A CATEGORICAL DESCRIPTION OF SIMPLE BETH COMPANIONS

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Abstract

A pp expansion of a quasivariety K is said to be *simple* when it is of the form $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. For instance, when K has the amalgamation property, all its pp expansions are simple. It is shown that the simple pp expansions of a quasivariety K coincide with the quasivarieties M for which the forgetful functor $U \colon \mathsf{M} \to \mathsf{K}$ is well defined and induces an isomorphism from M to a mono-reflective subcategory of K. As a consequence, if a quasivariety K possesses a simple Beth companion M, then M is the unique (up to term equivalence) quasivariety whose monomorphisms are regular that, moreover, satisfy the categorical description of simple pp expansions of K given above.

1. Expansions of quasivarieties

We denote the class operators of closure under isomorphisms, subalgebras, direct products, finite direct products, and ultraproduts by $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}^{<\omega}$, and $\mathbb{P}_{\mathrm{u}}$, respectively. A class of similar algebras is said to be a *quasivariety* when it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{u}}$ or, equivalently, when it can be axiomatized by a set of *quasiequations*, that is, formulas of the form $\bigsqcap \Phi \to \varphi$, where $\Phi \cup \{\varphi\}$ is a finite set of equations and $\bigsqcap$ the conjunction symbol (see, e.g., [BS12, Thm. V.2.25]).

Let K be a quasivariety. A congruence θ of an algebra $\boldsymbol{A}$ is a K-*congruence* when $\boldsymbol{A}/\theta \in \mathsf{K}$. For every $X \subseteq A \times A$ there exists the least K-congruence of $\boldsymbol{A}$ containing X, which we denote by $\mathsf{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(X)$ (see, e.g., [Gor98, Sec. 1.4.4, p. 39]). Every quasivariety can be viewed as a category whose objects are its members and whose arrows are the homomorphisms between them.

Remark 1.1. In every quasivariety, monomorphisms coincide with embeddings, i.e., injective homomorphisms (see, e.g., [McK96, p. 222]). ⊠

1.1. **The basic adjunction.** We denote the language of a quasivariety K by $\mathscr{L}_{\mathsf{K}}$. Given a quasivariety K and a language $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{K}}$, we denote the $\mathscr{L}$-reduct of a member $\boldsymbol{A}$ of K by $\boldsymbol{A}{\restriction}_{\mathscr{L}}$. Given a pair of quasivarieties K and M, the forgetful functor $U \colon \mathsf{M} \to \mathsf{K}$ is well defined if and only if $\mathscr{L}_{\mathsf{K}} \subseteq \mathscr{L}_{\mathsf{M}}$ and $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ for every $\boldsymbol{A} \in \mathsf{M}$.

Definition 1.2. Let K and M be a pair of quasivarieties. We say that M is an *expansion* of K when the forgetful functor $U \colon \mathsf{M} \to \mathsf{K}$ is well defined.

Let K be a quasivariety. We recall that for each nonempty set X the free algebra $\boldsymbol{T}_{\mathsf{K}}(X)$ over K with set free generators X belongs to K (see, e.g., [BS12, Thm. II.10.12]). We will often identify a term $t(x_1, \dots, x_n)$ of K with $x_1, \dots, x_n \in X$ with its equivalence class in $\boldsymbol{T}_{\mathsf{K}}(X)$.

Consider an expansion M of a quasivariety K. The forgetful functor $U\colon \mathsf{M} \to \mathsf{K}$ has a left adjoint F which can be described as follows. With each $\boldsymbol{A} \in \mathsf{K}$ we associate a set of elements of pair of terms of K as follows:

$$\mathsf{diag}^+(\boldsymbol{A}) = \{\langle t(a_1,\dots,a_n), s(a_1,\dots,a_n)\rangle : t(x_1,\dots,x_n) \text{ and } s(x_1,\dots,x_n) \text{ are terms of } \mathsf{K},\\ a_1,\dots,a_n \in A, \text{ and } t^{\boldsymbol{A}}(a_1,\dots,a_n) = s^{\boldsymbol{A}}(a_1,\dots,a_n)\}.$$

As M is an expansion of K, we can view $\mathsf{diag}^+(\boldsymbol{A})$ as a subset of $T_{\mathsf{M}}(A) \times T_{\mathsf{M}}(A)$. Then the algebra

$$F(\boldsymbol{A}) = \boldsymbol{T}_{\mathsf{M}}(A)/\theta_{\boldsymbol{A}}, \text{ where } \theta_{\boldsymbol{A}} = \mathsf{Cg}_{\mathsf{M}}^{\boldsymbol{T}_{\mathsf{M}}(A)}(\mathsf{diag}^+(\boldsymbol{A}))$$

belongs to M because θ is an M-congruence of $\boldsymbol{T}_{\mathsf{M}}(A)$ by definition. Furthermore, for each homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ let $F(h)\colon F(\boldsymbol{A}) \to F(\boldsymbol{B})$ be the homomorphism defined for all $a_1,\dots,a_n \in A$ and terms $t(x_1,\dots,x_n)$ of M as $F(h)(t(a_1,\dots,a_n)/\theta_{\boldsymbol{A}}) = t(h(a_1),\dots,h(a_n))/\theta_{\boldsymbol{B}}$. We call $F\colon \mathsf{K} \to \mathsf{M}$ the *free extension functor* associated with $U\colon \mathsf{M} \to \mathsf{K}$.

We denote the unit and the counit of the adjunction $F \dashv U$ by $\eta\colon \mathsf{id}_{\mathsf{K}} \to UF$ and $\epsilon\colon FU \to \mathsf{id}_{\mathsf{M}}$, respectively. For every $\boldsymbol{A} \in \mathsf{K}$ the map $\eta_{\boldsymbol{A}}\colon \boldsymbol{A} \to UF(\boldsymbol{A})$ is defined for every $a \in A$ as $\eta_{\boldsymbol{A}}(a) = a/\theta_{\boldsymbol{A}}$. Moreover, for every $\boldsymbol{B} \in \mathsf{M}$ the map $\epsilon_{\boldsymbol{B}}\colon FU(\boldsymbol{B}) \to \boldsymbol{B}$ is defined for all $b_1,\dots,b_n \in B$ and terms $t(x_1,\dots,x_n)$ of M as $\epsilon_{\boldsymbol{B}}(t(b_1,\dots,b_n)/\theta_{U(\boldsymbol{B})}) = t^{\boldsymbol{B}}(b_1,\dots,b_n)$.

1.2. **Implicit operations.** An *implicit operation* of a quasivariety K is a family of partial functions on the members of K that is globally preserved by homomorphisms and definable by a formula (see [CKM25, Sec. 3]). More precisely, an n-ary *operation* of K is a sequence $f = \langle f^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K}\rangle$, where each $f^{\boldsymbol{A}}\colon \mathsf{dom}(f^{\boldsymbol{A}}) \to A$ is a partial n-ary function on A with domain $\mathsf{dom}(f^{\boldsymbol{A}}) \subseteq A^n$ that is globally preserved by the homomorphisms between members of K. The latter means that for every homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $\langle a_1,\dots,a_n\rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ we have

$$\langle h(a_1),\dots,h(a_n)\rangle \in \mathsf{dom}(f^{\boldsymbol{B}}) \text{ and } h(f^{\boldsymbol{A}}(a_1,\dots,a_n)) = f^{\boldsymbol{B}}(h(a_1),\dots,h(a_n)).$$

For $n \geqslant 1$, an n-ary operation f of K is said to be *implicit* when it is defined by some first order formula $\varphi(x_1,\dots,x_n,y)$, in the sense that for all $\boldsymbol{A} \in \mathsf{K}$ and $a_1,\dots,a_n,b \in A$,

$$\boldsymbol{A} \vDash \varphi(a_1,\dots,a_n,b) \iff \langle a_1,\dots,a_n\rangle \in \mathsf{dom}(f^{\boldsymbol{A}}) \text{ and } f^{\boldsymbol{A}}(a_1,\dots,a_n) = b.$$

For instance, "taking inverses" is an implicit operation of the class of all monoids because it can be defined by the conjunction of equations $\varphi(x,y) = (xy \approx 1) \sqcap (yx \approx 1)$ and monoid homomorphisms preserve inverses when they exist.

Notably, implicit operations admit a description in terms of *primitive positive formulas* (for short, *pp formulas*), that is, formulas of the form $\exists x_1,\dots,x_n\varphi$, where φ is a conjunction of equations. More precisely, if f is an implicit operation of a quasivariety K, there exist implicit operations $f_1,\dots,f_n$ of K definable by pp formulas such that $f^{\boldsymbol{A}} = f_1^{\boldsymbol{A}} \cup \dots \cup f_n^{\boldsymbol{A}}$ for every $\boldsymbol{A} \in \mathsf{K}$ (see [CKM25, Cor. 3.10]). Consequently, the pp definable implicit operations of K form the building blocks of all implicit operations of K and, therefore, we restrict our attention to them. The next result simplifies the task of determining whether a function can be defined by a pp formula (see [CV15, Thm. 6.3(4)]).

Theorem 1.3. *Let* K *be a class of algebras,* $f \in \mathscr{L}_{\mathsf{K}}$*, and* $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{K}}$*. Then there exists a pp formula of* $\mathscr{L}$ *that defines* f *in* K *if and only if* f *is preserved by every homomorphism* $h\colon \boldsymbol{A}{\restriction}_{\mathscr{L}} \to \boldsymbol{B}{\restriction}_{\mathscr{L}}$ *with* $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{P}_{\mathrm{u}}\mathbb{P}^{<\omega}(\mathsf{K})$*.*

In general, the implicit operations of a quasivariety K need not be componentwise total. Therefore, we say that an implicit operation f of K is *extendable* when for all $\boldsymbol{A} \in \mathsf{K}$ and $\langle a_1, \dots, a_n\rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ there exists an algebra $\boldsymbol{B} \in \mathsf{K}$ extending $\boldsymbol{A}$ such that $\langle a_1, \dots, a_n\rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$. The class of extendable implicit operations of K will be denoted by $\mathsf{ext}(\mathsf{K})$, and that of pp definable extendable implicit operations of K by $\mathsf{ext}_{\mathrm{PP}}(\mathsf{K})$. The next result justifies the term "extendable" (see [CKM25, Prop. 8.1 and Thm. 8.4]).

Theorem 1.4. *Let* K *be a quasivariety and* $\boldsymbol{A} \in \mathsf{K}$*. Then there exists* $\boldsymbol{B} \in \mathsf{K}$ *with* $\boldsymbol{A} \leqslant \boldsymbol{B}$ *such that* $f^{\boldsymbol{B}}$ *is total and extends* $f^{\boldsymbol{A}}$ *for each* $f \in \mathsf{ext}(\mathsf{K})$*.*

The task of constructing extendable implicit operations is simplified by the following observation (see [CKM25, Cor. 3.11]).

Proposition 1.5. *Let* K *be a quasivariety and* $\varphi(x_1, \dots, x_n, y)$ *a pp formula. If every* $\boldsymbol{A}$ *can be extended to some* $\boldsymbol{B} \in \mathsf{K}$ *on which* φ *defines a total* n*-ary function, then* φ *defines an* n*-ary member of* $\mathsf{ext}_{\mathrm{PP}}(\mathsf{K})$*.*

In order to add a family of implicit operations $\mathcal{F} \subseteq \mathsf{ext}_{\mathrm{PP}}(\mathsf{K})$ to a quasivariety K, we proceed as follows. Let $\mathscr{L}_{\mathcal{F}}$ the language obtained by adding to $\mathscr{L}_{\mathsf{K}}$ a new n-ary function symbol g_f for each n-ary $f \in \mathcal{F}$. Then we expand every member $\boldsymbol{A}$ of K in which $\{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a family of total functions to an algebra $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ in the language $\mathscr{L}_{\mathcal{F}}$ by interpreting g_f as $f^{\boldsymbol{A}}$ for each $f \in \mathcal{F}$. The *pp expansion* of K induced by $\mathcal{F}$ is $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. We will make use of the following observation (see [CKM25, Prop. 10.2]).

Proposition 1.6. *Let* M *be a pp expansion of a quasivariety* K*. Then* $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ *for every* $\boldsymbol{A} \in \mathsf{M}$*.*

2. Simple Beth companions

In this note, we shall focus on the following kind of pp expansions.

Definition 2.1. A pp expansion of a quasivariety K is said to be *simple* when it is of the form $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ for some $\mathcal{F} \subseteq \mathsf{ext}_{\mathrm{PP}}(\mathsf{K})$.

Simple pp expansions are relatively common, as witnessed by the following.

Theorem 2.2. *Every pp expansion of a quasivariety with the amalgamation property is simple.*

Another source of pp expansions derives from the following kind of implicit operations.

Definition 2.3. Let K be a quasivariety and $f \in \mathsf{imp}_{\mathrm{PP}}(\mathsf{K})$. We say that f has *unique witnesses* when it can be defined by pp formula

$$\exists z_1, \dots, z_m \varphi(z_1, \dots, z_m, x_1, \dots, x_n, y)$$

such that for all $\boldsymbol{A} \in \mathsf{K}$ and $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ there exists a unique tuple $\langle b_1, \dots, b_m \rangle \in A^m$ satisfying

$$\boldsymbol{A} \vDash \varphi(b_1, \dots, b_m, a_1, \dots, a_n, f^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

We denote by $\mathsf{imp}_{\mathrm{PP!}}(\mathsf{K})$ the subset of $\mathsf{imp}_{\mathrm{PP}}(\mathsf{K})$ consisting of the definable implicit operations with unique witnesses. We also let $\mathsf{ext}_{\mathrm{PP!}}(\mathsf{K}) = \mathsf{ext}_{\mathrm{PP}}(\mathsf{K}) \cap \mathsf{imp}_{\mathrm{PP!}}(\mathsf{K})$.

We also recall that a pp expansion M of a quasivariety K is said to be a *Beth companion* of K when monomorphisms are regular in M (see [CKM25, Rmk. 6.4 & Thm. 11.6]). Although a quasivariety K may lack a Beth companion (see, e.g., [CKM25, Thm. 14.17] and [CKM26, Thm. 6.1]), when it possesses one, it must be essentially unique, as we proceed to illustrate.

Let M_1 and M_2 be a pair of pp expansions of a quasivariety K. For $i = 1, 2$ let T_i be the set of terms of M_i with variables in $\{x_n : n \in \mathbb{N}\}$. Let $\rho\colon \mathscr{L}_{\mathsf{M}_2} \to T_1$ be a map that preserves the arities. For each $\mathscr{L}_{\mathsf{M}_1}$-algebra $\boldsymbol{A}$ let $\rho(\boldsymbol{A})$ be the $\mathscr{L}_{\mathsf{M}_2}$-algebra with universe A such that $f^{\rho(\boldsymbol{A})} = \rho(f)^{\boldsymbol{A}}$ for each function symbol f in $\mathscr{L}_{\mathsf{M}_2}$. Similarly, given an arity-preserving map $\tau\colon \mathscr{L}_{\mathsf{M}_1} \to T_2$ and an $\mathscr{L}_{\mathsf{M}_2}$-algebra $\boldsymbol{B}$, we define an $\mathscr{L}_{\mathsf{M}_1}$-algebra $\tau(\boldsymbol{B})$. We say that M_1 and M_2 are *faithfully term equivalent relative to* K if there exist arity-preserving maps $\tau\colon \mathscr{L}_{\mathsf{M}_1} \to T_2$ and $\rho\colon \mathscr{L}_{\mathsf{M}_2} \to T_1$ such that $\tau(f) = f(x_1, \dots, x_n)$ and $\rho(f) = f(x_1, \dots, x_n)$ for each n-ary function symbol f in $\mathscr{L}_{\mathsf{K}}$, and for all $\boldsymbol{A} \in \mathsf{M}_1$ and $\boldsymbol{B} \in \mathsf{M}_2$ we have

(i) $\rho(\boldsymbol{A}) \in \mathsf{M}_2$;
(ii) $\tau(\boldsymbol{B}) \in \mathsf{M}_1$;
(iii) $\tau\rho(\boldsymbol{A}) = \boldsymbol{A}$;
(iv) $\rho\tau(\boldsymbol{B}) = \boldsymbol{B}$.

When they exist, Beth companions are essentially unique in the following sense.

Theorem 2.4 ([CKM25, Thm. 11.7])**.** *All the Beth companions of a quasivariety* K *are faithfully term equivalent relative to* K.

We will make use of the following characterization of Beth companions (see [CKM25, Thm. 11.6]).

Theorem 2.5. *The following are equivalent for a pp expansion* M *of a quasivariety* K*:*

(i) M *is a Beth companion of* K*;*
(ii) *for all* $\boldsymbol{A} \in \mathsf{M}$*,* $f \in \mathsf{imp}(\mathsf{K})$*, and* $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}})$ *there exists a term* t *of* M *such that*

$$t^{\boldsymbol{A}}(a_1, \dots, a_n) = f^{\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \dots, a_n).$$

As Beth companions are pp expansions, the concept of "simplicity" applies to them as well.

Definition 2.6. A Beth companion of a quasivariety K is said to be *simple* when it is a simple pp expansion of K.

Every Beth companion obtained by adding extendable pp definable implicit operations with unique witnesses is simple, as we proceed to illustrate.

Theorem 2.7. *Every Beth companion of a quasivariety* K *that is induced by a subset of* $\mathsf{ext}_{\mathrm{PP!}}(\mathsf{K})$ *is simple.*

Proof. Let $\mathcal{F} \subseteq \mathsf{ext}_{\text{PP!}}(\mathsf{K})$ and consider the corresponding pp expansion $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. Assume that $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is a Beth companion of K. We need to prove that it is simple. To this end, it suffices to show that $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}]) \subseteq \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$, for in this case $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}]) = \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$.

Consider $\boldsymbol{A} \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. Then $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ by Proposition 1.6. We will show that $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}$ is total and coincides with $g_f^{\boldsymbol{A}}$ for every $f \in \mathcal{F}$ because, in this case, we would get that $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}]$ is well defined and $\boldsymbol{A} = \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}] \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$, as desired.

To this end, consider an n-ary $f \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathsf{ext}_{\text{PP!}}(\mathsf{K})$ by assumption, the operation f is defined by a pp formula

$$\psi = \exists z_1, \dots, z_m \varphi(z_1, \dots, z_m, x_1, \dots, x_n, y)$$

of $\mathscr{L}_{\mathsf{K}}$ such that

$$\begin{gathered}\text{for all } \boldsymbol{C} \in \mathsf{K} \text{ and } \langle c_1, \dots, c_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}}) \text{ there exists a unique tuple} \\ \langle d_1, \dots, d_m \rangle \in C^m \text{ satisfying } \boldsymbol{C} \vDash \varphi(d_1, \dots, d_m, c_1, \dots, c_n, f^{\boldsymbol{C}}(c_1, \dots, c_n)).\end{gathered} \tag{1}$$

for all $\boldsymbol{C} \in \mathsf{K}$ and $\langle c_1, \dots, c_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$ there exists a unique tuple $\langle d_1, \dots, d_m \rangle \in C^m$ satisfying

$$\boldsymbol{C} \vDash \varphi(d_1, \dots, d_m, c_1, \dots, c_n, f^{\boldsymbol{C}}(c_1, \dots, c_n)).$$

Consider $a_1, \dots, a_n \in A$. As $\boldsymbol{A} \in \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$, there exists $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ with $\boldsymbol{A} \leqslant \boldsymbol{B}$. From $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ it follows that $g_f^{\boldsymbol{B}}$ is defined by ψ. Together with $\boldsymbol{A} \leqslant \boldsymbol{B}$, this guarantees the existence of a tuple $\langle b_1, \dots, b_m \rangle \in B^m$ such that

$$\boldsymbol{B} \vDash \varphi(b_1, \dots, b_m, a_1, \dots, a_n, g_f^{\boldsymbol{A}}(a_1, \dots, a_n)). \tag{2}$$

To conclude the proof, it will be enough to show that $b_1, \dots, b_m \in A$.

For suppose this is the case. As φ is a formula of $\mathscr{L}_{\mathsf{K}}$, $\boldsymbol{A} \leqslant \boldsymbol{B}$, and $a_1, \dots, a_n, g_f^{\boldsymbol{A}}(a_1, \dots, a_n) \in A$, the above display and the assumption that $b_1, \dots, b_m \in A$ imply

$$\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \vDash \varphi(b_1, \dots, b_m, a_1, \dots, a_n, g_f^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

Since ψ defines f, we obtain $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}})$ and $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \dots, a_n) = g_f^{\boldsymbol{A}}(a_1, \dots, a_n)$. Hence, we conclude that $f^{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}$ is total and coincides with $g_f^{\boldsymbol{A}}$, as desired.

Therefore, it only remains to show that $b_1, \dots, b_m \in A$. Consider a positive $k \leqslant m$. We will show that $b_k \in A$. Consider the pp formula

$$\psi_k(x_1, \dots, x_n, y, z_k) = \exists z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_m \varphi(z_1, \dots, z_m, x_1, \dots, x_n, y)$$

of $\mathscr{L}_{\mathsf{K}}$. We will show that ψ_k defines a member of $\mathsf{imp}_{\text{PP}}(\mathsf{K})$. Since pp formulas are preserved by homomorphisms, it suffices to prove that ψ_k defines a partial function on the members of K. To this end, consider $\boldsymbol{C} \in \mathsf{K}$ and $c_1, \dots, c_{n+1}, d, e \in C$ such that

$$\boldsymbol{C} \vDash \psi_k(c_1, \dots, c_{n+1}, d) \sqcap \psi_k(c_1, \dots, c_{n+1}, e).$$

By the definition of ψ_k there exist $p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_m, q_1, \dots, q_{k-1}, q_{k+1}, \dots, q_m \in C$ such that

$$\begin{gathered}\boldsymbol{C} \vDash \varphi(p_1, \dots, p_{k-1}, d, p_{k+1}, \dots, p_m, c_1, \dots, c_n, c_{n+1}); \\ \boldsymbol{C} \vDash \varphi(q_1, \dots, q_{k-1}, e, q_{k+1}, \dots, q_m, c_1, \dots, c_n, c_{n+1}).\end{gathered}$$

Together with the assumption that ψ defines f, the above display yields $\langle c_1, \ldots, c_n \rangle \in \mathsf{dom}(f^{\boldsymbol{C}})$ and $c_{n+1} = f^{\boldsymbol{C}}(c_1, \ldots, c_n)$. Consequently, (1) ensures that $e = d$. Hence, we conclude that ψ_k defines a member f_k of $\mathsf{imp}_{\mathrm{PP}}(\mathsf{K})$.

Recall that $\boldsymbol{B}$ belongs to the pp expansion $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ of K. Then $\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ by Proposition 1.6. From (2) and the assumption that ψ_k defines f_k it follows that

$$f_k^{\boldsymbol{B}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \ldots, a_n, g_f^{\boldsymbol{A}}(a_1, \ldots, a_n)) = b_k.$$

As $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ is a Beth companion of K by assumption, we can apply Theorem 2.5, obtaining a term $t(x_1, \ldots, x_{n+1})$ of $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ such that

$$t^{\boldsymbol{B}}(a_1, \ldots, a_n, g_f^{\boldsymbol{A}}(a_1, \ldots, a_n)) = b_k.$$

Since $\boldsymbol{A} \leqslant \boldsymbol{B}$ by assumption and $a_1, \ldots, a_n, g_f^{\boldsymbol{A}}(a_1, \ldots, a_n) \in A$, we conclude that $b_k \in A$. ⊠

3. The main result

We recall that a full subcategory M of a category K is said *reflective* when the inclusion functor $i \colon \mathsf{M} \to \mathsf{K}$ has a left adjoint. In this case, the counit of the resulting adjunction is a natural isomorphism (see, e.g., [AHS06, Thm. 19.14(4)]). When, in addition, the unit is componentwise a monomorphism, we say that M is a *mono-reflective* subcategory of K (see, e.g., [AHS06, Def. 16.1]). Lastly, given a functor $F \colon \mathsf{K} \to \mathsf{M}$, we denote the direct image of K under F, viewed as a subcategory of M, by $F[\mathsf{K}]$. Clearly, $F[\mathsf{K}]$ is full if and only if so is F.

Our main result is the following categorical description of simple pp expansions.

Theorem 3.1. *Let* M *be an expansion of a quasivariety* K. *Then the following are equivalent:*

(i) M *is a simple pp expansion of* K*;*
(ii) *the unit of the adjunction* $F \dashv U$ *is componentwise a monomorphism and the counit is a natural isomorphism;*
(iii) *the forgetful functor* $U \colon \mathsf{M} \to U[\mathsf{M}]$ *is an isomorphism from* M *to a mono-reflective subcategory of* K.

The following description of simple Beth companion is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let* M *be an expansion of a quasivariety* K. *Then* M *is a simple Beth companion of* K *if and only if monomorphisms are regular in* M *and any of the equivalent conditions in Theorem 3.1 holds.*

We shall now prove Theorem 3.1.

Proof. (iii)⇒(ii): Straightforward. (ii)⇒(iii): We begin by showing that the forgetful functor $U \colon \mathsf{M} \to U[\mathsf{M}]$ is an isomorphism. As it is always bijective on objects, it suffices to show that it is fully faithful. The latter holds because the counit of the adjunction $F \dashv U$ is a natural isomorphism by assumption and this always guarantees that the full faithfulness of the right adjoint (see, e.g., [AHS06, Thm. 19.14(4)]).

Next, we prove that $U[\mathsf{M}]$ is a mono-reflective subcategory of K. First, $U[\mathsf{M}]$ is a full subcategory of K because $U \colon \mathsf{M} \to \mathsf{K}$ is full. As the unit of the adjunction $F \dashv U$ is componentwise a monomorphism and $U\colon \mathsf{M} \to U[\mathsf{M}]$ is an isomorphism, we deduce that $U[\mathsf{M}]$ is a mono-reflective subcategory of K.

(ii)⇒(i): We begin with the following observation.

Claim 3.3. *For every n-ary $f \in \mathscr{L}_{\mathsf{M}}$ there exists a pp formula $\varphi_f(x_1, \dots, x_n, y)$ of $\mathscr{L}_{\mathsf{K}}$ such that for all $\boldsymbol{A} \in \mathsf{M}$ and $a_1, \dots, a_n, b \in A$,*

$$f^{\boldsymbol{A}}(a_1, \dots, a_n) = b \Longleftrightarrow \boldsymbol{A} \vDash \varphi_f(a_1, \dots, a_n, b).$$

Proof of the Claim. As M is a quasivariety, it is closed under $\mathbb{P}$ and $\mathbb{P}_{\mathsf{u}}$. Therefore, in view of Theorem 1.3, it suffices to show that $U\colon \mathsf{M} \to \mathsf{K}$ is full, which can be shown as in the proof of the implication (ii)⇒(iii). ⊠

Next, we verify the following.

Claim 3.4. *For each $f \in \mathscr{L}_{\mathsf{M}}$ the formula φ_f defines a member f^* of $\mathsf{ext}_{\text{PP}}(\mathsf{K})$.*

Proof of the Claim. From Claim 3.3 it follows that the pp formula $\varphi_f(x_1, \dots, x_n, y)$ defines a total n-ary function on each member of M. As φ_f is a formula of $\mathscr{L}_{\mathsf{K}}$, it also defines a total n-ary function on each member of $\{U(\boldsymbol{B}) : \boldsymbol{B} \in \mathsf{M}\}$. Consequently, in view of Proposition 1.5, it suffices to show that every member $\boldsymbol{A}$ of K embeds into an algebra of the form $U(\boldsymbol{B})$ with $\boldsymbol{B} \in \mathsf{M}$. To this end, consider $\boldsymbol{A} \in \mathsf{K}$. By assumption the map $\epsilon_{\boldsymbol{A}}\colon \boldsymbol{A} \to UF(\boldsymbol{A})$ is a monomorphism with $F(\boldsymbol{A}) \in \mathsf{M}$. By Remark 1.1 we obtain that $\boldsymbol{A}$ embeds into $UF(\boldsymbol{A})$. Thus, taking $\boldsymbol{B} = F(\boldsymbol{A})$, we are done. ⊠

By Claim 3.4 the set $\mathcal{F} = \{f^* : f \in \mathscr{L}_{\mathsf{M}} - \mathscr{L}_{\mathsf{K}}\}$ induces a pp expansion $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ of K. To conclude the proof, it only remains to show that this pp expansion is simple and coincides with M. The next claim establishes both facts at once.

Claim 3.5. *We have $\mathsf{M} = \mathsf{K}[\mathscr{L}_{\mathcal{F}}] = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$.*

Proof of the Claim. As M is closed under $\mathbb{S}$ because it is a quasivariety, it suffices to show that $\mathsf{M} = \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. We begin with the inclusion from left to right. Consider $\boldsymbol{A} \in \mathsf{M}$. As $U(\boldsymbol{A}) \in \mathsf{K}$, it suffices to show that $U(\boldsymbol{A})[\mathscr{L}_{\mathcal{F}}]$ is defined and coincides with $\boldsymbol{A}$ or, equivalently, that $f^{*U(\boldsymbol{A})}$ is total and coincides with $f^{\boldsymbol{A}}$ for each $f \in \mathscr{L}_{\mathsf{M}} - \mathscr{L}_{\mathsf{K}}$. To this end, consider an n-ary $f \in \mathscr{L}_{\mathsf{M}} - \mathscr{L}_{\mathsf{K}}$ and $a_1, \dots, a_n \in A$. By Claim 3.3 we have $\boldsymbol{A} \vDash \varphi_f(a_1, \dots, a_n, f^{\boldsymbol{A}}(a_1, \dots, a_n))$. Since φ_f is a formula in $\mathscr{L}_{\mathsf{K}}$, this yields $U(\boldsymbol{A}) \vDash \varphi_f(a_1, \dots, a_n, f^{\boldsymbol{A}}(a_1, \dots, a_n))$. As φ_f defines f^* by Claim 3.4, we conclude that $\langle a_1, \dots, a_n \rangle \in f^{*U(\boldsymbol{A})}$ and $f^{*U(\boldsymbol{A})}(a_1, \dots, a_n) = f^{\boldsymbol{A}}(a_1, \dots, a_n)$. Hence, $f^{*U(\boldsymbol{A})}$ is total and coincides with $f^{\boldsymbol{A}}$.

Next, we prove the inclusion $\mathsf{K}[\mathscr{L}_{\mathcal{F}}] \subseteq \mathsf{M}$. Consider $\boldsymbol{A} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. By the definition of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ we have $\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$ and $\boldsymbol{A} = \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}]$. It will be enough to prove that the map $h\colon \boldsymbol{A} \to F(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}})$ defined for every $a \in A$ as $h(a) = \eta_{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a)$ is an embedding. For suppose this is the case. Then $\boldsymbol{A} \in \mathbb{IS}(F(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}))$. Since $F(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}) \in \mathsf{M}$ and M is closed under $\mathbb{IS}$ because it is a quasivariety, we conclude that $\boldsymbol{A} \in \mathsf{M}$, as desired.

Recall that $\eta_{\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}$ is by assumption a monomorphism (i.e., an embedding by Remark 1.1). The definition of h guarantees that it is a well-defined injective map that preserves the operations in $\mathscr{L}_{\mathsf{K}}$. It only remains to prove that h preserves the operations in $\mathscr{L}_{\mathsf{M}} - \mathscr{L}_{\mathsf{K}}$ as well. Consider an n-ary $f \in \mathscr{L}_{\mathsf{M}} - \mathscr{L}_{\mathsf{K}}$ and $a_1, \dots, a_n \in A$. Since $\boldsymbol{A} = \boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}[\mathscr{L}_{\mathcal{F}}]$ and f^* is defined by φ_f by Claim 3.4 we have

$$\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}} \vDash \varphi_f(a_1, \dots, a_n, f^{*\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \dots, a_n)).$$

As φ_f is a pp formula in $\mathscr{L}_{\mathsf{K}}$ and h preserves the operations of $\mathscr{L}_{\mathsf{K}}$, we obtain that h preserves φ. Together with the above display, this implies

$$F(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}) \vDash \varphi_f(h(a_1), \dots, h(a_n), h(f^{*\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \dots, a_n))).$$

By Claim 3.3 and $F(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}) \in \mathsf{M}$ we conclude that

$$h(f^{*\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}}}(a_1, \dots, a_n)) = f^{F(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}})}(h(a_1), \dots, h(a_n)).$$

Hence, $h \colon \boldsymbol{A} \to F(\boldsymbol{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}})$ is an embedding, as desired. ☒

(i)⇒(ii): Let M be a simple pp expansion of K of the form $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ for some $\mathcal{F} \subseteq \mathsf{ext}_{\text{PP}}(\mathsf{K})$. We begin by showing that each component of the unit η is a monomorphism. Consider $\boldsymbol{A} \in \mathsf{K}$. Since $\mathcal{F} \subseteq \mathsf{ext}_{\text{PP}}(\mathsf{K})$, there exists $\boldsymbol{B} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ for which the inclusion map $i \colon \boldsymbol{A} \to U(\boldsymbol{B})$ is a well-defined embedding. As $F \dashv U$, there exists a homomorphism $g \colon F(\boldsymbol{A}) \to \boldsymbol{B}$ such that $U(g) \circ \eta_{\boldsymbol{A}} = i$ (see, e.g., [AHS06, Prop. 19.7(2)]). Since i is an embedding, so is $\eta_{\boldsymbol{A}}$. Consequently, $\eta_{\boldsymbol{A}}$ is a monomorphism by Remark 1.1.

Next, we prove that the counit ϵ is a natural isomorphism. To this end, it suffices to show that $U \colon \mathsf{M} \to \mathsf{K}$ is fully faithful (see, e.g., [AHS06, Thm. 19.14(4)]). As the forgetful functor is always faithful, it only remains to show that it is full. Consider $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{M} = \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ and a homomorphism $h \colon U(\boldsymbol{A}) \to U(\boldsymbol{B})$. From [CKM25, Prop. 9.5] it follows that h is also a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Hence, U is full, as desired. ☒

We close this note by observing that the property of "being simple" is preserved by faithful term equivalences between pp expansions.

Corollary 3.6. *Let* K *be a quasivariety and* $\mathsf{M}_1, \mathsf{M}_2$ *a pair of pp expansions of* K *that are faithfully term equivalent relative to* K*. The* M_1 *is simple if and only if so is* M_2*.*

Proof. Let $\tau \colon \mathscr{L}_{\mathsf{M}_1} \to T_2$ and $\rho \colon \mathscr{L}_{\mathsf{M}_2} \to T_1$ witness the faithful term equivalence, and assume that M_1 is a simple pp expansion. Then Theorem 3.1 implies that the forgetful functor $U_1 \colon \mathsf{M}_1 \to \mathsf{K}$ restricts to an isomorphism from M_1 to a mono-reflective subcategory $U_1[\mathsf{M}_1]$ of K. Let $U_2 \colon \mathsf{M}_2 \to \mathsf{K}$ be the forgetful functor. Let also $\tau \colon \mathsf{M}_2 \to \mathsf{M}_1$ be the functor that maps $\boldsymbol{A} \in \mathsf{M}_2$ to $\tau(\boldsymbol{A}) \in \mathsf{M}_1$ and is the identity on morphisms. Similarly, define $\rho \colon \mathsf{M}_1 \to \mathsf{M}_2$ as the functor that maps $\boldsymbol{A} \in \mathsf{M}_1$ to $\rho(\boldsymbol{A}) \in \mathsf{M}_2$ and is the identity on morphisms. Since τ and ρ witness a term equivalence relative to K, we have that these two functors are isomorphisms of categories that are inverses of each other and satisfy $U_1 \circ \tau = U_2$. It follows that $U_1[\mathsf{M}_1] = U_2[\mathsf{M}_2]$, and so $U_2[\mathsf{M}_2]$ is a mono-reflective subcategory of K. Moreover, as $\tau \colon \mathsf{M}_2 \to \mathsf{M}_1$ and $U_1 \colon \mathsf{M}_1 \to U_1[\mathsf{M}_1]$ are isomorphisms and $U_1 \circ \tau = U_2$ (the latter because the term equivalence witnessed by τ and ρ is faithful), we obtain that $U_2 \colon \mathsf{M}_2 \to U_2[\mathsf{M}_2]$ is an isomorphism as well. Therefore, Theorem 3.1 yields that M_2 is a simple pp expansion of K. ☒

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