) = r_p^{\boldsymbol{D}_i}((h(a))(i))$, whence (3) holds. ⊠

3.3. **Subdirect irreducibility.** Let K be a quasivariety. A congruence θ of an algebra $\boldsymbol{A} \in \mathsf{K}$ is said to be a K-*congruence* of $\boldsymbol{A}$ when $\boldsymbol{A}/\theta \in \mathsf{K}$. When ordered under inclusion, the set of K-congruences of $\boldsymbol{A}$ forms an algebraic lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, whose minimum is the identity relation id_A on A (see, e.g., [18, Prop. 1.4.7 & Cor. 1.4.11]). We recall that an element a of a bounded lattice $\boldsymbol{A}$ is *meet irreducible* when it is not the maximum of $\boldsymbol{A}$ and for all $b, c \in A$ such that $a = b \wedge c$ either $a = b$ or $a = c$ holds. Then, we say that a member $\boldsymbol{A}$ of K is *relatively finitely subdirectly irreducible* when id_A is meet irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ (see, e.g., [7, p. 659]). The class of relatively finitely subdirectly irreducible members of K will be denoted by K_{RFSI}. When K is a variety, the class K_{RFSI} will be denoted by K_{FSI}. The importance of K_{RFSI} derives from Birkhoff and Maltsev's subdirect decomposition theorem, which ensures that $\mathsf{K} = \mathbb{ISP}(\mathsf{K}_{\text{RFSI}})$ for every quasivariety K (see, e.g., [18, Thm. 3.1.1]).

Theorem 3.17 ([17, Lem. 1.5])**.** *Let* K *be a class of similar algebras. Then* $\mathbb{Q}(\mathsf{K})_{\text{RFSI}} \subseteq \mathbb{ISP}_{\mathrm{u}}(\mathsf{K})$*.*

Next, we recall that an algebra $\boldsymbol{A}$ is *simple* when it has exactly two congruences, namely, id_A and $A \times A$. For instance, every field is simple and, therefore, so is every field expansion. We also recall that ICM is a variety by Theorem 3.14. This explains why, in the next result, we write $\mathsf{ICM}_{\text{FSI}}$ instead of $\mathsf{ICM}_{\text{RFSI}}$.

Proposition 3.18. *The following equalities hold:*

$$\mathsf{RCR} = \mathbb{Q}(\mathsf{WRF}), \quad \mathsf{ICM} = \mathbb{Q}(\mathsf{ICF}), \quad \mathsf{RCR}_{\text{RFSI}} = \mathsf{ID} = \mathbb{S}(\mathsf{F}), \quad \mathsf{ICM}_{\text{FSI}} = \mathsf{ICF}.$$

Proof. We begin by proving $\mathsf{RCR} = \mathbb{Q}(\mathsf{WRF})$. Observe that

$$\mathsf{RCR} = \mathbb{ISP}(\mathsf{F}) \subseteq \mathbb{ISPS}(\mathsf{WRF}) = \mathbb{ISP}(\mathsf{WRF}) \subseteq \mathbb{ISP}(\mathsf{F}) = \mathsf{RCR},$$

where the first and the last steps hold by definition, the second by Corollary 3.3, the third because $\mathbb{ISP}(\mathsf{K})$ is closed under $\mathbb{S}$ for every class of algebras K (see, e.g., [8, Thm. V.2.20]), and the fourth because $\mathsf{WRF} \subseteq \mathsf{F}$. Therefore, $\mathsf{RCR} = \mathbb{ISP}(\mathsf{WRF})$. Since WRF is closed under $\mathbb{P}_{\text{u}}$ by Proposition 3.10, this yields $\mathsf{RCR} = \mathbb{ISPP}_{\text{u}}(\mathsf{WRF})$. By Theorem 3.12 we conclude that $\mathsf{RCR} = \mathbb{Q}(\mathsf{WRF})$, as desired.

Next, observe that

$$\mathsf{ICM} = \mathbb{ISP}(\mathsf{ICF}) = \mathbb{ISPP}_{\text{u}}(\mathsf{ICF}) = \mathbb{Q}(\mathsf{ICF}),$$

where the first equality holds by definition, the second because ICF is closed under $\mathbb{P}_{\text{u}}$ by Proposition 3.10, and the third by Theorem 3.12.

Lastly, recall that $\mathsf{ID} = \mathbb{S}(\mathsf{F})$ by definition. As $\mathsf{RCR}_{\text{RFSI}} = \mathsf{ID}$ (see, e.g., [10, p. 410]), it only remains to show that $\mathsf{ICM}_{\text{FSI}} = \mathsf{ICF}$. From $\mathsf{ICM} = \mathbb{Q}(\mathsf{ICF})$ and Theorem 3.17 it follows that $\mathsf{ICM}_{\text{FSI}} \subseteq \mathbb{ISP}_{\text{u}}(\mathsf{ICF}) = \mathsf{ICF}$, where the last equality holds by Proposition 3.10. To prove the reverse inclusion, consider $\boldsymbol{A} \in \mathsf{ICF} \subseteq \mathsf{ICM}$. As $\boldsymbol{A}$ has a field reduct, it is simple. Consequently, id_A is meet irreducible in $\mathsf{Con}_{\mathsf{ICM}}(\boldsymbol{A})$ and $\boldsymbol{A} \in \mathsf{ICM}_{\text{FSI}}$. ⊠

4. Primitive positive expansions

As we mentioned, in order to complete a quasivariety so that its monomorphisms become regular, it is sensible to expand it with enough implicit operations. In the context of quasivarieties, implicit operations admit a description in terms of *primitive positive formulas* (for short, *pp formulas*), that is, formulas of the form $\exists x_1, \dots, x_n \varphi$, where φ is a conjunction of equations. More precisely, if f is an implicit operation of a quasivariety K, there exist implicit operations $f_1, \dots, f_n$ of K definable by pp formulas such that $f^{\boldsymbol{A}} = f_1^{\boldsymbol{A}} \cup \dots \cup f_n^{\boldsymbol{A}}$ for every $\boldsymbol{A} \in \mathsf{K}$ (see [12, Cor. 3.10]). Consequently, the pp definable implicit operations of K form the building blocks of all implicit operations of K and, therefore, we restrict our attention to them.

In general, the implicit operations of a quasivariety need not be total. This motivates the following concept (see [12, Sec. 8]).

Definition 4.1. Let K be a quasivariety. An implicit operation f of K is *extendable* when for all $\boldsymbol{A} \in \mathsf{K}$ and $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ there exists an algebra $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$. The class of extendable implicit operations of K will be denoted by $\mathsf{ext}(\mathsf{K})$, and that of pp definable extendable implicit operations of K by $\mathsf{ext}_{\text{PP}}(\mathsf{K})$.

The term "extendable" in the above definition derives from the fact that every member of a quasivariety K can be upgraded to one in which all the extendable implicit operations are total in the sense that for every $\boldsymbol{A} \in \mathsf{K}$ there exists $\boldsymbol{B} \in \mathsf{K}$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$ such that $f^{\boldsymbol{B}}$ is total and extends $f^{\boldsymbol{A}}$ for each $f \in \mathsf{ext}(\mathsf{K})$ (see [12, Prop. 8.1 & Thm. 8.4]). This property of an implicit operation becomes crucial if we want to add it to K.

We will show that "taking weak inverses" and "taking weak prime roots" can be viewed as extendable implicit operations of RCR. To this end, we consider the following pp formulas in the language of RCR for every prime p:

$$\mathsf{inv}(x, y) = (x^2 y \approx x) \sqcap (xy^2 \approx y) \text{ and } \exists \mathsf{root}_p(x, y) = \exists z (\mathsf{inv}(p, z) \sqcap (y^p \approx (1 - zp)x)).$$

Theorem 4.2. *The following conditions hold for every prime p and $\boldsymbol{A} \in \mathsf{WRF}$:*

(i) *the formula $\mathsf{inv}(x, y)$ defines a unary $g \in \mathsf{ext}_{\text{PP}}(\mathsf{RCR})$ such that $g^{\boldsymbol{A}}$ is total and*
$$g^{\boldsymbol{A}}(a) = a^* \text{ for every } a \in A;$$

(ii) *the formula $\exists\mathsf{root}_p(x, y)$ defines a unary $f_p \in \mathsf{ext}_{\text{PP}}(\mathsf{RCR})$ such that $f_p^{\boldsymbol{A}}$ is total and*
$$f_p^{\boldsymbol{A}}(a) = r_p(a) \text{ for every } a \in A.$$

The proof of Theorem 4.2 is based on the next observation (see [12, Cor. 3.11 & Prop. 8.11(ii)]).

Proposition 4.3. *Let K be an elementary class of similar algebras and $\varphi(x_1, \dots, x_n, y)$ a pp formula satisfying the following conditions:*

(i) $\mathsf{K} \vDash (\varphi(x_1, \dots, x_n, y) \sqcap \varphi(x_1, \dots, x_n, z)) \to y \approx z$;

(ii) *for all $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \dots, a_n \in A$ there exist $\boldsymbol{B} \in \mathbb{Q}(\mathsf{K})$ and $b \in B$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$ and $\boldsymbol{B} \vDash \varphi(a_1, \dots, a_n, b)$.*

Then φ defines an n-ary member of $\mathsf{ext}_{\text{PP}}(\mathbb{Q}(\mathsf{K}))$.

We are now ready to prove Theorem 4.2.

Proof. We begin with the following observation.

Claim 4.4. *For all $\boldsymbol{A} \in \mathsf{WRF}$ and $a, b \in A$,*
$$(\boldsymbol{A} \vDash \mathsf{inv}(a, b) \iff b = a^*) \quad \text{and} \quad (\boldsymbol{A} \vDash \exists\mathsf{root}_p(a, b) \iff b = r_p(a)).$$

Proof of the Claim. To prove the left hand side of the above display, observe that $b = a^*$ immediately implies $\boldsymbol{A} \vDash \mathsf{inv}(a, b)$. Conversely, suppose that $\boldsymbol{A} \vDash \mathsf{inv}(a, b)$, that is, $a^2 b = a$ and $ab^2 = b$. First, suppose that $a = 0$. Then $a = a^* = 0$. Consequently, from $ab^2 = b$ it follows that $a^* = 0 = ab^2 = b$. Next, we consider the case where $a \neq 0$ and, therefore, $a^* = a^{-1}$. Thus, from $a^2 b = a$ it follows that $b = a^{-2} a^2 b = a^{-2} a = a^{-1} = a^*$, establishing the left hand side of the above display.

Next we prove the right hand side. Since $\boldsymbol{A}$ is a field, we can expand it to a zero totalized field $\boldsymbol{A}^+$. We will prove that
$$\begin{aligned}\boldsymbol{A} \vDash \exists\mathsf{root}_p(a, b) &\iff \boldsymbol{A} \vDash \mathsf{inv}(p1, c) \sqcap (b^p \approx (1 - cp)a) \text{ for some } c \in A\\ &\iff b^p = (1 - p^* p)a\\ &\iff \boldsymbol{A}^+ \vDash \mathsf{root}_p(a, b)\\ &\iff b = r_p(a).\end{aligned}$$

The above equivalences are justified as follows: the first holds by the definition of $\exists\mathsf{root}_p$, the second by the left hand side of the display in the statement of the claim, the third by the definition of $\mathsf{root}_p(x, y)$, and the fourth by Proposition 3.6 and the assumption that $\boldsymbol{A}$ is weakly rooted. ⊠

Recall from Proposition 3.10 that WRF is an elementary class. Therefore, we can apply Claim 4.4, obtaining that condition (i) of Proposition 4.3 holds in WRF both for $\mathsf{inv}(x, y)$ and $\exists\mathsf{root}_p(x, y)$. The same is true for condition (ii) of Proposition 4.3 by Claim 4.4 and the fact that for every $\boldsymbol{A} \in \mathsf{WRF}$ and $a \in A$ the elements a^* and $r_p(a)$ exist in $\boldsymbol{A}$. Therefore, from Proposition 4.3 it follows that there exist unary $g, f_p \in \mathsf{ext}_{\text{PP}}(\mathbb{Q}(\mathsf{WRF}))$ defined by $\mathsf{inv}(x, y)$ and $\exists\mathsf{root}_p(x, y)$, respectively. As $\mathsf{RCR} = \mathbb{Q}(\mathsf{WRF})$ by Proposition 3.18, we obtain $g, f_p \in \mathsf{ext}_{\text{PP}}(\mathsf{RCR})$. Lastly, let $\boldsymbol{A} \in \mathsf{WRF}$ and recall that a^* and $r_p(a)$ exist in $\boldsymbol{A}$ for every $a \in A$. Together with Claim 4.4, this implies that $g^{\boldsymbol{A}}$ and $f_p^{\boldsymbol{A}}$ are total and such that $g^{\boldsymbol{A}}(a) = a^*$ and $f_p^{\boldsymbol{A}}(a) = r_p(a)$ for every $a \in A$. ⊠

In order to add a family of implicit operations $\mathcal{F} \subseteq \mathsf{ext}_{\mathrm{PP}}(\mathsf{K})$ to a quasivariety K, we proceed as follows. Let $\mathscr{L}$ be the language of K and $\mathscr{L}_{\mathcal{F}}$ the language obtained by adding to $\mathscr{L}$ a new n-ary function symbol g_f for each n-ary $f \in \mathcal{F}$. Then, we expand every member $\boldsymbol{A}$ of K in which $\{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a family of total functions to an algebra $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ in the language $\mathscr{L}_{\mathcal{F}}$ by interpreting g_f as $f^{\boldsymbol{A}}$ for each $f \in \mathcal{F}$. In addition, for every $\mathsf{N} \subseteq \mathsf{K}$ let

$$\mathsf{N}[\mathscr{L}_{\mathcal{F}}] = \{\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}] : \boldsymbol{A} \in \mathsf{N} \text{ and } \{f^{\boldsymbol{A}} : f \in \mathcal{F}\} \text{ is a family of total functions}\}.$$

The next definition captures the idea of "adding the implicit operations in $\mathcal{F}$ to K".

Definition 4.5. Let K be a quasivariety.

(i) Given $\mathcal{F} \subseteq \mathsf{ext}_{\mathrm{PP}}(\mathsf{K})$, the *pp expansion* of K induced by $\mathcal{F}$ is $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$.
(ii) A pp expansion of K is said to be *simple* when it is of the form $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ for some $\mathcal{F} \subseteq \mathsf{ext}_{\mathrm{PP}}(\mathsf{K})$.

Notably, every pp expansion of a quasivariety is also a quasivariety (see [12, Thm. 10.3(ii)]). Simple pp expansions admit a transparent categorical description, as we proceed to recall. An isomorphism-closed full subcategory C of a category D is said *mono-reflective* when the inclusion functor $i \colon \mathsf{C} \to \mathsf{D}$ has a left adjoint and the unit of resulting adjunction is componentwise a monomorphism (see, e.g., [1, Def. 16.1]).

Theorem 4.6 ([13, Thm. 3.1])**.** *Let K and M be a pair of quasivarieties. Then M is a simple pp expansion of K if and only if the forgetful functor $U \colon \mathsf{M} \to \mathsf{K}$ is well defined and induces an isomorphism from M to a mono-reflective subcategory of K.*

For the present purpose, the pp expansions of interest are those obtained by adding enough implicit operations so that every monomorphism becomes regular.

Definition 4.7. Let K be a quasivariety.

(i) A pp expansion of K is said to be a *Beth companion* of K when its monomorphisms are regular.
(ii) A Beth companion of K is said to be *simple* when it is simple as a pp expansion of K.

While a quasivariety need not have a Beth companions, up to term equivalence it may possess only one (see [12, Thm. 11.7]). For this reason, we talk about *the* Beth companion of K (when it exists). The aim of this paper is to establish that ICM is the Beth companion of RCR and that, moreover, this Beth companion is simple (see Theorem 6.1). As a first step, in this section we will prove the following result. Let

$$\mathcal{F} = \{g\} \cup \{f_p : p \text{ is prime}\},$$

where g and f_p are the members of $\mathsf{ext}_{\mathrm{PP}}(\mathsf{RCR})$ given by Theorem 4.2.

Theorem 4.8. *The pp expansion of RCR induced by $\mathcal{F}$ is simple and coincides with $\mathsf{ICM} = \mathsf{RCR}[\mathscr{L}_{\mathcal{F}}]$.*

To this end, we will employ the next observation (see [12, Thm. 10.5]).

Proposition 4.9. *Let K be a class of similar algebras and $\mathcal{F} \subseteq \mathsf{ext}_{\mathrm{PP}}(\mathbb{Q}(\mathsf{K}))$. Assume that $f^{\boldsymbol{A}}$ is total for every $f \in \mathcal{F}$ and $\boldsymbol{A} \in \mathsf{K}$. Then $\mathbb{Q}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is the pp expansion of $\mathbb{Q}(\mathsf{K})$ induced by $\mathcal{F}$.*

We are now ready to prove Theorem 4.8.

Proof. We begin by proving that the pp expansion of RCR induced by $\mathcal{F}$ coincides with ICM. To this end, recall from Theorem 4.2 that $g^{\boldsymbol{A}} = (\,)^*$ and $f_p^{\boldsymbol{A}} = r_p$ are total for all $\boldsymbol{A} \in \mathsf{WRF}$ and prime p. Moreover, $\mathsf{RCR} = \mathbb{Q}(\mathsf{WRF})$ by Proposition 3.18. Therefore, we can apply Proposition 4.9 to $\mathsf{K} = \mathsf{WRF}$ and $\mathcal{F}$, obtaining that $\mathbb{Q}(\mathsf{WRF}[\mathscr{L}_{\mathcal{F}}])$ is the pp expansion of RCR induced by $\mathcal{F}$. As both $\mathsf{WRF}[\mathscr{L}_{\mathcal{F}}]$ and ICF consist of the members of WRF expanded with the operations $(\,)^*$ and $\{r_p : p \text{ prime}\}$, we conclude that $\mathsf{WRF}[\mathscr{L}_{\mathcal{F}}] = \mathsf{ICF}$. Thus, $\mathbb{Q}(\mathsf{ICF})$ is the pp expansion of RCR induced by $\mathcal{F}$. Moreover, it coincides with ICM by Proposition 3.18. Therefore, ICM is the pp expansion of RCR induced by $\mathcal{F}$, as desired.

In order to prove that this pp expansion is simple, it suffices to show that $\mathsf{ICM} = \mathsf{RCR}[\mathscr{L}_{\mathcal{F}}]$. As $\mathsf{ICM} = \mathbb{S}(\mathsf{RCR}[\mathscr{L}_{\mathcal{F}}])$, it only remains to show that $\mathsf{RCR}[\mathscr{L}_{\mathcal{F}}]$ is closed under subalgebras. To this end, consider $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{RCR}[\mathscr{L}_{\mathcal{F}}]$. We have to show that $f^{\boldsymbol{A}^-}$ is total for every $f \in \mathcal{F}$. To this end, will use repeatedly the fact that the maps $g^{\boldsymbol{B}^-}$ and $f_p^{\boldsymbol{B}^-}$ are total and and defined for every $a \in B$ as

$$g^{\boldsymbol{B}^-}(a) = a^{*\boldsymbol{B}} \text{ and } f_p^{\boldsymbol{B}^-}(a) = r_p^{\boldsymbol{B}}(a),$$

which holds because $\boldsymbol{B} \in \mathsf{RCR}[\mathscr{L}_{\mathcal{F}}]$.

First, let $f = g$ and $a \in A$. Since $\boldsymbol{B} \in \mathsf{RCR}[\mathscr{L}_{\mathcal{F}}]$ and $\mathsf{inv}(x, y)$ defines g by Theorem 4.2, we obtain $\boldsymbol{B} \vDash \mathsf{inv}(a, a^{*\boldsymbol{B}})$. As $a \in A$ and $\boldsymbol{A} \leqslant \boldsymbol{B}$, we have $a^{*\boldsymbol{B}} \in A$. Since $\mathsf{inv}(a, a^{*\boldsymbol{B}})$ is a conjunction of equations without free variables, $a, a^{*\boldsymbol{B}} \in A$, and $\boldsymbol{A} \leqslant \boldsymbol{B}$, we obtain $\boldsymbol{A} \vDash \mathsf{inv}(a, a^{\boldsymbol{B}*})$. Thus, $a \in \mathsf{dom}(g^{\boldsymbol{A}^-})$ because $\mathsf{inv}(x, y)$ defines g.

Next, we consider the case where $f = f_p$ for some prime p. Let $a \in A$. Since $\boldsymbol{B} \in \mathsf{RCR}[\mathscr{L}_{\mathcal{F}}]$ and $\exists\mathsf{root}_p(x, y)$ defines f_p by Theorem 4.2, we obtain $\boldsymbol{B} \vDash \exists\mathsf{root}_p(a, r_p^{\boldsymbol{B}}(a))$. By the definition of $\exists\mathsf{root}_p(x, y)$ there exists $c \in B$ such that

$$\boldsymbol{B} \vDash \mathsf{inv}(p, c) \sqcap (r_p^{\boldsymbol{B}}(a)^p \approx (1 - cp)a).$$

As $\mathsf{inv}(x, y)$ defines g, the above display amounts to

$$\boldsymbol{B} \vDash \mathsf{inv}(p, p^{*\boldsymbol{B}}) \sqcap (r_p^{\boldsymbol{B}}(a)^p \approx (1 - p^{*\boldsymbol{B}}p)a).$$

Observe that $a, p \in A$. Together with $\boldsymbol{A} \leqslant \boldsymbol{B}$, this ensures $r_p^{\boldsymbol{B}}(a), p^{*\boldsymbol{B}} \in A$. As equations are preserved by subalgebras, the above display implies

$$\boldsymbol{A} \vDash \mathsf{inv}(p, p^{*\boldsymbol{B}}) \sqcap (r_p^{\boldsymbol{B}}(a)^p \approx (1 - p^{*\boldsymbol{B}}p)a).$$

Hence, we conclude that $a \in \mathsf{dom}(f_p^{\boldsymbol{A}^-})$ because $\exists\mathsf{root}_p(x, y)$ defines f_p. ⊠

Definition 4.10. We denote by $\boldsymbol{A}^-$ the ring reduct of an implicitly closed meadow.

From [12, Prop. 10.2] and Theorem 4.8 we deduce the following.

Corollary 4.11. *The following conditions hold:*

(i) *for every $\boldsymbol{A} \in \mathsf{ICM}$ we have $\boldsymbol{A}^- \in \mathsf{RCR}$;*
(ii) *for every $\boldsymbol{A} \in \mathsf{RCR}$ there exists $\boldsymbol{B} \in \mathsf{ICM}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}^-$.*

We will also make use of the following observation.

Corollary 4.12. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{ICM}$. Every ring homomorphism $h\colon \boldsymbol{A}^- \to \boldsymbol{B}^-$ is also a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ of implicitly closed meadows.*

Proof. Recall from Theorem 4.8 that $\mathsf{ICM} = \mathsf{RCR}[\mathscr{L}_{\mathcal{F}}]$. By [12, Prop. 9.5] the latter yields the desired conclusion. ⊠

5. Amalgamation

Definition 5.1. Given a class K of similar algebras, we say that

(i) a tuple $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, h_1, h_2 \rangle$ is a *span* in K when $h_1 \colon \boldsymbol{A} \to \boldsymbol{B}$ and $h_2 \colon \boldsymbol{A} \to \boldsymbol{C}$ is a pair of embeddings with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathsf{K}$;

(ii) a span $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, h_1, h_2 \rangle$ in K has an *amalgam* in K when there exists a pair of embeddings $g_1 \colon \boldsymbol{B} \to \boldsymbol{D}$ and $g_2 \colon \boldsymbol{C} \to \boldsymbol{D}$ with $\boldsymbol{D} \in \mathsf{K}$ such that $g_1 \circ h_1 = g_2 \circ h_2$;

(iii) a member $\boldsymbol{A}$ of K is an *amalgamation base* for K when every span in K of the form $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, h_1, h_2 \rangle$ has an amalgam in K;

(iv) K has the *amalgamation property* when every span in K has an amalgam in K.

The aim of this section is to establish the following.

Theorem 5.2. ICM *has the amalgamation property.*

This result contrasts with the case of RCR, a class that lacks the amalgamation property (see, e.g., [16, p. 426], where reduced commutative rings are called "commutative semiprime rings with identity"). The next concept will be instrumental for proving Theorem 5.2 (see [30, Def. 4]).

Definition 5.3. A commutative ring $\boldsymbol{A}$ is said to be *(von Neumann) regular* when for every $a \in A$ there exists $b \in A$ such that $a = a^2 b$.

It is immediate that every regular commutative ring is reduced. Moreover, we have the following.

Proposition 5.4. *Let* $\boldsymbol{A} \in \mathsf{ICM}$*. Then* $\boldsymbol{A}^-$ *is a regular commutative ring.*

Proof. Clearly, $\boldsymbol{A}^-$ is a commutative ring. To prove that it is regular, consider $a \in A$ and let $b = a^*$. From Theorem 3.14 it follows that $a = a^2 a^* = a^2 b$. ⊠

For the present purpose, the importance of regular rings derives from the following fact (see, e.g., [16, Thm. 1.6]).

Theorem 5.5. *The class of regular commutative rings has the amalgamation property.*

We are now ready to prove Theorem 5.2.

Proof. Let $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, h_1, h_2 \rangle$ be a span in ICM. We need to show that it has an amalgam in ICM. First, observe that $\langle \boldsymbol{A}^-, \boldsymbol{B}^-, \boldsymbol{C}^-, h_1, h_2 \rangle$ is a span in the class of regular commutative rings by Proposition 5.4. By Theorem 5.5 there exist a regular commutative ring $\boldsymbol{D}$ and a pair of ring embeddings $g_1 \colon \boldsymbol{B}^- \to \boldsymbol{D}$ and $g_2 \colon \boldsymbol{C}^- \to \boldsymbol{D}$ such that $g_1 \circ h_1 = g_2 \circ h_2$. Next, recall that every regular commutative ring is reduced, whence $\boldsymbol{B}^-, \boldsymbol{C}^-, \boldsymbol{D} \in \mathsf{RCR}$. In view of Corollary 4.11(ii), we may assume that $\boldsymbol{D} = \boldsymbol{E}^-$ for some $\boldsymbol{E} \in \mathsf{ICM}$, and view g_1 and g_2 as ring embeddings $g_1 \colon \boldsymbol{B}^- \to \boldsymbol{E}^-$ and $g_2 \colon \boldsymbol{C}^- \to \boldsymbol{E}^-$ with $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{E} \in \mathsf{ICM}$. Hence, we can apply Corollary 4.12, obtaining that g_1 and g_2 can also be viewed as homomorphisms $g_1 \colon \boldsymbol{B} \to \boldsymbol{E}$ and $g_2 \colon \boldsymbol{C} \to \boldsymbol{E}$ of implicitly closed meadows. Since $g_1 \circ h_1 = g_2 \circ h_2$, we conclude that $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, h_1, h_2 \rangle$ has an amalgam in ICM. ⊠

Recall that every regular commutative ring is reduced.

Proposition 5.6. *Every regular commutative ring is an amalgamation base for* RCR.

Proof. Let $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, h_1, h_2 \rangle$ be a span in RCR with $\boldsymbol{A}$ a regular commutative ring. We need to show that it has an amalgam in RCR. From Corollary 4.11(ii) and Proposition 5.4 it follows that there exists a pair of regular commutative rings $\boldsymbol{B}'$ and $\boldsymbol{C}'$ such that $\boldsymbol{B} \leqslant \boldsymbol{B}'$ and $\boldsymbol{C} \leqslant \boldsymbol{C}'$. Therefore,

$\langle \boldsymbol{A}, \boldsymbol{B}', \boldsymbol{C}', h_1, h_2 \rangle$ is a span in the class of regular commutative rings. By Theorem 5.5 there exists a pair of embeddings $g_1 \colon \boldsymbol{B}' \to \boldsymbol{D}$ and $g_2 \colon \boldsymbol{C}' \to \boldsymbol{D}$ with $\boldsymbol{D}$ a regular commutative ring such that $g_1 \circ h_1 = g_2 \circ h_2$. Clearly, the restrictions $g_1^* \colon \boldsymbol{B} \to \boldsymbol{D}$ and $g_2^* \colon \boldsymbol{C} \to \boldsymbol{D}$ are also embeddings such that $g_1^* \circ h_1 = g_2^* \circ h_2$. As $\boldsymbol{D}$ is reduced (because it is regular), we are done. ⊠

6. The main result

The aim of this section is to prove our main result, which takes the following form.

Theorem 6.1. ICM *is the Beth companion of* RCR *and, as such, it is simple.*

As a consequence of Theorem 6.1, we obtain a categorical description of implicitly closed meadows.

Corollary 6.2. *Up to term equivalence,* ICM *is the unique quasivariety* K *in which monomorphisms are regular for which the forgetful functor* $U \colon \mathsf{K} \to \mathsf{RCR}$ *is well defined and induces an isomorphism from* K *to a mono-reflective subcategory of* RCR*.*

Proof. As Beth companions are unique up to term equivalence (see [12, Thm. 11.7]), Theorem 6.1 implies that ICM is (up to term equivalence) the only simple pp expansion of RCR in which monomorphisms are regular. Therefore, the desired results follows from Theorem 4.6. ⊠

The remainder of this section is devoted to proving Theorem 6.1. Recall from Theorem 4.8 that ICM is a simple pp expansion of RCR. Therefore, the proof of Theorem 6.1 reduces to verifying the following.

Theorem 6.3. *Monomorphisms are regular in* ICM*.*

The next concepts will be instrumental in proving Theorem 6.3. We begin by reviewing the notion of a dominion due to Isbell (see [19]).

Definition 6.4. Given a class of similar algebras K and $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$, the *dominion* of $\boldsymbol{A}$ in $\boldsymbol{B}$ relative to K is the set

$$\begin{aligned} \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \{ b \in B : g(b) = h(b) \text{ for every pair of homomorphisms} \\ g, h \colon \boldsymbol{B} \to \boldsymbol{C} \text{ with } \boldsymbol{C} \in \mathsf{K} \text{ such that } g{\restriction}_A = h{\restriction}_A \}. \end{aligned}$$

We will also rely on the following concept.

Definition 6.5. A variety K is *congruence permutable* when $\theta \circ \phi = \phi \circ \theta$ for all $\boldsymbol{A} \in \mathsf{K}$ and $\theta, \phi \in \mathsf{Con}(\boldsymbol{A})$, where $\circ$ is the operation of composition of relations.

Remark 6.6. We recall that every variety with a group reduct is congruence permutable (see, e.g., [8, p. 79, Example. 1]).
In particular, ICM is congruence permutable. ⊠

The proof of Theorem 6.3 is facilitated by the following result (see [12, Cor. 7.16]).

Theorem 6.7. *Let* K *be a congruence permutable variety with the amalgamation property. Then monomorphisms are regular in* K *if and only if* $\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = A$ *for all* $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}_{\text{FSI}}$*.*

As ICM is a congruence permutable variety with the amalgamation property (see Remark 6.6 and Theorem 5.2), it falls within the scope of Theorem 6.7. Keeping in mind that $\mathsf{ICM}_{\text{FSI}} = \mathsf{ICF}$ (see Proposition 3.18), this implies that the proof of Theorem 6.3 reduces to verifying the following.

Proposition 6.8. *For all* $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{ICF}$ *we have* $\mathsf{d}_{\mathsf{ICM}}(\boldsymbol{A}, \boldsymbol{B}) = A$*.*

To this end, we will utilize the next observation (see [12, Prop. 4.11]).

Proposition 6.9. *Let* K *be a class of similar algebras closed under finite direct products and* $\boldsymbol{A} \leqslant \boldsymbol{B} \leqslant \boldsymbol{C}$ *with* $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{K}$*. If* $\boldsymbol{B}$ *is an amalgamation base for* K*, then*

$$\mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B}) = \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{C}) \cap B.$$

We are now ready to prove Proposition 6.8. This will establish Theorems 6.1 and 6.3 as well.

Proof. Consider $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{ICF}$. As the inclusion $A \subseteq \mathsf{d}_{\mathsf{ICM}}(\boldsymbol{A}, \boldsymbol{B})$ is an immediate consequence of the definition of a dominion, we detail only the proof of the reverse inclusion. To this end, we reason by contraposition. Consider $b \in B - A$. We will show that $b \notin \mathsf{d}_{\mathsf{ICM}}(\boldsymbol{A}, \boldsymbol{B})$.

As $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{ICF}$ by assumption and ICF is closed under subalgebras by Proposition 3.10, we obtain $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{ICF}$. Together with $\boldsymbol{A} \leqslant \boldsymbol{B}$, this yields that

$$\boldsymbol{A}^- \leqslant \boldsymbol{B}^- \text{ is a field extension with } \boldsymbol{A}^-, \boldsymbol{B}^- \in \mathsf{WRF}. \tag{4}$$

We begin with the following observation.

Claim 6.10. *We have* $b \notin \mathsf{d}_{\mathsf{RCR}}(\boldsymbol{A}^-, \boldsymbol{B}^-)$.

Proof of the Claim. Observe that $\boldsymbol{A}^-(b)$ is regular because it is a field. Consequently, it is an amalgamation base for RCR by Proposition 5.6. This allows us to apply Proposition 6.9, obtaining $\mathsf{d}_{\mathsf{RCR}}(\boldsymbol{A}^-, \boldsymbol{A}^-(b)) = \mathsf{d}_{\mathsf{RCR}}(\boldsymbol{A}^-, \boldsymbol{B}^-) \cap A^-(b)$. Therefore, to establish the claim, it suffices to show that $b \notin \mathsf{d}_{\mathsf{RCR}}(\boldsymbol{A}^-, \boldsymbol{A}^-(b))$.

To this end, we consider the field extension $\boldsymbol{A}^- \leqslant \boldsymbol{A}^-(b)$ (see (4) if necessary). We distinguish two cases depending on whether b is algebraic or transcendental over $\boldsymbol{A}^-$. First, suppose that b is algebraic over $\boldsymbol{A}^-$. Let μ_b be the minimal polynomial of b over $\boldsymbol{A}^-$. Since $b \notin A$ by assumption, we have $\mathsf{deg}(\mu_b) \geqslant 2$. Recall that $\boldsymbol{A}^- \in \mathsf{WRF}$ by (4). Therefore, Theorem 3.4 yields that $\boldsymbol{A}^-$ is perfect. Moreover, the field extension $\boldsymbol{A}^- \leqslant \boldsymbol{A}^-(b)$ is algebraic because b is algebraic over $\boldsymbol{A}^-(b)$ by assumption (see, e.g., [24, F5 p. 18]). As $\boldsymbol{A}^-$ is perfect, this yields that $\boldsymbol{A}^- \leqslant \boldsymbol{A}^-(b)$ is separable. Since $\mathsf{deg}(\mu_b) \geqslant 2$, there exist a pair of distinct roots c and d of μ_b in $\mathsf{acl}(\boldsymbol{A}^-)$. By Proposition 2.8(i) there also exists a pair of embeddings $g, h \colon \boldsymbol{A}^-(b) \to \boldsymbol{A}^-(c, d)$ such that

$$g{\restriction}_A = h{\restriction}_A \quad \text{and} \quad g(b) = c \neq d = h(b).$$

As $\boldsymbol{A}^-(c, d)$ is a field, the above display yields $b \notin \mathsf{d}_{\mathsf{RCR}}(\boldsymbol{A}^-, \boldsymbol{A}^-(b))$, as desired.

Next, we consider the case where b is transcendental over $\boldsymbol{A}^-$. By Proposition 2.8(ii) there exists an isomorphism $g_1 \colon \boldsymbol{A}^-(b) \to \boldsymbol{A}^-(x)$ such that $g_1(b) = x$ and $g_1(a) = a$ for every $a \in A$. By the same token there exists an isomorphism $h_1 \colon \boldsymbol{A}^-(b) \to \boldsymbol{A}^-(y)$ such that $h_1(b) = y$ and $h_1(a) = a$ for every $a \in A$. Observe that $\boldsymbol{A}^-[x], \boldsymbol{A}[y]^- \leqslant \boldsymbol{A}^-[x, y] \leqslant \boldsymbol{A}^-(x, y)$. Therefore, by Proposition 2.4 there exists a pair of embeddings $g_2 \colon \boldsymbol{A}^-(x) \to \boldsymbol{A}^-(x, y)$ and $h_2 \colon \boldsymbol{A}^-(y) \to \boldsymbol{A}^-(x, y)$ such that $g_2(x) = x$, $h_2(y) = y$, and $g_2(a) = h_2(a) = a$ for every $a \in A$. Consequently, $g = g_2 \circ g_1$ and $h = h_2 \circ h_1$ form a pair of embeddings $g, h \colon \boldsymbol{A}^-(b) \to \boldsymbol{A}^-(x, y)$ satisfying

$$g{\restriction}_A = h{\restriction}_A \quad \text{and} \quad g(b) = x \neq y = h(b).$$

As $\boldsymbol{A}^-(x, y)$ is a field, the above display yields $b \notin \mathsf{d}_{\mathsf{RCR}}(\boldsymbol{A}^-, \boldsymbol{A}^-(b))$. ⊠

By Claim 6.10 there exists a pair of homomorphisms $g, h \colon \boldsymbol{B}^- \to \boldsymbol{D}$ with $\boldsymbol{D} \in \mathsf{RCR}$ such that $g{\restriction}_A = h{\restriction}_A$ and $g(b) \neq h(b)$. In view of Corollary 4.11(ii), we may assume that $\boldsymbol{D} = \boldsymbol{E}^-$ for some $\boldsymbol{E} \in \mathsf{ICM}$. Therefore, we can apply Corollary 4.12, obtaining that g and h can be viewed as homomorphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{E}$ of implicitly closed meadows. Since $g{\restriction}_A = h{\restriction}_A$ and $g(b) \neq h(b)$, we conclude that $b \notin \mathsf{d}_{\mathsf{ICM}}(\boldsymbol{A}, \boldsymbol{B})$. ⊠

We have thus established that adding the implicit operations of "taking weak inverses" and "taking weak prime roots" to the reduced commutative rings produces the canonical completion of this class in which every monomorphism (and every epimorphism) becomes regular. This completion is equationally axiomatizable (see Theorem 3.14) and has the amalgamation property (see Theorem 5.2). We close this section by highlighting that it also forms a discriminator variety.

Remark 6.11. A variety is *discriminator* when it is of the form $\mathbb{V}(\mathsf{K})$ for some class K with a term $t(x, y, z)$ such that for all $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c \in A$,

$$t^{\boldsymbol{A}}(a, b, c) = \begin{cases} c & \text{if } a = b; \\ a & \text{otherwise} \end{cases} \tag{5}$$

(see, e.g., [8, Sec. IV.9]). The importance of discriminator varieties derives from the fact that they have many desirable properties such as a representation theorem in terms of Boolean products with subdirectly irreducible factors (see, e.g., [8, Thm. IV.9.4]) or the fact that they are *arithmetical*, in the sense that they satisfy a general form of the Chinese Remainder Theorem (see, e.g., [21, p. 35 & Thm. 2.2.1]).

Recall that $\mathsf{ICM} = \mathbb{ISP}(\mathsf{ICF})$ by definition and that ICM is a variety by Theorem 3.14. Therefore, $\mathsf{ICM} = \mathbb{V}(\mathsf{ICF})$. Furthermore, it is straightforward to check that the term

$$t(x, y, z) = x(x-y)(x-y)^* + z(1-(x-y)(x-y)^*)$$

satisfies (5) for every $\boldsymbol{A} \in \mathsf{ICF}$ (see [11, p. 300]). Consequently, ICM is a discriminator variety. ⊠

7. Dominions

We close this paper with a description of dominions in every class K such that $\mathsf{F} \subseteq \mathsf{K} \subseteq \mathsf{RCR}$.

Definition 7.1. Let $\boldsymbol{A} \in \mathsf{RCR}$. The set of prime ideals of $\boldsymbol{A}$ will be denoted by $\mathsf{Spec}(\boldsymbol{A})$.

Let $\boldsymbol{A} \in \mathsf{RCR}$ and $I \in \mathsf{Spec}(\boldsymbol{A})$. We recall that $\mathsf{frac}(\boldsymbol{A}/I)$ is a field (see Proposition 2.3) and that $\mathsf{acl}(\mathsf{frac}(\boldsymbol{A}/I))^+$ is an implicitly closed field by Proposition 3.9.

Definition 7.2. Let $\boldsymbol{A} \in \mathsf{RCR}$. Then

(i) for every $I \in \mathsf{Spec}(\boldsymbol{A})$ let
$$\mathsf{icf}_I(\boldsymbol{A}) = \mathsf{acl}(\mathsf{frac}(\boldsymbol{A}/I))^+;$$
(ii) for all $I, J \in \mathsf{Spec}(\boldsymbol{A})$ let
$$\mathsf{icf}_{I\times J}(\boldsymbol{A}) = \mathsf{icf}_I(\boldsymbol{A}) \times \mathsf{icf}_J(\boldsymbol{A}).$$

Observe that for all $\boldsymbol{A} \in \mathsf{RCR}$ and $I \in \mathsf{Spec}(\boldsymbol{A})$,

$$\boldsymbol{A}/I \leqslant \mathsf{acl}(\mathsf{frac}(\boldsymbol{A}/I)) = \mathsf{icf}_I(\boldsymbol{A})^-.$$

Our aim is to prove the following.

Theorem 7.3. *Let K be a class such that $\mathsf{F} \subseteq \mathsf{K} \subseteq \mathsf{RCR}$. Moreover, let $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$ and $b \in B$. Then $b \in \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$ if and only if for all $I, J \in \mathsf{Spec}(\boldsymbol{B})$ we have*

$$\langle b + I, b + J\rangle \in \mathsf{Sg}^{\mathsf{icf}_{I\times J}(\boldsymbol{B})}(\{\langle a + I, a + J\rangle : a \in A\}).$$

Before proving Theorem 7.3, we show that this result admits the following improvement in the case where $\mathsf{K} = \mathsf{F}$. Recall that $\mathsf{acl}(\boldsymbol{A})^+$ is an implicitly closed field for every field $\boldsymbol{A}$ (see Proposition 3.9).

Corollary 7.4. *Let $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{F}$. Then*

$$\mathsf{d}_{\mathsf{F}}(\boldsymbol{A}, \boldsymbol{B}) = B \cap \mathsf{Sg}^{\mathsf{acl}(\boldsymbol{B})^{+}}(A).$$

Proof. We will use repeatedly the assumption that $\boldsymbol{B}$ is a field. First, observe that the map $h\colon \boldsymbol{B} \to \mathsf{frac}(\boldsymbol{B}/\{0\})$ defined as $h(b) = b/\{0\}$ for every $b \in B$ is an isomorphism by the definitions of a quotient ring and a fraction field. Together with the definition of $\mathsf{icf}_{\{0\}}(\boldsymbol{B})$, this ensures the existence of an isomorphism $g\colon \mathsf{acl}(\boldsymbol{B})^{+} \to \mathsf{icf}_{\{0\}}(\boldsymbol{B})$ such that $g(b) = b/\{0\}$ for every $b \in B$. Next, observe that the only prime ideal of $\boldsymbol{B}$ is $\{0\}$. Therefore, Theorem 7.3 translates to the following: for every $b \in B$,

$$b \in \mathsf{d}_{\mathsf{F}}(\boldsymbol{A}, \boldsymbol{B}) \iff \langle b, b\rangle \in \mathsf{Sg}^{\mathsf{acl}(\boldsymbol{B})^{+} \times \mathsf{acl}(\boldsymbol{B})^{+}}(\{\langle a, a\rangle : a \in A\}).$$

As the right hand side of the above display is equivalent to $b \in \mathsf{Sg}^{\mathsf{acl}(\boldsymbol{B})^{+}}(A)$, we are done. ⊠

The proof of Theorem 7.3 hinges upon the next observation.

Proposition 7.5. *Let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be a ring homomorphism with $\boldsymbol{A} \in \mathsf{RCR}$ and $\boldsymbol{B} \in \mathsf{F}$. Then there exist $I \in \mathsf{Spec}(\boldsymbol{A})$ and a homomorphism $g\colon \mathsf{icf}_I(\boldsymbol{A}) \to \mathsf{acl}(\boldsymbol{B})^{+}$ of implicitly closed meadows such that $g(a + I) = h(a)$ for every $a \in A$.*

Proof. Since $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, we have $h[\boldsymbol{A}] \leqslant \boldsymbol{B}$. As $\boldsymbol{B}$ is a field, this yields that $h[\boldsymbol{A}]$ is an integral domain. Consequently, the kernel I of h is a prime ideal of $\boldsymbol{A}$ (see Proposition 2.3). Moreover, the map $i\colon \boldsymbol{A}/I \to \boldsymbol{B}$ defined for every $a \in A$ as $i(a + I) = h(a)$ is a ring embedding. Applying Proposition 2.4, we obtain a ring embedding $j\colon \mathsf{frac}(\boldsymbol{A}/I) \to \boldsymbol{B}$ such that

$$j(a + I) = h(a) \text{ for every } a \in A. \tag{6}$$

By Proposition 2.11 the map j extends to a ring embedding $g\colon \mathsf{acl}(\mathsf{frac}(\boldsymbol{A}/I)) \to \mathsf{acl}(\boldsymbol{B})$. Since $\mathsf{acl}(\mathsf{frac}(\boldsymbol{A}/I))^{+}, \mathsf{acl}(\boldsymbol{B})^{+} \in \mathsf{ICM}$ by Proposition 3.9, from Corollary 4.12 it follows that g can be viewed as a homomorphism $g\colon \mathsf{acl}(\mathsf{frac}(\boldsymbol{A}/I))^{+} \to \mathsf{acl}(\boldsymbol{B})^{+}$ of implicitly closed meadows. As $\mathsf{icf}_I(\boldsymbol{A}) = \mathsf{acl}(\mathsf{frac}(\boldsymbol{A}))^{+}$ by definition and g extends j, the validity of (6) concludes the proof. ⊠

We are now ready to prove Theorem 7.3.

Proof. We begin by proving the implication from left to right. To this end, we reason by contraposition. Suppose that there exist $I, J \in \mathsf{Spec}(\boldsymbol{B})$ such that $\langle b + I, b + J\rangle \notin C$, where

$$\boldsymbol{C} = \mathsf{Sg}^{\mathsf{icf}_{I \times J}(\boldsymbol{B})}(\{\langle a + I, a + J\rangle : a \in A\}).$$

We need to show that $b \notin \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$. Notice that $\langle b+I, b+J\rangle \in \mathsf{icf}_{I \times J}(\boldsymbol{B})$ and $\boldsymbol{C} \leqslant \mathsf{icf}_{I \times J}(\boldsymbol{B}) \in \mathsf{ICM}$. Since monomorphisms are regular in ICM by Theorem 6.3, there exist $\boldsymbol{D} \in \mathsf{ICM}$ and a pair of homomorphisms $g, h\colon \mathsf{icf}_{I \times J}(\boldsymbol{B}) \to \boldsymbol{D}$ of implicitly closed meadows such that

$$g{\restriction}_C = h{\restriction}_C \quad \text{and} \quad g(\langle b + I, b + J\rangle) \neq h(\langle b + I, b + J\rangle). \tag{7}$$

Claim 7.6. *We may assume that $\boldsymbol{D} \in \mathsf{ICF}$.*

Proof of the Claim. Recall that $\mathsf{ICM} = \mathbb{ISP}(\mathsf{ICF})$ by definition. Therefore, there exists an embedding $f\colon \boldsymbol{D} \to \prod_{k \in K} \boldsymbol{D}_k$, where $\boldsymbol{D}_k \in \mathsf{ICF}$ for each $k \in K$. From (7) and the assumption that f is an embedding it follows that there exists $j \in K$ such that

$$(p_j \circ f \circ g){\restriction}_C = (p_j \circ f \circ h){\restriction}_C \;\text{ and }\; (p_j \circ f \circ g)(\langle b + I, b + J\rangle) \neq (p_j \circ f \circ h)(\langle b + I, b + J\rangle),$$

where $p_j\colon \prod_{k \in K} \boldsymbol{D}_k \to \boldsymbol{D}_j$ is the projection onto the j-th coordinate. Therefore, we may replace $\boldsymbol{D}$ by $\boldsymbol{D}_j$ in the main proof. As $\boldsymbol{D}_j \in \mathsf{ICF}$, we are done. ⊠

In view of Claim 7.6, we have $\boldsymbol{D}^- \in \mathsf{F}$. As $\mathsf{F} \subseteq \mathsf{K}$ by assumption, we obtain $\boldsymbol{D}^- \in \mathsf{K}$. Moreover, observe that g and h can be viewed as ring homomorphisms $g, h \colon \mathsf{icf}_{I\times J}(\boldsymbol{B})^- \to \boldsymbol{D}^-$. Lastly, let $f \colon \boldsymbol{B} \to \mathsf{icf}_{I\times J}(\boldsymbol{B})^-$ be the ring homomorphism defined as $f(a) = \langle a+I, a+J\rangle$ for every $a \in B$. Therefore, $g \circ f, h \circ f \colon \boldsymbol{B} \to \boldsymbol{D}^-$ are ring homomorphisms with $\boldsymbol{D}^- \in \mathsf{K}$. Since the definitions of f and C ensure that $f[A] \subseteq C$, the left hand side of (7) yields $(g \circ f){\restriction}_A = (h \circ f){\restriction}_A$. On the other hand, the definition of f and the right hand side of (7) guarantee that

$$g(f(b)) = g(\langle b+I, b+J\rangle) \neq h(\langle b+I, b+J\rangle) = h(f(b)).$$

Hence, we conclude that $b \notin \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$, as desired.

Next, we prove the implication from right to left. Suppose, with a view to contradiction, that $\langle b+I, b+J\rangle \in \mathsf{Sg}^{\mathsf{icf}_{I\times J}(\boldsymbol{B})}(\{\langle a+I, a+J\rangle : a \in A\})$ for all $I, J \in \mathsf{Spec}(\boldsymbol{B})$ and that $b \notin \mathsf{d}_{\mathsf{K}}(\boldsymbol{A}, \boldsymbol{B})$. The latter implies that there exists a pair of ring homomorphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ with $\boldsymbol{C} \in \mathsf{K}$ such that

$$g{\restriction}_A = h{\restriction}_A \quad \text{and} \quad g(b) \neq h(b). \tag{8}$$

An argument analogous to the one detailed in the proof of Claim 7.6 (with the only difference that the role of the equality $\mathsf{ICM} = \mathbb{ISP}(\mathsf{ICF})$ in the proof should now be played by $\mathsf{RCR} = \mathbb{ISP}(\mathsf{F})$) shows that $\boldsymbol{C}$ may be chosen in F. Consequently, we can apply Proposition 7.5, obtaining $I, J \in \mathsf{Spec}(\boldsymbol{B})$ and a pair of homomorphisms of implicitly closed meadows $g^* \colon \mathsf{icf}_I(\boldsymbol{B}) \to \mathsf{acl}(\boldsymbol{C})^+$ and $h^* \colon \mathsf{icf}_J(\boldsymbol{B}) \to \mathsf{acl}(\boldsymbol{C})^+$ such that for every $c \in B$,

$$g^*(c+I) = g(c) \quad \text{and} \quad h^*(c+J) = h(c). \tag{9}$$

As $\langle b+I, b+J\rangle \in \mathsf{Sg}^{\mathsf{icf}_{I\times J}(\boldsymbol{B})}(\{\langle a+I, a+J\rangle : a \in A\})$ by assumption, there exist $a_1, \dots, a_n \in A$ and a term $t(x_1, \dots, x_n)$ of ICM such that

$$\langle b+I, b+J\rangle = t^{\mathsf{icf}_{I\times J}(\boldsymbol{B})}(\langle a_1+I, a_1+J\rangle, \dots, \langle a_n+I, a_n+J\rangle). \tag{10}$$

Recall that $\boldsymbol{C} \leqslant \mathsf{acl}(\boldsymbol{C})$ by the definition of an algebraic closure. Therefore, $g[B] \cup h[B] \subseteq C \subseteq \mathsf{acl}(C)^+$. We will show that

$$g(b) = t^{\mathsf{acl}(\boldsymbol{C})^+}(g(a_1), \dots, g(a_n)) \quad \text{and} \quad h(b) = t^{\mathsf{acl}(\boldsymbol{C})^+}(h(a_1), \dots, h(a_n)). \tag{11}$$

By symmetry it suffices to prove the left hand side of the above display. To this end, observe that

$$\begin{aligned} g(b) &= g^*(b+I) = g^*(t^{\mathsf{icf}_I(\boldsymbol{B})}(a_1+I, \dots, a_n+I)) \\ &= t^{\mathsf{acl}(\boldsymbol{C})^+}(g^*(a_1+I), \dots, g^*(a_n+I)) = t^{\mathsf{acl}(\boldsymbol{C})^+}(g(a_1), \dots, g(a_n)), \end{aligned}$$

where the first and the last equalities hold by the left hand side of (9), the second by (10) and the definition of $\mathsf{icf}_{I\times J}(\boldsymbol{B})$, and the third because $g^* \colon \mathsf{icf}_I(\boldsymbol{B}) \to \mathsf{acl}(\boldsymbol{C})^+$ is a homomorphism of implicitly closed meadows and $t(x_1, \dots, x_n)$ a term of ICM. This establishes (11). Lastly, from (11), the left hand side of (8), and the fact that $a_1, \dots, a_n \in A$ it follows that

$$g(b) = t^{\mathsf{acl}(\boldsymbol{C})^+}(g(a_1), \dots, g(a_n)) = t^{\mathsf{acl}(\boldsymbol{C})^+}(h(a_1), \dots, h(a_n)) = h(b),$$

a contradiction with the right hand side of (8). ⊠

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