BI-INTERMEDIATE LOGICS OF TREES AND CO-TREES

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ABSTRACT. A bi-Heyting algebra validates the Gödel-Dummett axiom $(p \to q) \vee (q \to p)$ iff the poset of its prime filters is a disjoint union of co-trees (i.e., order duals of trees). Bi-Heyting algebras of this kind are called *bi-Gödel algebras* and form a variety that algebraizes the extension bi-GD of bi-intuitionistic logic axiomatized by the Gödel-Dummett axiom. In this paper we initiate the study of the lattice $\Lambda(\text{bi-GD})$ of extensions of bi-GD.

We develop the methods of Jankov-style formulas for bi-Gödel algebras and use them to prove that there are exactly continuum many extensions of bi-GD. We also show that all these extensions can be uniformly axiomatized by canonical formulas. Our main result is a characterization of the locally tabular extensions of bi-GD. We introduce a sequence of co-trees, called the *finite combs*, and show that a logic in $\Lambda(\text{bi-GD})$ is locally tabular iff it contains at least one of the Jankov formulas associated with the finite combs. It follows that there exists the greatest nonlocally tabular extension of bi-GD and consequently, a unique pre-locally tabular extension of bi-GD. These results contrast with the case of the intermediate logic axiomatized by the Gödel-Dummett axiom, which is known to have only countably many extensions, all of which are locally tabular.

1. INTRODUCTION

Bi-intuitionistic logic bi-IPC is the conservative extension of intuitionistic logic IPC obtained by adding a new binary connective $\leftarrow$ to the language, called the *co-implication* (or exclusion, or subtraction), which is dual to $\to$. Thus, bi-IPC enjoys a symmetry that IPC lacks (each connective $\wedge, \to, \bot$, has its dual $\vee, \leftarrow, \top$, respectively).

The Kripke semantics of bi-IPC [53] provides a transparent interpretation of co-implication: given a Kripke model $\mathfrak{M}$, a point x in $\mathfrak{M}$, and formulas ϕ, ψ, we define

$$\mathfrak{M}, x \models \phi \leftarrow \psi \iff \exists y \leqslant x \ (\mathfrak{M}, y \models \phi \text{ and } \mathfrak{M}, y \not\models \psi).$$

This new connective gives bi-IPC significantly greater expressivity than IPC. For instance, if the points of a Kripke frame are interpreted as states in time, the language of bi-IPC is expressive enough to reason about the past, something that is not possible in IPC. With this example in mind, Wolter [59] extended Gödel's embedding of IPC into S4 to an embedding of bi-IPC into tense-S4. In particular, he proved a version of the Blok-Esakia Theorem [11, 26] stating that the lattice $\Lambda(\text{bi-IPC})$ of *bi-intermediate logics* (i.e., consistent axiomatic extensions* of bi-IPC) is isomorphic to that of consistent normal tense logics containing Grz.t (see also [16, 57]).

The greater symmetry of bi-IPC compared to IPC is reflected in the fact that bi-IPC is algebraized in the sense of [13] by the variety bi-HA of *bi-Heyting algebras* [52], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, the lattice $\Lambda(\text{bi-IPC})$ is dually isomorphic to that of nontrivial varieties of bi-Heyting algebras. The latter, in turn, is amenable to the methods of universal algebra and duality theory since the category of bi-Heyting algebras is dually equivalent to that of *bi-Esakia spaces* [25], see also [7].

The theory of bi-Heyting algebras was developed in a series of papers by Rauszer and others motivated by the connection with bi-intuitionistic logic (see, e.g., [2, 41, 51, 52, 53, 55]). However, bi-Heyting algebras also arise naturally in other fields of research such as topos theory [46, 47, 54]. Furthermore, the lattice of open sets of an Alexandrov space is always a bi-Heyting algebra, and so is the lattice of subgraphs of an arbitrary graph (see, e.g., [58]). Similarly, every quantum system can be associated with a complete bi-Heyting algebra [20].

The lattice $\Lambda(\text{IPC})$ of *intermediate logics* (i.e., consistent extensions of IPC) has been thoroughly investigated (see, e.g., [17]). On the other hand, the lattice $\Lambda(\text{bi-IPC})$ of bi-intermediate logics

*From now on we will use *extension* as a synonym of *axiomatic extension*.

lacks such an in-depth analysis, but for some recent developments see, e.g., [1, 10, 31, 32, 56]. In this paper we contribute to filling this gap by studying a simpler, yet nontrivial, sublattice of $\Lambda(\text{bi-IPC})$: the lattice of consistent extensions of the *bi-intuitionistic Gödel-Dummett logic*

$$\text{bi-GD} := \text{bi-IPC} + (p \to q) \lor (q \to p).$$

The choice of bi-GD as a case study was motivated by some of its properties that make it an interesting logic on its own. Firstly, bi-GD is the bi-intermediate logic of *co-trees* (i.e., order duals of trees), that is, it is complete in the sense of Kripke semantics with respect to the class of co-trees. Furthermore, because of the symmetric nature of bi-intuitionistic logic, our results on extensions of bi-GD can be rephrased in a straightforward manner as results on the extensions of the bi-intermediate logic of trees

$$\text{bi-IPC} + \neg[(q \leftarrow p) \land (p \leftarrow q)]$$

by replacing in what follows every formula φ by its dual $\neg\varphi^\partial$, where φ^∂ is the formula obtained from φ by replacing each occurrence of $\land, \lor, \alpha \to \beta, \alpha \leftarrow \beta, \bot, \top$ by $\lor, \land, \beta \leftarrow \alpha, \beta \to \alpha, \top, \bot$ respectively, and every algebra or Kripke frame by its order dual. We chose to study bi-GD instead of its dual to be coherent with and build upon the extensive literature on the *(intuitionistic) Gödel-Dummett logic*

$$\text{LC} := \text{IPC} + (p \to q) \lor (q \to p),$$

also known as the *intuitionistic linear calculus* (see, e.g., [21, 30, 33, 34]).

Secondly, the properties of bi-GD and its extensions diverge significantly from those of their intuitionistic fragments. For instance, the bi-intermediate logic of *chains* (i.e., linearly ordered posets), or the *bi-intuitionistic linear calculus*

$$\text{bi-LC} := \text{bi-IPC} + (p \to q) \lor (q \to p) + \neg[(q \leftarrow p) \land (p \leftarrow q)],$$

is a proper extension of bi-GD, as shown in Theorem 3.10 (see also Theorem 4.25 for a different axiomatization of bi-LC). This contrasts with the intuitionistic case, where LC is both the intermediate logic of chains and of co-trees [33], and suggests that the language of bi-IPC is more appropriate to study tree-like structures than that of IPC. Moreover, we show in Theorem 4.16 that the lattice $\Lambda(\text{bi-GD})$ of consistent extensions of bi-GD is not a chain and has the cardinality of the continuum, whereas the lattice $\Lambda(\text{LC})$ of consistent extensions of LC is known to be a chain of order type $(\omega + 1)^\partial$ (see, e.g., [17]). Finally, while it is a well-known fact that LC is locally tabular [34], it is an immediate consequence of Corollary 5.31 that bi-GD is not.

Thirdly, extensions of bi-GD admit a form of a classical *reductio ad absurdum* (Theorem 4.1). Recall that a deductive system $\vdash$ is said to have a *classical inconsistency lemma* if, for every nonnegative integer n, there exists a finite set of formulas $\Psi_n(p_1, \dots, p_n)$ which satisfies the equivalence

$$\Gamma \cup \Psi_n(\varphi_1, \dots, \varphi_n) \text{ is inconsistent in } \vdash \iff \Gamma \vdash \{\varphi_1, \dots, \varphi_n\}, \tag{1}$$

for all sets of formulas $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$ [50] (see also [15, 45, 44]). As expected, the only intermediate logic having a classical inconsistency lemma is CPC (with $\Psi_n(p_1, \dots, p_n) := \{\neg(p_1 \land \dots \land p_n)\}$). This contrasts with the case of bi-intermediate logics, where every member of $\Lambda(\text{bi-GD})$ has a classical inconsistency lemma witnessed by

$$\Psi_n := \{\sim\neg\sim(p_1 \land \dots \land p_n)\}$$

(here, $\neg\varphi$ and $\sim\varphi$ are shorthands for $\varphi \to \bot$ and $\top \leftarrow \varphi$, respectively). Accordingly, logics in $\Lambda(\text{bi-GD})$ exhibit a certain balance between the classical and intuitionistic behavior of negation connectives.

The main contributions of the paper can be summarized as follows. In order to classify extensions of bi-GD, we develop theories of Jankov, subframe and canonical formulas for them. We then employ Jankov formulas to obtain a characterization of splittings in $\Lambda(\text{bi-GD})$ and to show that this lattice has the cardinality of the continuum (Theorems 4.11 and 4.16), cf. [8]. Moreover, we show that canonical formulas provide a uniform axiomatization for all the extensions of bi-GD (Theorem 4.7). Lastly, subframe formulas can be used to describe the fine structure of co-trees, by governing the embeddability of finite co-trees into arbitrary co-forests (Lemma 4.23).

By combining the defining properties of subframe and Jankov formulas, we establish the main result of this paper: a characterization of locally tabular extensions of bi-GD (Theorem 5.1). More precisely, we show that an extension L of bi-GD is locally tabular iff L contains at least one of the Jankov formulas associated with *finite combs* (a particular class of co-trees defined in Figure 5). It follows that the logic of finite combs is the greatest nonlocally tabular extension of bi-GD. Recall that a logic is called *pre-locally tabular* if it is not locally tabular, but all of its proper extensions are. It is a consequence of our main result that the logic of finite combs is the only pre-locally tabular extension of bi-GD (Corollary 5.31), and that bi-GD is not locally tabular.

2. Preliminaries

In this section, we review the basic concepts and results that we will need throughout this paper. For a deeper study of bi-IPC and bi-Heyting algebras, see, e.g., [43, 51, 52, 53, 58]. As a main source for universal algebra we use [3, 14]. Henceforth, $|X|$ denotes the cardinality of a set X, ω denotes the set of nonnegative integers, $\mathbb{Z}^+$ the set of positive integers, and given $n \in \omega$, the notation $i \leqslant n$ will always mean either $i \in \{0, \dots, n\}$ or $i \in \{1, \dots, n\}$, depending on the context.

2.1. **Bi-intuitionistic propositional logic.** Given a formula ϕ, we write $\neg\phi$ and $\sim\phi$ as a shorthand for $\phi \to \bot$ and $\top \leftarrow \phi$. The *bi-intuitionistic propositional calculus* bi-IPC is the least set of formulas in the language $\wedge, \vee, \to, \leftarrow, \bot, \top$, built up from a denumerable set *Prop* of variables, that contains IPC and the eight axioms below, and which is moreover closed under modus ponens, uniform substitutions, and the *double negation rule* "from ϕ infer $\neg \sim\phi$".

1. $p \to (q \vee (p \leftarrow q))$,
2. $(p \leftarrow q) \to \sim(p \to q)$,
3. $((p \leftarrow q) \leftarrow r) \to (p \leftarrow q \vee r)$,
4. $\neg(p \leftarrow q) \to (p \to q)$,
5. $(p \to (q \leftarrow q)) \to \neg p$,
6. $\neg p \to (p \to (q \leftarrow q)$,
7. $((p \to p) \leftarrow q) \to \sim q$,
8. $\sim q \to ((p \to p) \leftarrow q)$,

It turns out that bi-IPC is a conservative extension of IPC. Furthermore, we may identify the *classical propositional calculus* CPC with the proper extension of bi-IPC obtained by adding the *law of excluded middle* $p \vee \neg p$. Notably, in CPC the co-implication $\leftarrow$ is term-definable by the other connectives, since $(p \leftarrow q) \leftrightarrow (p \wedge \neg q) \in$ CPC. Consequently, the double negation rule becomes superfluous, as it translates to "from ϕ infer ϕ".

A set of formulas L closed under the three inference rules listed above is called a *super-bi-intuitionistic logic* if it contains bi-IPC. Given a formula ϕ and a super-bi-intuitionistic logic L, we say that ϕ is a *theorem* of L, denoted by $L \vdash \phi$, if $\phi \in L$. Otherwise, write $L \nvdash \phi$. We call L *consistent* if $L \nvdash \bot$ and *inconsistent* otherwise. Given another super-bi-intuitionistic logic L', we say that L' is an *extension* of L if $L \subseteq L'$. Consistent extensions of bi-IPC are called *bi-intermediate logics*, and it can be shown that a super-bi-intuitionistic logic L is a bi-intermediate logic iff bi-IPC $\subseteq L \subseteq$ CPC. Finally, given a set of formulas Σ, we denote by $L + \Sigma$ the least (with respect to inclusion) bi-intuitionistic logic containing $L \cup \Sigma$. If Σ is a singleton $\{\phi\}$, we simply write $L + \phi$. Given another formula ψ, we say that ϕ and ψ are *L-equivalent* if $L \vdash \phi \leftrightarrow \psi$.

2.2. **Varieties of algebras.** We denote by $\mathbb{H}, \mathbb{S}, \mathbb{I}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$ the class operators of closure under homomorphic images, subalgebras, isomorphic copies, direct products, and ultraproducts, respectively. A variety V is a class of (similar) algebras closed under homomorphic images, subalgebras, and (direct) products. By Birkhoff's Theorem, varieties coincide with classes of algebras that can be axiomatized by sets of equations (see, e.g., [14, Thm. II.11.9]). The smallest variety $\mathbb{V}(\mathsf{K})$ containing a class K of algebras is called the *variety generated by* K and coincides with $\mathbb{HSP}(\mathsf{K})$. If $\mathsf{K} = \{\mathbf{A}\}$, we simply write $\mathbb{V}(\mathbf{A})$.

Given an algebra $\mathbf{A}$, we denote by $Con(\mathbf{A})$ its congruence lattice. An algebra $\mathbf{A}$ is said to be *subdirectly irreducible*, or SI for short, (resp. *simple*) if $Con(\mathbf{A})$ has a second least element (resp.

has exactly two elements: the identity relation Id_A and the total relation A^2). Consequently, every simple algebra is subdirectly irreducible.

Given a class K of algebras, we denote by $\mathsf{K}^{<\omega}$, K_{SI}, and $\mathsf{K}_{SI}^{<\omega}$ the classes of finite members of K, SI members of K, and SI members of K which are finite, respectively. In view of the Subdirect Decomposition Theorem, if K is a variety, then $\mathsf{K} = \mathbb{V}(\mathsf{K}_{SI})$ (see, e.g., [14, Thm. II.8.6]).

Definition 2.1. A variety V is said to:

(i) be *semi-simple* if its SI members are simple;
(ii) be *locally finite* if its finitely generated members are finite;
(iii) have the *finite model property* (FMP for short) if it is generated by its finite members;
(iv) be *congruence distributive* if every member of V has a distributive lattice of congruences;
(v) have *equationally definable principal congruences* (EDPC for short) if there exists a conjunction $\Phi(x, y, z, v)$ of finitely many equations such that for every $\mathbf{A} \in \mathsf{V}$ and all $a, b, c, d \in A$,
$$(c, d) \in \Theta^{\mathbf{A}}(a, b) \iff \mathbf{A} \models \Phi(a, b, c, d),$$
where $\Theta^{\mathbf{A}}(a, b)$ is the least congruence of $\mathbf{A}$ that identifies a and b;
(vi) be a *discriminator variety* if there exists a *discriminator term* $t(x, y, z)$ for V, i.e., a ternary term such that for every $\mathbf{A} \in \mathsf{V}_{SI}$ and all $a, b, c \in A$, we have
$$t^{\mathbf{A}}(a, b, c) = \begin{cases} c & \text{if } a = b, \\ a & \text{if } a \neq b. \end{cases}$$

The next result collects some of the relations between these properties.

Proposition 2.2. *If V is a variety and K a class of algebras, then the following conditions hold:*

(i) *if V is locally finite, then its subvarieties have the FMP;*
(ii) *V has the FMP iff $\mathsf{V} = \mathbb{V}(\mathsf{V}_{SI}^{<\omega})$;*
(iii) *if V has EDPC, then V is congruence distributive and $\mathbb{HS}(\mathsf{K}) = \mathbb{SH}(\mathsf{K})$ for all $\mathsf{K} \subseteq \mathsf{V}$;*
(iv) (Jónsson's Lemma) *if $\mathbb{V}(\mathsf{K})$ is congruence distributive, then $\mathbb{V}(\mathsf{K})_{SI} \subseteq \mathbb{HSP}_{\mathrm{U}}(\mathsf{K})$;*
(v) *if V is discriminator, then it is semi-simple and it has EDPC.*

Proof. Condition (i) holds because every variety is generated by its finitely generated members (see, e.g., [3, Thm. 4.4]), while condition (ii) is an immediate consequence of the definition of the FMP together with the Subdirect Decomposition Theorem. The first part of condition (iii) was established in [38] and the second in [19]. For condition (iv), see, e.g., [14, Thm. VI.6.8]. Lastly, for the first part of condition (v) see, e.g., [14, Lem. IV.9.2(b)] and for the second [12, Exa. 6 p. 200]. ☐

The following result provides a useful description of locally finite varieties of finite type (for a proof, see [4]).

Theorem 2.3. *A variety V of a finite type is locally finite iff*
$$\forall m \in \omega, \exists k(m) \in \omega, \forall \mathbf{A} \in \mathsf{V}_{SI} \ \big(\mathbf{A} \text{ is } m\text{-generated} \implies |A| \leqslant k(m)\big).$$

2.3. **Bi-Heyting algebras.** A *poset* is a pair $\mathcal{X} = (X, \leqslant)$, where X is a set and $\leqslant$ a partial order. Given a subset U of a poset $\mathcal{X}$, we let $max(U)$ be the set the *maximal* elements of U viewed as a subposet of $\mathcal{X}$, and if U has a *maximum* (i.e., a greatest element), we denote it by $MAX(U)$. Similarly, we define $min(U)$ and $MIN(U)$. We denote the *upset generated* by U by
$$\uparrow U := \{x \in X : \exists u \in U \ (u \leqslant x)\},$$
and if $U = \uparrow U$, then U is called an *upset*. If $U = \{u\}$, we simply write $\uparrow u$ and call it a *principal upset*. We define the *downsets* of $\mathcal{X}$ and the $\downarrow$ operator in a similar way. A set that is both an upset and a downset is an *updownset*. We denote the set of upsets of $\mathcal{X}$ by $Up(\mathcal{X})$, of downsets by $Do(\mathcal{X})$, and of updownsets by $UpDo(\mathcal{X})$. We will always use the convention that the "arrow operators" defined above bind stronger than other set theoretic operations. For example, the expressions $\uparrow U \smallsetminus V$ and $\downarrow U \cap V$ are to be read as $(\uparrow U) \smallsetminus V$ and $(\downarrow U) \cap V$, respectively. Given two distinct points $x, y \in X$, if $x \leqslant y$ and no point of $\mathcal{X}$ lies between them (i.e., if $x \leqslant z \leqslant y$

implies either $x = z$ or $y = z$, for every $z \in X$), we call x an *immediate predecessor* of y, call y an *immediate successor* of x, and denote this by $x \prec y$.

Definition 2.4. Let $\mathbf{A}$ be an algebra whose $(\wedge, \vee, 0, 1)$-reduct is a bounded distributive lattice and consider the following equations:

(i) $x \to x \approx 1$,
(ii) $x \wedge (x \to y) \approx x \wedge y$,
(iii) $y \wedge (x \to y) \approx y$,
(iv) $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$,
(v) $x \leftarrow x \approx 0$,
(vi) $x \vee (y \leftarrow x) \approx x \vee y$,
(vii) $y \vee (y \leftarrow x) \approx y$,
(viii) $(y \vee z) \leftarrow x \approx (y \leftarrow x) \vee (z \leftarrow x)$.

If $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ and $\mathbf{A}$ validates the equations (i)-(iv), we call it a *Heyting algebra* and use the abbreviation $\neg a := a \to 0$, for each $a \in A$.

If $\mathbf{A} = (A, \wedge, \vee, \leftarrow, 0, 1)$ and $\mathbf{A}$ validates the equations (v)-(viii), we call it a *co-Heyting algebra* and use the abbreviation $\sim a := 1 \leftarrow a$, for each $a \in A$.

Finally, if $\mathbf{A} = (A, \wedge, \vee, \to, \leftarrow, 0, 1)$ and $\mathbf{A}$ validates the equations (i)-(viii), we call it a *bi-Heyting algebra*. We denote the class of bi-Heyting algebras by bi-HA.

Remark 2.5. We can think of bi-Heyting algebras as symmetric Heyting algebras, in the sense that they are Heyting algebras whose order duals are also Heyting algebras. In other words, Heyting algebras that can also be viewed as co-Heyting algebras. We note that in a bi-Heyting algebra $\mathbf{A}$, for every $a, b, c \in A$, the elements $a \to b, a \leftarrow b \in A$ satisfy the *residuation laws*:

$$\big(c \leqslant a \to b \iff a \wedge c \leqslant b\big) \text{ and } \big(a \leftarrow b \leqslant c \iff a \leqslant b \vee c\big).$$

Furthermore, since the class of bounded distributive lattices is a variety, it follows immediately from Birkhoff's Theorem that the three classes of algebras defined above are also varieties.

Next we list some useful properties of bi-Heyting algebras which follow easily from the definition of these structures (for a proof, see, e.g., [43]).

Proposition 2.6. *If* $\mathbf{A} \in$ bi-HA *and* $a, b, c \in A$*, then:*

1. $a \to b = \bigvee\{d \in A : a \wedge d \leqslant b\}$,
2. $a \to b = 1 \iff a \leqslant b$,
3. $\neg a = 1 \iff a = 0$,
4. $a \wedge \neg a = 0$,
5. $a \leftarrow b = \bigwedge\{d \in A : a \leqslant d \vee b\}$,
6. $a \leftarrow b = 0 \iff a \leqslant b$,
7. $\sim a = 0 \iff a = 1$,
8. $a \vee \sim a = 1$.

Example 2.7. Here we present some standard examples of bi-Heyting algebras.

(i) Every finite Heyting algebra $\mathbf{A}$ can be viewed as a bi-Heyting algebra, simply by defining $a \leftarrow b := \bigwedge\{d \in A : a \leqslant d \vee b\}$. Since this is a meet of finitely many elements, the operation $\leftarrow$ is well defined on $\mathbf{A}$, and it can be easily shown that it satisfies the dual residuation law presented in Remark 2.5.
(ii) Every Boolean algebra $\mathbf{A}$ can be viewed as a bi-Heyting algebra, by defining the co-implication as $a \leftarrow b := a \wedge \neg b$.
(iii) If $\mathcal{X}$ is a poset, then $(Up(\mathcal{X}), \cap, \cup, \to, \leftarrow, \varnothing, X)$ is a bi-Heyting algebra, whose implications are defined by

$$U \to V := X \smallsetminus {\downarrow}(U \smallsetminus V) \text{ and } U \leftarrow V := {\uparrow}(U \smallsetminus V).$$

A *valuation* on a bi-Heyting algebra $\mathbf{A}$ is a bi-Heyting homomorphism $v \colon \mathbf{Fm} \to \mathbf{A}$, where $\mathbf{Fm}$ denotes the *algebra of formulas* of the language of bi-IPC. Clearly, any map $v \colon Prop \to \mathbf{A}$ (where $Prop$ denotes the denumerable set of propositional variables of our language) can be extended uniquely to a valuation on $\mathbf{A}$, hence we also call such maps *valuations* on $\mathbf{A}$. We say that a formula ϕ is *valid* on $\mathbf{A}$, denoted by $\mathbf{A} \models \phi$, if $v(\phi) = 1$ for all valuations v on $\mathbf{A}$. On the other hand, if $v(\phi) \neq 1$ for some valuation v on $\mathbf{A}$, we say that $\mathbf{A}$ *refutes* ϕ (via v), and write $\mathbf{A} \not\models \phi$. If K is a class of bi-Heyting algebras such that $\mathbf{A} \models \phi$ for all $\mathbf{A} \in$ K, we write K $\models \phi$. Otherwise, write K $\not\models \phi$.

Using the well-known Lindenbaum-Tarski construction (see, e.g., [17, 29]) we obtain the following equivalence: $\mathsf{bi\text{-}IPC} \vdash \phi$ iff $\mathsf{bi\text{-}HA} \models \phi$. This phenomenon, known as the algebraic completeness of bi-IPC, can be extended to all other super-bi-intuitionistic logics. Let L be such a logic, and denote the *variety of* L by $\mathsf{V}_L := \{\mathbf{A} \in \mathsf{bi\text{-}HA} \colon \mathbf{A} \models L\}$. On the other hand, given a subvariety $\mathsf{V} \subseteq \mathsf{bi\text{-}HA}$, we denote its *logic* by $L_{\mathsf{V}} := Log(\mathsf{V}) = \{\phi \in Fm \colon \mathsf{V} \models \phi\}$. Again using the standard Lindenbaum-Tarski construction, it can be shown that L is *sound* and *complete* with respect to V_L, i.e., for all formulas ϕ, we have $L \vdash \phi$ iff $\mathsf{V}_L \models \phi$. It follows that this correspondence between extensions of bi-IPC and subvarieties of bi-Heyting algebras is one-to-one, and therefore the following theorem can now be easily proved.

Theorem 2.8. *If L is a super-bi-intuitionistic logic, then the lattice of extensions of L is dually isomorphic to the lattice of subvarieties of V_L. Equivalently, if V is a variety of bi-Heyting algebras, then the lattice of subvarieties of V is dually isomorphic to the lattice of extensions of L_{V}.*

2.4. **Bi-Esakia spaces.** Given an ordered topological space $\mathcal{X}$, we denote its set of: open sets by $Op(\mathcal{X})$, closed sets by $Cl(\mathcal{X})$, clopen sets by $Clop(\mathcal{X})$, clopen upsets by $ClopUp(\mathcal{X})$, and closed updownsets by $ClUpDo(\mathcal{X})$.

Definition 2.9. Let $\mathcal{X} = (X, \leqslant)$ and $\mathcal{Y} = (Y, \leqslant)$ be posets, $f \colon X \to Y$ a map between them, and consider the following conditions:

(i) ***Order preserving:*** $\forall x, z \in X \; \big(x \leqslant z \implies f(x) \leqslant f(z)\big)$;
(ii) ***Up:*** $\forall x \in X, \forall y \in Y \; \big(f(x) \leqslant y \implies \exists z \in {\uparrow}x \; (f(z) = y)\big)$;
(iii) ***Down:*** $\forall x \in X, \forall y \in Y \; \big(y \leqslant f(x) \implies \exists z \in {\downarrow}x \; (f(z) = y)\big)$.

When f is order preserving, we call it: a *p-morphism* if it satisfies the up condition, a *co-p-morphism* if it satisfies the down condition, and a *bi-p-morphism* if it satisfies both the up and down conditions. In all three cases, we use the notation $f \colon \mathcal{X} \to \mathcal{Y}$.

If $f \colon \mathcal{X} \to \mathcal{Y}$ is a surjective bi-p-morphism (or p-morphism, or co-p-morphism), then we say that $\mathcal{Y}$ is a *bi-p-morphic image* (or *p-morphic image*, or *co-p-morphic image*, respectively) of $\mathcal{X}$ (via f), and denote this by $f \colon \mathcal{X} \twoheadrightarrow \mathcal{Y}$.

Proposition 2.10. *If $f \colon \mathcal{X} \to \mathcal{Y}$ is a bi-p-morphism, then the following conditions hold:*

(i) *$f[{\uparrow}x] = {\uparrow}f(x)$ and $f[{\downarrow}x] = {\downarrow}f(x)$, for every $x \in \mathcal{X}$;*
(ii) *$f[max(\mathcal{X})] \subseteq max(\mathcal{Y})$ and $f[min(\mathcal{X})] \subseteq min(\mathcal{Y})$;*
(iii) *if both $MAX(\mathcal{X})$ and $MAX(\mathcal{Y})$ exist, then $f\big(MAX(\mathcal{X})\big) = MAX(\mathcal{Y})$ and f is necessarily surjective.*

Proof. Condition (i) follows immediately from the definition of bi-p-morphisms, while the other two are direct consequences of (i). □

Definition 2.11. Let $\mathcal{X} = (X, \tau, \leqslant)$ be an ordered topological space and consider the following conditions:

(i) (X, τ) is compact;
(ii) *Priestley separation axiom* (PSA for short)[†]:

$$\forall x, y \in X \; \big(x \nleqslant y \implies \exists V \in ClopUp(\mathcal{X}) \; (x \in V \text{ and } y \notin V)\big);$$

(iii) $\forall U \in Clop(\mathcal{X}) \; \big({\downarrow}U \in Clop(\mathcal{X})\big)$;
(iv) $\forall U \in Clop(\mathcal{X}) \; \big({\uparrow}U \in Clop(\mathcal{X})\big)$.

We call $\mathcal{X}$: an *Esakia space* if it satisfies conditions (i)-(iii); a *co-Esakia space* if it satisfies conditions (i), (ii), and (iv); and a *bi-Esakia space* if it satisfies conditions (i)-(iv).

A map $f \colon X \to Y$ is a *bi-Esakia morphism* (or an *Esakia morphism*, or a *co-Esakia morphism*), denoted by $f \colon \mathcal{X} \to \mathcal{Y}$, if it is a continuous bi-p-morphism (or p-morphism, or co-p-morphism, respectively) between bi-Esakia spaces (or Esakia spaces, or co-Esakia spaces, respectively). When f is surjective, we call $\mathcal{Y}$ a *bi-Esakia image* (or *Esakia image*, or *co-Esakia image*, respectively)

[†]Given V as in the display below, we will often say that V *separates* x from y.

and denote this by $f\colon \mathcal{X} \twoheadrightarrow \mathcal{Y}$. If f is moreover bijective, then $\mathcal{X}$ and $\mathcal{Y}$ are said to be *isomorphic*, denoted by $\mathcal{X} \cong \mathcal{Y}$.

Finally, when U and V are subsets of X, we define:

$$\begin{aligned} U \to V &:= X \smallsetminus {\downarrow}(U \smallsetminus V) = \{x \in X\colon {\uparrow}x \cap U \subseteq V\}, \\ U \leftarrow V &:= {\uparrow}(U \smallsetminus V) = \{x \in X\colon {\downarrow}x \cap U \nsubseteq V\}, \\ \neg U &:= U \to \varnothing = X \smallsetminus {\downarrow}U, \\ {\sim}U &:= X \leftarrow U = {\uparrow}(X \smallsetminus U). \end{aligned}$$

If $\mathcal{X}$ is an Esakia space (resp. co-Esakia space), then $\to$ (resp. $\leftarrow$) is a well-defined binary operation on $ClopUp(\mathcal{X})$.

Before we present some equivalent conditions to the definition of bi-Esakia spaces, we recall that a topological space $\mathcal{X}$ is a *Stone space* if it is *0-dimensional* (i.e., it has a basis of clopen sets), compact, and Hausdorff. We recall as well that when $\mathcal{X}$ is an ordered topological space, its order relation $\leqslant$ is said to be *point closed* when ${\uparrow}x$ is a closed set, for each $x \in X$.

Theorem 2.12. *If $\mathcal{X} = (X, \tau, \leqslant)$ is an ordered topological space, then the following conditions are equivalent:*

(i) *$\mathcal{X}$ is a bi-Esakia space;*
(ii) *$\mathcal{X}$ is a Stone space and for each subset $U \subseteq X$, if U is closed then both ${\downarrow}U$ and ${\uparrow}U$ are closed, and if U is open then both ${\downarrow}U$ and ${\uparrow}U$ are open;*
(iii) *$\mathcal{X}$ is a Stone space, $\leqslant$ is point closed, and for each clopen set $U \subseteq X$, both ${\downarrow}U$ and ${\uparrow}U$ are clopen;*
(iv) *$\mathcal{X}$ is a compact space that satisfies the PSA and for each open set $U \subseteq X$, both ${\downarrow}U$ and ${\uparrow}U$ are open.*

Proof. (i) $\Longrightarrow$ (ii) Suppose that $\mathcal{X}$ is a bi-Esakia space. Since, by definition, bi-Esakia spaces are always Esakia spaces, it follows from [27, Thm. 3.1.2] that $\mathcal{X}$ is a Stone space in which the downset generated by each closed (resp. open) subset is again closed (resp. open). Furthermore, the aforementioned result also ensures that in an Esakia space, the upset generated by each closed subset is closed as well. Thus, it only remains to show that if $U \subseteq X$ is open, then ${\uparrow}U$ is an open set. Accordingly, suppose that $y \leqslant x$, for some $y \in U$. As U is open and $\mathcal{X}$ 0-dimensional, the set U contains a clopen neighbourhood V of y. By the definition of bi-Esakia spaces, ${\uparrow}V$ is also a clopen set. It now follows that ${\uparrow}V$ is a clopen neighbourhood of x (since $y \leqslant x$ and $y \in V$), which is moreover contained in ${\uparrow}U$. So ${\uparrow}U$ is indeed open.

(ii) $\Longrightarrow$ (iii) It suffices to prove point closedness. To this end, consider $x \in X$. As Stone spaces are Hausdorff, the singleton $\{x\}$ is closed. Therefore, ${\uparrow}x$ is closed by assumption.

(iii) $\Longrightarrow$ (iv) We assume that $\mathcal{X}$ satisfies (iii). In particular, that $\mathcal{X}$ is a Stone space, hence compact by definition. It is moreover 0-dimensional. Consequently, every open subset $U \subseteq X$ can be written as a union $\bigcup_{i\in I} V_i$ of clopen sets. Again using our assumption, both ${\downarrow}V_i$ and ${\uparrow}V_i$ are also clopen, for each $i \in I$. Since the equalities ${\downarrow}U = \bigcup_{i\in I} {\downarrow}V_i$ and ${\uparrow}U = \bigcup_{i\in I} {\uparrow}V_i$ clearly hold true, we conclude that both ${\downarrow}U$ and ${\uparrow}U$ are open.

Next we show that $\mathcal{X}$ satisfies the PSA. To this end, suppose that $x, y \in X$ are such that $x \nleqslant y$. Since, by assumption, $\leqslant$ is point closed, we know that ${\uparrow}x$ is a closed upset that contains x but omits y. Thus, $X \setminus {\uparrow}x$ is an open downset that separates y from x. As previously mentioned, $\mathcal{X}$ is a 0-dimensional space, so there exists a clopen neighbourhood V of y contained in $X \setminus {\uparrow}x$. Furthermore, ${\downarrow}V$ must be clopen by assumption, and it is clear that ${\downarrow}V$ is a subset of the downset $X \setminus {\uparrow}x$. Since the latter set omits x, so does ${\downarrow}V$, and we conclude that $X \setminus {\downarrow}V$ is a clopen upset that separates x from y, as desired.

(iv) $\Longrightarrow$ (i) Using our definition of bi-Esakia spaces 2.11, it is clear that to establish this implication it suffices to show that if $\mathcal{X}$ satisfies (iv), then both ${\downarrow}U$ and ${\uparrow}U$ are closed, for each clopen subset $U \subseteq X$. Accordingly, we suppose that U is a clopen set satisfying $x \notin {\downarrow}U$, for some $x \in X$, and show that x has an open neighbourhood disjoint from ${\downarrow}U$. By the PSA, there exists a clopen downset V_z that separates z from x, for each $z \in U$. It follows that $\{V_z\}_{z\in U}$ is an open cover of the closed set U, and whose union omits x. Because we assumed that $\mathcal{X}$ is compact, this yields $U \subseteq \bigcup_{i=1}^{n} V_{z_i} \subseteq X \setminus \{x\}$, for some $z_1, \dots, z_n \in U$. As $\bigcup_{i=1}^{n} V_{z_i}$ is a finite union of clopen downsets, it is also a clopen downset, so it must also contain $\downarrow U$. Therefore, $X \setminus (\bigcup_{i=1}^{n} V_{z_i})$ is a clopen neighbourhood of x which is disjoint from $\downarrow U$, as desired.

The proof that $\uparrow U$ is closed for each clopen set $U \subseteq X$ is analogous hence we omit it. □

Example 2.13. Every finite poset can be viewed as a bi-Esakia space, when equipped with the discrete topology. In fact, since bi-Esakia spaces are Hausdorff, this is the only way to view a finite poset as a bi-Esakia space. Furthermore, since maps between spaces equipped with the discrete topology are always continuous, it follows that every bi-p-morphism between finite posets can be regarded as a bi-Esakia morphism.

The next result collects some useful properties of bi-Esakia spaces.

Proposition 2.14. *The following conditions hold for a bi-Esakia space $\mathcal{X}$:*
(i) *if $x \in X$, then there are $y \in min(\mathcal{X})$ and $z \in max(\mathcal{X})$ satisfying $y \leqslant x \leqslant z$;*
(ii) *$\neg \sim U = \{x \in X \colon \downarrow\uparrow x \subseteq U\}$, for each $U \in ClopUp(\mathcal{X})$.*

Proof. Condition (i) is a well-known result for Esakia spaces (see, e.g., [27, Thm. 3.2.1]), hence we will only provide here a proof for (ii). Let $U \in ClopUp(\mathcal{X})$. By spelling out the definition of $\neg \sim U$, we have

$$\neg \sim U = X \smallsetminus \downarrow\uparrow(X \smallsetminus U) = \{x \in X \colon \forall y \in X \, (\exists z \in X \smallsetminus U \, (z \leqslant y) \implies x \nleqslant y)\}.$$

Suppose that $x \in \neg \sim U$ and let $u \in \downarrow\uparrow x$. So, there must exist a $y \in \uparrow x$ satisfying $u \leqslant y$. By the above display, $x \in \neg \sim U$ and $x \leqslant y$ entail $z \nleqslant y$, for every $z \in X \smallsetminus U$. As $u \leqslant y$, it now follows that $u \in U$. This shows $\downarrow\uparrow x \subseteq U$, and we have proved $\neg \sim U \subseteq \{x \in X \colon \downarrow\uparrow x \subseteq U\}$.

To prove the reverse inclusion, suppose $\downarrow\uparrow x \subseteq U$, for some $x \in X$. Let $y \in X$ be such that there exists a $z \in (X \smallsetminus U) \cap \downarrow y$. If $x \leqslant y$, then we would have $z \in \downarrow\uparrow x \subseteq U$, a contradiction. Thus $x \nleqslant y$, and we conclude $x \in \neg \sim U$, again by the above display. □

The celebrated Esakia duality restricts to a duality between the category of bi-Heyting algebras and bi-Heyting homomorphisms, and that of bi-Esakia spaces and bi-Esakia morphisms [25] (for a proof, see [43]). Here, we will just recall the contravariant functors which establish this duality. Given a bi-Heyting algebra $\mathbf{A}$, we denote its *bi-Esakia dual* by $\mathbf{A}_* := (A_*, \tau, \subseteq)$, where A_* is the set of prime filters of $\mathbf{A}$ and τ is the topology generated by the subbasis

$$\{\varphi(a) \colon a \in A\} \cup \{A_* \smallsetminus \varphi(a) \colon a \in A\},$$

where $\varphi(a) := \{F \in A_* \colon a \in F\}$. Notably, we have that $ClopUp(\mathbf{A}_*) = \{\varphi(a) \colon a \in A\}$. Furthermore, if $f \colon \mathbf{A} \to \mathbf{B}$ is a bi-Heyting homomorphism, then its dual is the restricted inverse image map $f_* := f^{-1}[-] \colon \mathbf{B}_* \to \mathbf{A}_*$.

Conversely, if $\mathcal{X}$ is a bi-Esakia space then we denote its *bi-Heyting* (or *algebraic*) *dual* by $\mathcal{X}^* := (ClopUp(\mathcal{X}), \cap, \cup, \to, \leftarrow, \emptyset, X)$, and if $f \colon \mathcal{X} \to \mathcal{Y}$ is a bi-Esakia morphism, then its dual is the restricted inverse image map $f^* := f^{-1}[-] \colon \mathcal{Y}^* \to \mathcal{X}^*$. We note that $\mathbf{A}$ and $(\mathbf{A}_*)^*$ are isomorphic as bi-Heyting algebras, while $\mathcal{X}$ and $(\mathcal{X}^*)_*$ are isomorphic as bi-Esakia spaces.

Next we define the three standard methods of generating new bi-Esakia spaces from old ones. Let $\mathcal{X} = (X, \tau, R)$, $\mathcal{Y} = (Y, \pi, S)$, $\mathcal{X}_1 = (X_1, \tau_1, R_1), \dots, \mathcal{X}_n = (X_n, \tau_n, R_n)$ be bi-Esakia spaces. We say that:
(i) $\mathcal{Y}$ is a *bi-generated subframe* of $\mathcal{X}$ if $Y \in ClUpDo(\mathcal{X})$, π is the subspace topology, and $S = Y^2 \cap R$;
(ii) $\mathcal{Y}$ is a *bi-Esakia (morphic) image* of $\mathcal{X}$, denoted by $\mathcal{X} \twoheadrightarrow \mathcal{Y}$, if there exists a surjective bi-Esakia morphism from $\mathcal{X}$ onto $\mathcal{Y}$;
(iii) $\mathcal{X} = \biguplus_{i=1}^{n} \mathcal{X}_i$ is the *disjoint union* of the collection $\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$ if (X, R) is the disjoint union $\biguplus_{i=1}^{n}(X_i, R_i)$ of the various posets and (X, τ) is the topological sum of the (X_i, τ_i).

As is the case with the analogous notions for Esakia spaces, the above definitions can be translated (using the bi-Esakia duality) into the terminology of bi-Heyting algebras (for a proof, see [43]).

Proposition 2.15. *Let* $\{\mathbf{A}, \mathbf{B}\} \cup \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$ *and* $\{\mathcal{X}_1, \ldots, \mathcal{X}_n\}$ *be finite sets of bi-Heyting algebras and bi-Esakia spaces, respectively. The following conditions hold:*

(i) $\mathbf{B}$ *is a homomorphic image of* $\mathbf{A}$ *iff* $\mathbf{B}_*$ *is (isomorphic to) a bi-generated subframe of* $\mathbf{A}_*$*;*
(ii) $\mathbf{B}$ *is (isomorphic to) a subalgebra of* $\mathbf{A}$ *iff* $\mathbf{B}_*$ *is a bi-Esakia image of* $\mathbf{A}_*$*;*
(iii) $(\prod_{i=1}^n \mathbf{A}_i)_* \cong \biguplus_{i=1}^n \mathbf{A}_{i*}$ *and* $(\biguplus_{i=1}^n \mathcal{X}_i)^* \cong \prod_{i=1}^n \mathcal{X}_i^*$.

Let $\mathcal{X}$ be a bi-Esakia space. A valuation V on $ClopUp(\mathcal{X})$ is also called a *valuation* on $\mathcal{X}$, and the pair $\mathfrak{M} := (\mathcal{X}, V)$ a *bi-Esakia model* (on $\mathcal{X}$). If $x \in X$ and ϕ is a formula, we say that ϕ *is* (or *holds*) *true* in x when $x \in V(\phi)$, and write $\mathfrak{M}, x \models \phi$. Moreover, we say that $\mathcal{X}$ *validates* ϕ, or that ϕ is *valid* in $\mathcal{X}$, when $V'(\phi) = X$, for all valuations V' on $\mathcal{X}$ (in other words, when $\mathcal{X}^* \models \phi$). Otherwise, write $\mathcal{X} \not\models \phi$ and say that $\mathcal{X}$ *refutes* ϕ. Since the validity of a formula is preserved under taking homomorphic images, subalgebras, and direct products of bi-Heyting algebras, it follows from the previous proposition that the validity of a formula is preserved under taking bi-generated subframes, bi-Esakia images, and finite disjoint unions of bi-Esakia spaces.

Finally, we present the Coloring Theorem, a result that provides a characterization of the finitely generated bi-Heyting algebras using properties of their bi-Esakia duals. To this end, we first need to recall the notions of bi-E-partitions and colorings on bi-Esakia spaces.

Definition 2.16. Let $\mathcal{X}$ be a bi-Esakia space and E an equivalence relation on X. We say that E is a *bi-E-partition* of $\mathcal{X}$ if it satisfies the following conditions:

- ***Up:*** $\forall x, y, w \in X \left(xEy \text{ and } w \in {\uparrow} y \implies \exists v \in {\uparrow} x \ (vEw)\right)$;
- ***Down:*** $\forall x, y, w \in X \left(xEy \text{ and } w \in {\downarrow} y \implies \exists v \in {\downarrow} x \ (vEw)\right)$;
- ***Refined:*** $\forall x, y \in X \left(\neg(xEy) \implies \exists U \in ClopUp(\mathcal{X}) \, (E[U] = U \text{ and } |U \cap \{x, y\}| = 1)\right)$.

A subset U of X satisfying $E[U] = U$ is called *E-saturated*. Using this terminology, the last condition above can be rephrased as "any two non-E-equivalent elements of $\mathcal{X}$ are separated by an E-saturated clopen upset". We call a bi-E-partition E of $\mathcal{X}$ *trivial* if $E = X^2$, and *proper* otherwise.

It is well known that the *E-partitions* of an Esakia space $\mathcal{X}$ (i.e., equivalence relations that are only required to satisfy the up and refined conditions defined above) are in a one-to-one correspondence with the Esakia images of $\mathcal{X}$ (for a proof, see, e.g., [27, Lem. 3.4.11]). This correspondence, which will be used without further reference in the next proof, can be easily extended to the setting of bi-Esakia spaces.

Proposition 2.17. *There is a one-to-one correspondence between the bi-E-partitions of a bi-Esakia space and its bi-Esakia images (modulo isomorphism).*

Proof. Let $\mathcal{X}$ be a bi-Esakia space and E a bi-E-partition of $\mathcal{X}$. Consider the quotient space $\mathcal{X}/E := (\mathcal{X}/E, \tau_E, \leqslant)$, where τ_E is the topology generated by the subbasis

$$\mathcal{S} := \{U/E \colon U \text{ is an } E\text{-saturated clopen of } \mathcal{X}\},$$

and $\leqslant$ is the induced order (i.e., $x/E \leqslant y/E$ iff $x' \leqslant y'$ for some $x' \in E(x)$ and $y' \in E(y)$). It is routine to check that $Clop(\mathcal{X}/E) = \mathcal{S}$.

Let us show that $\mathcal{X}/E$ is a bi-Esakia space. Since, by definition, every bi-E-partition of $\mathcal{X}$ is also an E-partition, we know that $\mathcal{X}/E$ is an Esakia space. Therefore, we only need to prove that in $\mathcal{X}/E$, upsets generated by clopen sets are also clopen. To this end, let U be an E-saturated clopen set of $\mathcal{X}$. We show that not only is ${\uparrow} U$ also E-saturated, but that $({\uparrow} U)/E = {\uparrow}(U/E)$. Since $\mathcal{X}$ is a bi-Esakia space, ${\uparrow} U$ must be clopen, thus the previous conditions will entail that ${\uparrow}(U/E)$ is a clopen set of $\mathcal{X}/E$, by our definition of $\mathcal{S}$.

Let $y \in E[{\uparrow} U]$, so there are $x' \in U$ and $y' \in X$ such that $x' \leqslant y' E y$. By the down condition of E, there is $x \in E(x')$ such that $x \leqslant y$. But U is E-saturated by assumption, hence $x \in U$, and $y \in {\uparrow} U$ now follows. This shows $E[{\uparrow} U] \subseteq {\uparrow} U$. As the inclusion ${\uparrow} U \subseteq E[{\uparrow} U]$ is trivial, the set ${\uparrow} U$ must be E-saturated. Because of this, it is easy to see that $({\uparrow} U)/E \subseteq {\uparrow}(U/E)$. To prove the reverse inclusion, consider $y/E \in {\uparrow}(U/E)$, so there exists $x/E \in U/E$ satisfying $x/E \leqslant y/E$. By our definition of $\mathcal{X}/E$, we have $x' \leqslant y'$ for some $x' \in E(X)$ and $y' \in E(y)$. Since

U is E-saturated, we have $x' \in U$. This forces $y' \in {\uparrow}U$, which in turn yields $y/E \in ({\uparrow}U)/E$. We can now conclude that ${\uparrow}(U/E) = ({\uparrow}U)/E$ is a clopen set of $\mathcal{X}/E$, as desired.

Finally, using the down condition of E, we can easily show that the quotient map $f\colon \mathcal{X} \twoheadrightarrow \mathcal{X}/E$ (which is known to be a surjective Esakia morphism) satisfies the homonymous condition of bi-p-morphisms. Thus, $\mathcal{X}/E$ is indeed a bi-Esakia image of $\mathcal{X}$.

Next we show that every bi-Esakia image of $\mathcal{X}$ gives rise to a bi-E-partition. Consider a surjective bi-Esakia morphism $f\colon \mathcal{X} \twoheadrightarrow \mathcal{Y}$. We know that the equivalence relation $E := \{(x,y) \in X^2 \colon f(x) = f(y)\}$ is an E-partition of $\mathcal{X}$. Using the down condition of f, it is easily verified that E is a bi-E-partition. It is routine to check that this correspondence is one to one. □

Remark 2.18. In view of Propositions 2.15 and 2.17, the bi-E-partitions of a bi-Esakia space correspond to the isomorphic copies of subalgebras of $\mathcal{X}_*$. We sketch this correspondence. If E is a bi-E-partition of $\mathcal{X}$, then the bi-Heyting algebra with universe $\{U/E \colon U \text{ is an } E\text{-saturated clopen upset of } \mathcal{X}\}$ embeds into $\mathcal{X}_*$. Conversely, if $\mathbf{B}$ is a subalgebra of $\mathcal{X}_*$, then the relation $E_{\mathbf{B}}$ on $\mathcal{X}$ defined by

$$xE_{\mathbf{B}}y \text{ iff } x \cap B = y \cap B$$

is a bi-E-partition of $\mathcal{X}$.

Let $\mathcal{X}$ be a bi-Esakia space and $p_1, \dots, p_n$ a finite number of fixed distinct propositional variables. Given a map $c\colon \{p_1, \dots, p_n\} \to ClopUp(\mathcal{X})$, we associate to each point $x \in \mathcal{X}$ the sequence $col(x) := (i_1, \dots, i_n)$ defined by

$$i_k := \begin{cases} 1 & \text{if } x \in c(p_k), \\ 0 & \text{if } x \notin c(p_k), \end{cases}$$

for $k \in \{1, \dots, n\}$. We call $col(x)$ the *color* of x (relative to $p_1, \dots, p_n$), the map c an *n-coloring* of $\mathcal{X}$, and the pair $(\mathcal{X}, c)$ an *n-colored* bi-Esakia space.

Now, if $\mathbf{A}$ is a bi-Heyting algebra endowed with some fixed elements $a_1, \dots, a_n$, then we can think of this structure as a pair $(\mathbf{A}, v)$, where $v\colon \{p_1, \dots, p_n\} \to A$ is the map defined by $v(p_i) := a_i$, for each $i \leqslant n$. Defining $c\colon \{p_1, \dots, p_n\} \to ClopUp(\mathbf{A}_*)$ by $c(p_i) := \{x \in A_* \colon a_i \in x\}$, for each $i \leqslant n$, yields an n-coloring of the bi-Esakia dual $\mathbf{A}_*$ of $\mathbf{A}$, and thus an n-colored bi-Esakia space $(\mathbf{A}_*, c)$.

We are now ready to prove the Coloring Theorem for bi-Heyting algebras. The Coloring Theorem for Heyting algebras was first stated in [28] (for a proof, see, e.g., [9, Thm. 3.1.5]). Our argument follows closely the one found in [9], but we include it for the sake of completeness. Notice the use of the notation $\mathbf{A} = \langle a_1, \dots, a_n \rangle$ for "$\mathbf{A}$ is generated as a bi-Heyting algebra by $\{a_1, \dots, a_n\}$".

Theorem 2.19 (Coloring Theorem)**.** *Let* $\mathbf{A}$ *be a bi-Heyting algebra,* $a_1, \dots, a_n \in \mathbf{A}$*, and* $(\mathcal{X}, c)$ *the corresponding n-colored bi-Esakia space. Then* $\mathbf{A} = \langle a_1, \dots, a_n \rangle$ *iff every proper bi-E-partition* E *of* $\mathcal{X}$ *identifies points of different colors.*

Proof. Suppose $\mathbf{A} = \langle a_1, \dots, a_n \rangle$ and that E is a proper bi-E-partition of $\mathcal{X}$. It follows from Remark 2.18 that $\{U/E \colon U \text{ is an } E\text{-saturated clopen upset of } \mathcal{X}\}$ is the universe of a bi-Heyting algebra which is isomorphic to a proper subalgebra of $\mathbf{A}$. This entails that there exists $i \leqslant n$ such that $c(p_i) = \{x \in X \colon a_i \in x\}$ is a clopen upset of $\mathcal{X}$ which is not E-saturated. In other words, there exists $x \in E[c(p_i)] \smallsetminus c(p_i)$. Since the clopen upsets $c(p_j)$ define our coloring, it is clear that x is identified with a point of a different color.

We now suppose that $\mathbf{B} := \langle a_1, \dots, a_n \rangle$ is a proper subalgebra of $\mathbf{A}$ and prove the right to left implication of the statement by contraposition. In view of Remark 2.18, the relation $E_{\mathbf{B}}$ is a proper bi-E-partition of $\mathcal{X}$. As every a_i is contained in $\mathbf{B}$, it follows easily from the definition of $E_{\mathbf{B}}$ that every $c(p_i)$ is $E_{\mathbf{B}}$-saturated. Because of this, if $xE_{\mathbf{B}}y$ then the equivalence $x \in c(p_i)$ iff $y \in c(p_i)$ holds for all $i \leqslant n$. Therefore, $E_{\mathbf{B}}$ can only identify points of the same color. □

3. THE BI-INTUITIONISTIC GÖDEL-DUMMETT LOGIC

The *bi-intuitionistic Gödel-Dummett logic* is the bi-intermediate logic

$$\mathsf{bi\text{-}GD} := \mathsf{bi\text{-}IPC} + (p \to q) \vee (q \to p),$$

and the formula $(p \to q) \vee (q \to p)$ is called the *Gödel-Dummett axiom* (or the *prelinearity axiom*). Over IPC, this formula axiomatizes the intuitionistic linear calculus LC, a logic that has been extensively studied in the literature (see, e.g., [21, 30, 33, 34]) and whose algebraic models (i.e., Heyting algebras which validate the Gödel-Dummett axiom) have been called *Gödel algebras*. In view of Theorem 2.8, the logic $\mathsf{bi\text{-}GD}$ corresponds (in fact, is algebraized by) the variety

$$\mathsf{bi\text{-}GA} := \mathsf{V}_{\mathsf{bi\text{-}GD}} = \{\mathbf{A} \in \mathsf{bi\text{-}HA} \colon \mathbf{A} \models (p \to q) \vee (q \to p)\},$$

whose elements will be called *bi-Gödel algebras*. Furthermore, there exists a dual isomorphism between the lattice $\Lambda(\mathsf{bi\text{-}GD})$ of consistent extensions of $\mathsf{bi\text{-}GD}$ and that of nontrivial subvarieties of $\mathsf{bi\text{-}GA}$. In order to describe the order topological duals of these algebras, we need to recall the definitions of *chains* (i.e., linearly ordered posets), of *co-trees* (i.e., posets with a greatest element, called the *co-root*, and whose principal upsets are chains) and of *co-forests* (i.e., possibly empty disjoint unions of co-trees). Moreover, a *bi-Esakia chain*, or *co-tree*, or *co-forest*, is a bi-Esakia space whose underlying poset is a chain, or co-tree, or co-forest, respectively. The notions of *Esakia chains*, *co-trees*, and *co-forests* are defined similarly.

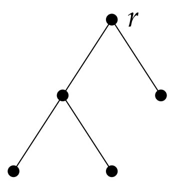

FIGURE 1. A co-tree.

In this section, we use the duality between Gödel algebras and Esakia co-forests (see, e.g., [33]) to achieve a duality between bi-Gödel algebras and bi-Esakia co-forests. This allows us to obtain the transparent description of the SI members of $\mathsf{bi\text{-}GA}$ as the algebraic duals of bi-Esakia co-trees. In contrast, SI Gödel algebras are the algebraic duals of *strongly rooted Esakia chains* (i.e., Esakia spaces whose underlying posets are bounded chains with an isolated least element) [33].

Theorem 3.1. *If* $\mathbf{A} \in \mathsf{bi\text{-}HA}$*, then* $\mathbf{A}$ *is a bi-Gödel algebra iff* $\mathbf{A}_*$ *is a bi-Esakia co-forest.*

Proof. Observe that a bi-Heyting algebra $\mathbf{A}$ validates the axiom $(p \to q) \vee (q \to p)$ iff its Heyting algebra reduct $\mathbf{A}^-$ validates the same axiom. Since the latter condition is equivalent to $\mathbf{A}^-_*$ being a co-forest [33, Thm. 2.4], and as $\mathbf{A}_* = \mathbf{A}^-_*$, the result follows. □

Example 3.2. Here we present a Gödel algebra which fails to be a bi-Gödel algebra, by depicting its Esakia dual. Notice that, in view of Example 2.7 and the definitions, any such algebra must be infinite. Let $\mathcal{X} := (X, \leqslant)$ be the countably infinite co-tree depicted in Figure 2, with co-root r and such that every other point is minimal. Using similar terminology as the one introduced in Section 4.3, we call $\mathcal{X}$ the ω*-co-fork*. We will show below that the co-tree $\mathcal{X}$ can be equipped

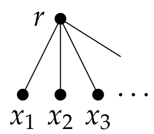

FIGURE 2. The ω-co-fork.

with an Esakia topology τ, hence $(X, \tau, \leqslant)^*$ is a Gödel algebra [33, Thm. 2.4]. However, we will also prove that this topology has clopen sets which generate upsets that are not clopen. Consequently, $(X, \tau, \leqslant)$ does not satisfy the definition of bi-Esakia spaces 2.11, so the bi-Esakia duality ensures that $(X, \tau, \leqslant)^*$ is not a bi-Heyting algebra (hence, in particular, not a bi-Gödel algebra).

Let τ be the topology on $\mathcal{X}$ generated by the subbasis

$$\mathcal{S} := \{N \colon N \subseteq min(\mathcal{X}) \text{ is finite}\} \cup \{M \cup \{r\} \colon M \subseteq min(\mathcal{X}) \text{ is co-finite}\}.$$

It is routine to check that $Clop(\mathcal{X}) = \mathcal{S}$. Using the Alexander Subbasis Theorem, it follows easily from the definition of $\mathcal{S}$ that τ is compact. Moreover, if $U \in \mathcal{S}$ then it is clear that ${\downarrow}U = U$ or ${\downarrow}U = X$, hence the downset generated by a clopen set is again clopen. To see that $\mathcal{X}$ satisfies the PSA, consider $x, y \in X$ such that $x \not\leqslant y$. Then $y \neq r$ holds, thus $X \smallsetminus \{y\} = min(\mathcal{X}) \smallsetminus \{y\} \cup \{r\} \in \mathcal{S}$ and $X \smallsetminus \{y\}$ is a clopen upset that separates x from y. We conclude that $\mathcal{X}$ is an Esakia space. Now, simply take $x \in min(\mathcal{X})$ and notice that $\{x\} \in \mathcal{S}$ while ${\uparrow}\{x\} = \{x, r\} \notin \mathcal{S}$. So $\mathcal{X}$ fails to be a bi-Esakia space, as desired.

Before we characterize the SI bi-Gödel algebras, let us recall the standard characterization of simple and SI bi-Heyting algebras by means of their bi-Esakia duals, as well as prove that the existence of a greatest prime filter of $\mathbf{A} \in \mathsf{bi\text{-}HA}$ is a sufficient condition for $\mathbf{A}$ to be simple.

Theorem 3.3. *If $\mathbf{A} \in \mathsf{bi\text{-}HA}$, then the following conditions are equivalent:*

(i) *$\mathbf{A}$ is SI;*
(ii) *$\big(Con(\mathbf{A}) \smallsetminus \{Id_A\}, \subseteq\big)$ has a least element;*
(iii) *$\big(ClUpDo(\mathbf{A}_*) \smallsetminus \{A_*\}, \subseteq\big)$ has a greatest element.*

Proof. Using Proposition 2.15, it can be shown that the lattice of congruences on $\mathbf{A}$ is dually isomorphic to that of closed updownsets of $\mathbf{A}_*$, yielding the result. □

Corollary 3.4. *If $\mathbf{A} \in \mathsf{bi\text{-}HA}$, then the following conditions are equivalent:*

(i) *$\mathbf{A}$ is simple;*
(ii) *$Con(\mathbf{A}) = \{Id_A, A^2\}$ and $Id_A \neq A^2$;*
(iii) *$ClUpDo(\mathbf{A}_*) = \{\emptyset, A_*\}$ and $\emptyset \neq A_*$.*

Proof. This result follows immediately from the definition of a simple algebra and the aforementioned dual isomorphism between the lattice of congruences on $\mathbf{A}$ and that of closed updownsets of $\mathbf{A}_*$. □

Proposition 3.5. *Let $\mathbf{A} \in \mathsf{bi\text{-}HA}$. If $\mathbf{A}_*$ has a greatest element, then $\mathbf{A}$ is simple.*

Proof. First, let us note that if $\mathbf{A}_*$ has a greatest element x, then $A_* \neq \emptyset$ and every nonempty upset of $\mathbf{A}_*$ contains x. Since we also have ${\downarrow}x = A_*$, it now follows that $UpDo(\mathbf{A}_*) = \{\emptyset, A_*\}$, hence also $ClUpDo(\mathbf{A}_*) = \{\emptyset, A_*\}$. Therefore, by Corollary 3.4, we have that $\mathbf{A}$ is simple. □

Remark 3.6. The converse of Proposition 3.5 fails in general because $Up(\mathcal{X})$ is a simple bi-Heyting algebra for every nonempty finite connected poset $\mathcal{X}$.

The next theorem lists equivalent conditions for a bi-Gödel algebra to be SI. In particular, it provides a transparent characterization of these algebras: they are exactly the bi-Heyting algebras whose duals are bi-Esakia co-trees. Recall that in a bounded distributive lattice $\mathbf{L}$, the element 0 is said to be *$\wedge$-irreducible* if $a \wedge b = 0$ implies $a = 0$ or $b = 0$, for all $a, b \in L$.

Theorem 3.7. *If $\mathbf{A} \in \mathsf{bi\text{-}GA}$, then the following conditions are equivalent:*

(i) *$\mathbf{A}$ is SI;*
(ii) *$\mathbf{A}_*$ is a bi-Esakia co-tree;*
(iii) *$\mathbf{A}_*$ has a greatest element;*
(iv) *$\mathbf{A}$ is simple;*
(v) *$\mathbf{A}$ is nontrivial and $\neg a = 0$, for all $a \in A \smallsetminus \{0\}$;*
(vi) *$\mathbf{A}$ is nontrivial and 0 is $\wedge$-irreducible.*

Proof. [(i) $\Longrightarrow$ (ii)] Let $\mathbf{A}$ be an SI bi-Gödel algebra. Since $\mathbf{A}_*$ is a bi-Esakia co-forest by Theorem 3.1, we can write $A_* = \biguplus\{{\downarrow}x \colon x \in max(\mathbf{A}_*)\}$ as a disjoint union of co-trees. As $\mathbf{A}$ is SI, Theorem 3.3 entails that $ClUpDo(\mathbf{A}_*) \smallsetminus \{A_*\}$ has a greatest element U. Since U is a proper downset, there must be a maximal point w of $\mathbf{A}_*$ not in U. Note that, since w is maximal and principal upsets are chains, it follows that ${\downarrow}w$ is an upset of $\mathbf{A}_*$. By definition, it is also a downset. Moreover, we know by Theorem 2.12 that ${\downarrow}w$ is closed, hence it is a closed updownset not contained in U. By the definition of U, it follows that ${\downarrow}w = A_*$, so $\mathbf{A}_*$ is indeed a bi-Esakia co-tree.

$\boxed{\text{(ii)} \implies \text{(iii)} \implies \text{(iv)} \implies \text{(i)}}$ The first implication follows directly from the definition of a co-tree, the second is an immediate consequence of Proposition 3.5, while the third follows from the fact that simple algebras are SI.

$\boxed{\text{(iii)} \implies \text{(v)}}$ Suppose that $\mathbf{A}_*$ has a greatest element, i.e., that $\mathbf{A}$ has a greatest prime filter r. It is an immediate consequence of the Prime Filter Theorem and the definition of r that $a \in A \smallsetminus \{0\}$ iff a is contained in some prime filter iff $a \in r$. If $a \in A \smallsetminus \{0\}$, then $a \wedge \neg a = 0 \notin r$ entails $\neg a \notin r$, i.e., $\neg a = 0$ by the previous remark.

$\boxed{\text{(v)} \implies \text{(vi)}}$ Suppose that $\mathbf{A}$ satisfies condition (v), and $a \wedge c = 0$, for some $a, c \in A$. If $a \neq 0$, then $\neg a = 0$, so $c \in \{b \in A \colon b \wedge a \leqslant 0\}$ now entails $c = 0 = \neg a = \bigvee\{b \in A \colon b \wedge a \leqslant 0\}$. Therefore, 0 is $\wedge$-irreducible.

$\boxed{\text{(vi)} \implies \text{(iii)}}$ If $\mathbf{A}$ is a nontrivial bi-Gödel algebra whose 0 is $\wedge$-irreducible, then it is easy to see that $A \smallsetminus \{0\}$ is not only a prime filter, but is in fact the greatest such filter. □

Corollary 3.8. *The following conditions hold:*

(i) $\mathsf{bi\text{-}GA}$ *is a semi-simple variety,*
(ii) $\mathsf{bi\text{-}GA}$ *is a discriminator variety,*
(iii) $\mathsf{bi\text{-}GA}$ *has EDPC.*

Proof. That $\mathsf{bi\text{-}GA}$ is semi-simple follows immediately from the equivalence (i) $\iff$ (iv) of Theorem 3.7.

To show that $\mathsf{bi\text{-}GA}$ is discriminator, we use the following characterization (which was proved in [58]): a variety V of bi-Heyting algebras is discriminator iff there exists some $n \in \omega$ such that

$$\mathsf{V} \models (\neg\sim)^{n+1}x \approx (\neg\sim)^n x,$$

where $(\neg\sim)^{n+1}x$ is recursively defined as $(\neg\sim)^{n+1}x := \neg\sim\big((\neg\sim)^n x\big)$. Clearly, it suffices to show that there exists an equation of this form which is satisfied by every SI bi-Gödel algebra. Accordingly, let $a \in \mathbf{A} \in \mathsf{bi\text{-}GA}_{SI}$. By Proposition 2.6.(7), we know that $\sim a = 0$ iff $a = 1$. So $\neg\sim 1 = \neg 0 = 1$, and using the equivalence (i) $\iff$ (v) of Theorem 3.7, we also have that $\neg\sim a = 0$ when $a \neq 1$. It is now clear that $\mathbf{A} \models (\neg\sim)^2 x \approx \neg\sim x$, as desired.

For the sake of completeness, we also provide a discriminator term for the variety $\mathsf{bi\text{-}GA}$, namely,

$$t(x,y,z) := \big((x+y) \wedge z\big) \vee \big(\neg(x+y) \wedge x\big),$$

where $x + y := \neg\big((x \leftarrow y) \vee (y \leftarrow x)\big)$. Let $\mathbf{A} \in \mathsf{bi\text{-}GA}_{SI}$ and $a, b \in A$. If $a = b$, then $a \leftarrow b = 0 = b \leftarrow a$, hence $a + b = \neg 0 = 1$. On the other hand, if $a \neq b$, we can assume without loss of generality that $a \nleqslant b$. By Proposition 2.6, this is equivalent to $a \leftarrow b \neq 0$, and therefore $0 < \big((a \leftarrow b) \vee (b \leftarrow a)\big)$. As $\mathbf{A}$ is SI, it follows from the equivalence (i) $\iff$ (v) of Theorem 3.7 that $a + b = \neg\big((a \leftarrow b) \vee (b \leftarrow a)\big) = 0$. This discussion yields

$$a + b = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

and it is now routine to check that the term t is a discriminator term for $\mathbf{A}$.

Finally, the fact that $\mathsf{bi\text{-}GA}$ has EDPC now follows from 2.2.(v). □

Corollary 3.9. *If* $\mathbf{A}$ *is a subalgebra of an SI bi-Gödel algebra, or is an ultraproduct of a family of SI bi-Gödel algebras, then* $\mathbf{A}$ *is also SI.*

Proof. It is well known that if V is a discriminator variety, then V_{SI} forms a universal class (see, e.g., [14, Thm. IX.9.4.(c)]). Since universal sentences are preserved under taking subalgebras and ultraproducts, the desired result now follows from the previous corollary.

To be more explicit, notice that Theorem 3.7 ensures that a bi-Gödel algebra $\mathbf{A}$ is SI iff $\mathbf{A}$ is nontrivial and 0 is $\wedge$-irreducible. It follows that $\mathsf{bi\text{-}GA}_{SI}$ can be axiomatized (over $\mathsf{bi\text{-}GD}$) by a single universal sentence:

$$0 \neq 1 \text{ and } \forall x, y \, \big(x \wedge y = 0 \implies (x = 0 \text{ or } y = 0)\big). \qquad \square$$

To finish this section, we prove that the *bi-intuitionistic linear calculus*

$$\mathsf{bi\text{-}LC} := \mathsf{bi\text{-}IPC} + (p \to q) \vee (q \to p) + \neg[(q \leftarrow p) \wedge (p \leftarrow q)]$$

is the bi-intermediate logic of linearly ordered posets (i.e., chains). Recall that a *tree* is the order dual of a co-tree (that is, a poset with a least element, called the *root*, and whose principal downsets are chains) and that a *forest* is a disjoint union of trees. Moreover, a *bi-Esakia tree* or *forest* is a bi-Esakia space whose underlying poset is a tree or forest, respectively.

Theorem 3.10. *If* $\mathbf{A} \in \mathsf{bi\text{-}HA}$, *then* $\mathbf{A} \in (\mathsf{V}_{\mathsf{bi\text{-}LC}})_{SI}$ *iff* $\mathbf{A}_*$ *is a nonempty bi-Esakia chain. Consequently,* $\mathsf{bi\text{-}LC}$ *is Kripke complete with respect to the class of chains.*

Proof. By order dualizing Theorem 3.1, it follows that a bi-Heyting algebra $\mathbf{A}$ validates the dual Gödel-Dummett axiom $\neg[(p \leftarrow q) \wedge (q \leftarrow p)]$ iff $\mathbf{A}_*$ is a bi-Esakia forest. Furthermore, by order dualizing Theorem 3.7, any such algebra $\mathbf{A}$ is SI iff $\mathbf{A}_*$ is a bi-Esakia tree. Now, simply note that our respective characterizations of the SI algebras which validate each of the axioms of $\mathsf{bi\text{-}LC}$ already ensure that $\mathbf{A} \in (\mathsf{V}_{\mathsf{bi\text{-}LC}})_{SI}$ iff $\mathbf{A}_*$ is both a bi-Esakia tree and co-tree iff $\mathbf{A}_*$ is a nonempty bi-Esakia chain (notice that, in view of Proposition 2.14.(i), any nonempty (bi-)Esakia chain must be bounded). The last part of the statement is now clear. $\square$

It is an immediate consequence of the bi-Esakia duality that if $\mathbf{A}$ is a bi-Heyting algebra, then $\mathbf{A}_*$ is a nonempty bi-Esakia chain iff $\mathbf{A}$ is nontrivial and its order is linear. Notably, every bounded distributive lattice $\mathbf{A}$ whose order is linear can be viewed as a bi-Heyting algebra by setting

$$a \to b := \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{if } b < a, \end{cases} \qquad \text{and} \qquad a \leftarrow b := \begin{cases} 0 & \text{if } a \leqslant b, \\ a & \text{if } b < a, \end{cases}$$

for each $a, b \in \mathbf{A}$. Consequently, the terms *linear bounded distributive lattice*, *linear Heyting algebra*, and *linear bi-Heyting algebra* are all synonyms. We denote the variety generated by these algebras by $\mathsf{bi\text{-}LA}$ and note that the above discussion, together with the previous theorem, yields:

Corollary 3.11. *The bi-intermediate logic* $\mathsf{bi\text{-}LC}$ *of chains is algebraized by the variety* $\mathsf{bi\text{-}LA}$.

4. Stable, Jankov and subframe formulas for bi-Gödel algebras

In this section, we develop the theories of stable canonical and subframe formulas for bi-Gödel algebras. For an overview of these formulas and their use in superintuitionistic and modal logics we refer to [8] and [17], respectively. We use stable canonical formulas to provide a uniform axiomatization of all extensions of $\mathsf{bi\text{-}GD}$. We then use Jankov formulas (a particular type of stable canonical formulas) to fully characterize the splitting logics of the lattice $\Lambda(\mathsf{bi\text{-}GD})$, and prove that $|\Lambda(\mathsf{bi\text{-}GD})| = 2^{\aleph_0}$. As for the subframe formulas, we utilize them to establish a straightforward axiomatization of some notable extensions of $\mathsf{bi\text{-}GD}$, such as the logic of co-trees of depth (respectively, width) less than an arbitrary $n \in \mathbb{Z}^+$. But our main use for these formulas will come in the following section, when we characterize the locally tabular extensions of $\mathsf{bi\text{-}GD}$.

4.1. Stable canonical formulas and Jankov formulas.

Let $\{\phi, \varphi, \psi\} \cup \Sigma$ be a set of formulas and let $\mathfrak{M} = (W, \leqslant, V)$ be a Kripke model (i.e., a poset $(W, \leqslant)$ equipped with a valuation V on the bi-Heyting algebra $Up(\mathcal{X})$ of its upsets). We write $\mathfrak{M}, x \models \phi$ when $x \in V(\phi)$, and $\mathfrak{M} \models \phi$ when $V(\phi) = W$. Moreover, if $\mathfrak{M}' \models \phi$ for all Kripke models $\mathfrak{M}' = (W, \leqslant, V')$ on $(W, \leqslant)$, we say that $(W, \leqslant)$ *validates* ϕ, denoted $(W, \leqslant) \models \phi$. When $(W, \leqslant)$ validates every formula in Σ, we write $(W, \leqslant) \models \Sigma$ and say that $(W, \leqslant)$ *validates* Σ.

Recall that the Kripke semantics of bi-IPC define

$$\mathfrak{M}, x \models \psi \leftarrow \varphi \iff \exists y \in {\downarrow} x \ (\mathfrak{M}, y \models \psi \text{ and } \mathfrak{M}, y \not\models \varphi),$$

and note the following equivalences, for an arbitrary $w \in \mathfrak{M}$:

$$\begin{aligned}\mathfrak{M}, w \models {\sim}\neg\phi &\iff \mathfrak{M}, w \models \top \leftarrow \neg\phi \iff \exists v \in {\downarrow} w \ (\mathfrak{M}, v \models \top \text{ and } \mathfrak{M}, v \not\models \neg\phi) \\ &\iff \exists v \in {\downarrow} w \ (\mathfrak{M}, v \not\models \neg\phi) \iff \exists u \in {\uparrow}{\downarrow} w \ (\mathfrak{M}, u \models \phi).\end{aligned}$$

Let us now assume that $\mathfrak{M}$ has a greatest element r. It is clear from the above display that $\mathfrak{M}, w \models {\sim}\neg\phi$ implies that ϕ holds in some point of $\mathfrak{M}$. Suppose now that $\mathfrak{M}, v \models \phi$, i.e., that $v \in V(\phi)$, for some $v \in \mathfrak{M}$. Since $V(\phi)$ is an upset and r the greatest element of $\mathfrak{M}$, it follows $\mathfrak{M}, r \models \phi$. Moreover, our assumption on r also yields $r \in {\uparrow}{\downarrow} w$. This, together with $\mathfrak{M}, r \models \phi$, now implies $\mathfrak{M}, w \models {\sim}\neg\phi$, again by the above display.

From this discussion, we can conclude that for an arbitrary $w \in \mathfrak{M}$, we have $\mathfrak{M}, w \models {\sim}\neg\phi$ iff ϕ holds in some point of $\mathfrak{M}$. Thus, in the setting of Kripke models with a greatest element (in particular, of Kripke models on co-trees), ${\sim}\neg$ can be viewed as an analogue to the notion of the universal diamond from modal logic.

Similarly, let us note that for an arbitrary $w \in \mathfrak{M}$, the followng equivalences hold:

$$\begin{aligned}\mathfrak{M}, w \models \neg{\sim}\phi &\iff \forall v \in {\uparrow} w \ (\mathfrak{M}, v \not\models {\sim}\phi) \iff \forall v \in {\uparrow} w \ (\mathfrak{M}, v \not\models \top \leftarrow \phi) \\ &\iff \forall u \in {\downarrow}{\uparrow} w \ (\mathfrak{M}, u \not\models \top \text{ or } \mathfrak{M}, u \models \phi) \iff \forall u \in {\downarrow}{\uparrow} w \ (\mathfrak{M}, u \models \phi).\end{aligned}$$

If we again assume that $\mathfrak{M}$ has a greatest element r, then clearly $W = {\downarrow} r = {\downarrow}{\uparrow} w$. Thus, the equivalences above imply that $\mathfrak{M}, w \models \neg{\sim}\phi$ iff ϕ holds everywhere in $\mathfrak{M}$. Therefore, in the setting of Kripke models with a greatest element (in particular, of Kripke models on co-trees), $\neg{\sim}$ can be viewed as an analogue of the notion of the universal box from modal logic.

This discussion not only provides some intuition for what follows, by highlighting a similarity with the construction of the Jankov-Fine formulas for modal frames, but it helps us show that extensions of bi-GD admit a metalogical classical inconsistency lemma as in condition (1). These types of lemmas were formally introduced and studied in [50], see also [15, 45, 44].

Let L be an extension of bi-GD. We define a consequence relation $\vdash_L$ in the following manner: given a set of formulas $\Sigma \cup \{\phi\}$, we write $\Sigma \vdash_L \phi$ iff $\mathfrak{M} \models \Sigma$ implies $\mathfrak{M} \models \phi$, for every Kripke model $\mathfrak{M} = (\mathcal{X}, V)$ on a co-tree $\mathcal{X}$ which validates L.

Theorem 4.1. *Let L be an extension of* bi-GD. *If $\Sigma \cup \{\phi\}$ is a set of formulas, then*

$$\Sigma \cup \{{\sim}\neg{\sim}\phi\} \vdash_L \bot \iff \Sigma \vdash_L \phi.$$

Consequently, L has a classical inconsistency lemma.

Proof. Assume $\Sigma \cup \{{\sim}\neg{\sim}\phi\} \vdash_L \bot$ and that $\mathfrak{M} \models \Sigma$, where $\mathfrak{M} = (\mathcal{X}, V)$ is a Kripke model on a co-tree $\mathcal{X}$ which validates L. Since $\mathcal{X}$ is a co-tree, $\mathfrak{M}$ is nonempty, hence $\mathfrak{M} \not\models \bot$. As $\Sigma \cup \{{\sim}\neg{\sim}\phi\} \vdash_L \bot$ and $\mathfrak{M} \models \Sigma$, this implies $\mathfrak{M} \not\models {\sim}\neg{\sim}\phi$, i.e., that there exists $w \in X$ such that $\mathfrak{M}, w \not\models {\sim}\neg{\sim}\phi$. It now follows that $\mathfrak{M}, w \models \neg{\sim}\phi$, and by our previous discussion on the connective $\neg{\sim}$, we conclude $\mathfrak{M} \models \phi$.

For the converse, let us suppose $\Sigma \vdash_L \phi$. We can show that $\Sigma \cup \{{\sim}\neg{\sim}\phi\} \vdash_L \bot$ holds by proving that $\mathfrak{M} \not\models \Sigma \cup \{{\sim}\neg{\sim}\phi\}$, for every model $\mathfrak{M} = (\mathcal{X}, V)$ such that $\mathcal{X}$ is a co-tree and $\mathcal{X} \models L$. Accordingly, let $\mathfrak{M}$ be such a model and suppose that $\mathfrak{M} \models \Sigma \cup \{{\sim}\neg{\sim}\phi\}$. By our assumption, $\mathfrak{M} \models \Sigma$ implies $\mathfrak{M} \models \phi$, and by our previous discussion on the connective ${\sim}\neg$, we know that $\mathfrak{M} \models {\sim}\neg{\sim}\phi$ entails the existence of a point $w \in X$ such that $\mathfrak{M}, w \models {\sim}\phi$. But now we have $\mathfrak{M}, v \not\models \phi$, for some $v \in {\downarrow} w$, contradicting $\mathfrak{M} \models \phi$. Therefore, $\mathfrak{M} \not\models \Sigma \cup \{{\sim}\neg{\sim}\phi\}$, as desired. □

In view of Theorem 4.1, extensions of bi-GD exhibit an appealing balance between the classical and intuitionistic behavior of negation connectives, adding more motivation for studying this system.

We will now extend the method of stable canonical formulas [8] to the setting of bi-GD. Let $\mathbf{A} \in \text{bi-GA}_{SI}^{<\omega}$ and $D \subseteq A^2$. For each $a \in A$, we introduce a fresh propositional variable

$p_a \in Prop$. Let Γ be the formula describing the Heyting algebra structure of $\mathbf{A}$ fully, while the behavior of the operator $\leftarrow$ is only given for elements of D, i.e.,

$$\Gamma := \bigwedge\{p_{a\vee b} \leftrightarrow (p_a \vee p_b)\colon (a,b) \in A^2\} \wedge \bigwedge\{p_{a\wedge b} \leftrightarrow (p_a \wedge p_b)\colon (a,b) \in A^2\} \wedge$$
$$\bigwedge\{p_{a\to b} \leftrightarrow (p_a \to p_b)\colon (a,b) \in A^2\} \wedge \bigwedge\{p_{a\leftarrow b} \leftrightarrow (p_a \leftarrow p_b)\colon (a,b) \in D\} \wedge$$
$$\wedge \{p_0 \leftrightarrow \bot\} \wedge \{p_1 \leftrightarrow \top\}.$$

We define the *stable canonical formula* associated with $\mathbf{A}$ and D by

$$\gamma(\mathbf{A}, D) := \neg \sim\Gamma \to \neg \bigwedge \{p_a \leftarrow p_b\colon (a,b) \in A^2 \text{ and } a \nleqslant b\}.$$

Moreover, given $\mathbf{B} \in$ bi-GA and a Heyting homomorphism $h\colon \mathbf{A} \to \mathbf{B}$, we call D a *$\leftarrow$-stable domain* of $\mathbf{A}$ if for all $(a,b) \in D$, we have $h(a \leftarrow b) = h(a) \leftarrow h(b)$. In this case, we say that h satisfies the *$\leftarrow$-stable domain condition* ($\text{SDC}_{\leftarrow}$ for short) for D.

Before we discuss the fundamental properties of the stable canonical formulas, we recall four elementary facts about bi-Heyting algebras, that will be used without further reference in what follows.

Lemma 4.2. *If $\mathbf{A} \in$ bi-HA and $a, b \in A$, then:*

(i) $\neg a = 1 \iff a = 0$,
(ii) $\neg \sim a = 1 \iff a = 1$,
(iii) $a \to b = 1 \iff a \leqslant b$,
(iv) $a \leftarrow b \neq 0 \iff a \nleqslant b$.

Lemma 4.3 (Stable Jankov Lemma)**.** *Let $\mathbf{B} \in$ bi-GA. If $\mathbf{A} \in \text{bi-GA}_{SI}^{<\omega}$ and $D \subseteq A^2$, then $\mathbf{B} \nvDash \gamma(\mathbf{A}, D)$ iff there exists a Heyting algebra embedding $h\colon \mathbf{A} \hookrightarrow \mathbf{C}$ satisfying the $\text{SDC}_{\leftarrow}$ for D, for some $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$.*

Proof. We start by proving the left to right implication. Suppose that a bi-Gödel algebra $\mathbf{B}$ refutes $\gamma(\mathbf{A}, D)$. By the Subdirect Decomposition Theorem, there exists $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$ such that $\mathbf{C} \nvDash \gamma(\mathbf{A}, D)$. Let v be a valuation on $\mathbf{C}$ refuting $\gamma(\mathbf{A}, D)$, i.e., such that

$$v(\gamma(\mathbf{A}, D)) = v(\neg \sim\Gamma) \to v(\neg \bigwedge \{p_a \leftarrow p_b\colon (a,b) \in A^2 \text{ and } a \nleqslant b\}) < 1.$$

Notice that this implies $0 < v(\neg \sim\Gamma)$, and that if $(\mathbf{C}_*, V)$ is the bi-Esakia model corresponding to $(\mathbf{C}, v)$ (via the bi-Esakia duality), then $0 < v(\neg \sim\Gamma)$ iff $V(\neg \sim\Gamma) \neq \emptyset$, i.e., $\neg \sim\Gamma$ holds true in some point of $(\mathbf{C}_*, V)$. By our previous discussion on the behaviour of $\neg \sim$ on models with a greatest element, the fact that $\mathbf{C}$ is SI now entails that $V(\Gamma) = C_*$, i.e., that $v(\Gamma) = 1$.

Let us now define a map $h\colon A \to C$ by setting $h(a) := v(p_a)$, for every $a \in A$. By the definitions of h and the formula Γ, it is easy to see that $v(\Gamma) = 1$ iff h is a Heyting homomorphism which satisfies the $\text{SDC}_{\leftarrow}$ for D, hence the only condition that remains to be shown is that h is injective. To this end, let $a \neq b \in A$ and suppose, without loss of generality, that $a \nleqslant b$. Notice that, since $v(\gamma(\mathbf{A}, D)) \neq 1$, we must have

$$v(\neg \bigwedge \{p_x \leftarrow p_y\colon x, y \in A \text{ and } x \nleqslant y\}) \neq 1,$$

which is equivalent to

$$\bigwedge \{v(p_x) \leftarrow v(p_y)\colon x, y \in A \text{ and } x \nleqslant y\}) \neq 0.$$

By the definition of h, it now follows that

$$\bigwedge \{h(x) \leftarrow h(y)\colon x, y \in A \text{ and } x \nleqslant y\}) \neq 0,$$

thus $h(a) \leftarrow h(b) \neq 0$, i.e., $h(a) \nleqslant h(b)$. We conclude that h is indeed a Heyting algebra embedding satisfying the $\text{SDC}_{\leftarrow}$ for D, as desired.

For the converse, let $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$, $h\colon \mathbf{A} \hookrightarrow \mathbf{C}$ be a Heyting algebra embedding that satisfies the $\text{SDC}_{\leftarrow}$ for D, and $v\colon Prop \to C$ a valuation satisfying $v(p_a) = h(a)$, for each $a \in A$. We shall prove that $\mathbf{C}$ refutes $\gamma(\mathbf{A}, D)$ via v, hence showing that $\mathbf{B} \nvDash \gamma(\mathbf{A}, D)$ holds (recall that the validity of a formula is preserved under taking homomorphic images).

By the definitions of h and Γ, it is clear that $v(\Gamma) = 1$, which is equivalent to $\neg{\sim}v(\Gamma) = v(\neg{\sim}\Gamma) = 1$. Now, let $a, b \in A$ be such that $a \nleqslant b$, i.e., $a \to b \neq 1$. Since h is a Heyting algebra embedding, it follows that $h(a \to b) = h(a) \to h(b) \neq 1$, i.e., $h(a) \nleqslant h(b)$, which in turn is equivalent to $h(a) \leftarrow h(b) = v(p_a) \leftarrow v(p_b) \neq 0$. Thus, we have

$$0 \notin \{v(p_a) \leftarrow v(p_b) \colon a, b \in A \text{ and } a \nleqslant b\}.$$

As $\mathbf{C}$ is SI by assumption, $0_{\mathbf{C}}$ is $\wedge$-irreducible (see Theorem 3.7), hence

$$v\big(\bigwedge\{p_a \leftarrow p_b \colon a, b \in A \text{ and } a \nleqslant b\}\big) \neq 0.$$

Consequently, $\neg v\big(\bigwedge\{p_a \leftarrow p_b \colon a, b \in A \text{ and } a \nleqslant b\}\big) \neq 1$, and we conclude that $v\big(\gamma(\mathbf{A}, D)\big) \neq 1$. Therefore, $\mathbf{C}$ refutes $\gamma(\mathbf{A}, D)$ via v. ☐

The following lemma is essential for what follows. Notice that it makes crucial use of the fact that the HA-reduct of bi-GA is locally finite [34], much like the analogous version of this result for Heyting algebras relies on the local finiteness of the lattice reduct of HA. Henceforth, we will make extensive use of the fact that every finite Gödel algebra (i.e., a Heyting algebra that validates the Gödel-Dummett axiom) can be regarded as a finite bi-Gödel algebra (see Example 2.7).

Lemma 4.4 (Stable Filtration Lemma)**.** *Let ϕ be a formula and $\mathbf{B} \in$ bi-GA. If $\mathbf{B} \not\models \phi$, then there exists a finite Heyting subalgebra $\mathbf{A}$ of $\mathbf{B}$ such that $\mathbf{A} \in$ bi-GA and $\mathbf{A} \not\models \phi$. If $\mathbf{B}$ is moreover SI, then so is $\mathbf{A}$.*

Proof. Suppose that $\mathbf{B} \not\models \phi$. Then $\phi^{\mathbf{B}}(\bar{a}) \neq 1$ for some tuple $\bar{a} \in B$. Let

$$\Sigma := \{\psi(\bar{a}) \colon \psi \text{ is a subformula of } \phi\}$$

and $\mathbf{A}$ be the Heyting subalgebra of $\mathbf{B}$ generated by Σ. Since Σ is finite and bi-GA has a locally finite HA-reduct, it follows that $\mathbf{A}$ is a finite Heyting algebra, hence also a finite bi-Heyting algebra (see Example 2.7), although not necessarily a bi-Heyting subalgebra of $\mathbf{B}$. Moreover, since bi-GA is axiomatized (relative to bi-IPC) by the Gödel-Dummett axiom, a $\leftarrow$-free formula, and $\mathbf{A}$ is a Heyting subalgebra of $\mathbf{B}$, then clearly $\mathbf{B} \in$ bi-GA implies $\mathbf{A} \in$ bi-GA.

As $\mathbf{A}$ is a bi-Heyting algebra, it has a well-defined $\leftarrow^{\mathbf{A}}$ operation. And although this operation need not coincide with $\leftarrow^{\mathbf{B}}$, it crucially does so when $a {\leftarrow^{\mathbf{B}}} b \in \Sigma$. To see this, just note that $A \subseteq B$ implies

$$a {\leftarrow^{\mathbf{B}}} b = \bigwedge\{c \in B \colon a \leqslant c \vee b\} \leqslant \bigwedge\{c \in A \colon a \leqslant c \vee b\} = a {\leftarrow^{\mathbf{A}}} b,$$

for all $a, b \in A$. Moreover, if $a {\leftarrow^{\mathbf{B}}} b \in A$, then clearly $a {\leftarrow^{\mathbf{B}}} b \in \{c \in A \colon a \leqslant c \vee b\}$, hence $a {\leftarrow^{\mathbf{A}}} b \leqslant a {\leftarrow^{\mathbf{B}}} b$. It now follows that if $a {\leftarrow^{\mathbf{B}}} b \in A$ (and in particular, if $a {\leftarrow^{\mathbf{B}}} b \in \Sigma$), then $a {\leftarrow^{\mathbf{B}}} b = a {\leftarrow^{\mathbf{A}}} b$. Therefore, using a simple argument by induction on the complexity of ϕ, we can conclude that $\phi^{\mathbf{B}}(\bar{a}) = \phi^{\mathbf{A}}(\bar{a})$, hence $\phi^{\mathbf{A}}(\bar{a}) \neq 1$. Thus, $\mathbf{A} \not\models \phi$, as desired.

Suppose now that $\mathbf{B}$ is SI, hence $0_{\mathbf{B}}$ is $\wedge$-irreducible by Theorem 3.7. Since $\mathbf{A}$ is a Heyting subalgebra of $\mathbf{B}$, it is nontrivial and $0_{\mathbf{A}}$ must also be $\wedge$-irreducible. Again using Theorem 3.7, we conclude that $\mathbf{A}$ is SI. ☐

Corollary 4.5. *The variety* bi-GA *has the FMP.*

Proof. This result follows easily from the previous lemma. ☐

Equipped with Lemmas 4.3 and 4.4, we can start our proof of the first main result of this subsection, which establishes a uniform axiomatization of all extensions of bi-GD by means of stable canonical formulas. This is in analogy with the intuitionistic case, see, e.g., [7, 17]. However, a similar axiomatization technique for arbitrary bi-intermediate logics cannot be obtained, as we discuss below.

Fix a formula $\phi \notin$ bi-GD and set $n := |Sub(\phi)|$. Since the HA-reduct of bi-GA is locally finite, there exists a bound $c(\phi) \in \omega$ on the size of n-generated Heyting algebras belonging to this reduct. Accordingly, let $\mathbf{A}_1, \dots, \mathbf{A}_{m(n)}$ be the list of (up to isomorphism) all n-generated SI bi-Gödel algebras such that $|A_i| \leqslant c(\phi)$ and $\mathbf{A}_i \not\models \phi$. Now, for each of these bi-Gödel algebras $\mathbf{A}_i$ refuting ϕ via a valuation v, we set

$$D^{\leftarrow} := \{(a,b) \in \Theta^2 \colon a \leftarrow b \in \Theta\},$$

where $\Theta := v[Sub(\phi)]$. Consider a new list $(\mathbf{A}_1, D_1^{\leftarrow}), \ldots, (\mathbf{A}_{k(n)}, D_{k(n)}^{\leftarrow})$ (notice that $k(n)$ need not be smaller than $m(n)$, since each $\mathbf{A}_i$ may refute ϕ through distinct valuations), whose elements we call the *refutation patterns* for ϕ. Keeping this discussion in mind, we have the following theorem:

Theorem 4.6. *If* $\mathbf{B}$ *is an SI bi-Gödel algebra, then:*

(i) $\mathbf{B} \not\models \phi$ *iff there exists* $i \leqslant k(n)$ *and a Heyting algebra embedding* $h \colon \mathbf{A}_i \hookrightarrow \mathbf{B}$ *satisfying the SDC*$_{\leftarrow}$ *for* $D_i^{\leftarrow}$*;*

(ii) $\mathbf{B} \models \phi \iff \mathbf{B} \models \bigwedge_{i=1}^{k(n)} \gamma(\mathbf{A}_i, D_i^{\leftarrow})$.

Proof. (i) Firstly, note that right to left implication follows immediately from $(\mathbf{A}_i, D_i^{\leftarrow}) \not\models \phi$ and the definition of $D_i^{\leftarrow}$, since if a Heyting algebra embedding $h \colon \mathbf{A}_i \hookrightarrow \mathbf{B}$ satisfies the SDC$_{\leftarrow}$ for $D_i^{\leftarrow}$, then we clearly have $\mathbf{B} \not\models \phi$. To prove the converse, suppose that $\mathbf{B} \not\models \phi$. As $\mathbf{B} \in \mathsf{bi\text{-}GA}_{SI}$, it follows from Lemma 4.4 that there is a finite Heyting subalgebra $\mathbf{A}$ of $\mathbf{B}$ such that $\mathbf{A} \in \mathsf{bi\text{-}GA}_{SI}$ and $\mathbf{A}$ refutes ϕ via some valuation v. Thus, there exists a Heyting embedding $h \colon \mathbf{A} \hookrightarrow \mathbf{B}$, and by looking at the proof of Lemma 4.4, we not only see that $\mathbf{A}$ is n-generated for $n = |Sub(\phi)|$ (as a Heyting algebra), but also that $a \leftarrow b \in v[Sub(\phi)]$ implies $h(a \leftarrow b) = h(a) \leftarrow h(b)$, for all $a, b \in A$. It is now easy to see that h satisfies the SDC$_{\leftarrow}$ for

$$D^{\leftarrow} := \{(a,b) \in v[Sub(\phi)]^2 \colon a \leftarrow b \in v[Sub(\phi)]\}.$$

Therefore, the pair $(\mathbf{A}, D^{\leftarrow})$ must be one of the $(\mathbf{A}_i, D_i^{\leftarrow})$ listed above, hence we showed that the right side of the desired equivalence holds, as desired.

(ii) This follows immediately from (i), together with the Stable Jankov Lemma 4.3. ☐

As a consequence, stable canonical formulas can be used to axiomatize extensions of bi-GD.

Theorem 4.7. *Every extension of* bi-GD *is axiomatizable by stable canonical formulas. Moreover, if* L *is finitely axiomatized, then* L *is axiomatizable by finitely many stable canonical formulas.*

Proof. Suppose that $L = \mathsf{bi\text{-}GD} + \{\phi_i \colon i \in I\}$, so we can assume without loss of generality that $\mathsf{bi\text{-}GD} \nvdash \phi_i$, for all $i \in I$. By the previous theorem, we know that for each $i \in I$ there is a list of refutation patterns $(\mathbf{A}_{i,1}, D_{i,1}^{\leftarrow}), \ldots, (\mathbf{A}_{i,k(i)}, D_{i,k(i)}^{\leftarrow})$ such that

$$\mathsf{bi\text{-}GD} + \phi_i = \mathsf{bi\text{-}GD} + \bigwedge_{j=1}^{k(i)} \gamma(\mathbf{A}_{i,j}, D_{i,j}^{\leftarrow}).$$

Thus, we have

$$L = \mathsf{bi\text{-}GD} + \{\phi_i \colon i \in I\} = \mathsf{bi\text{-}GD} + \{\bigwedge_{j=1}^{k(i)} \gamma(\mathbf{A}_{i,j}, D_{i,j}^{\leftarrow}) \colon i \in I\}.$$

The last part of the statement clearly follows from the previous equality. ☐

Corollary 4.8. *Let* $L' \subseteq L$ *be extensions of* bi-GD*. Then* L *is axiomatizable over* L' *by stable canonical formulas. Moreover, if* L *is finitely axiomatized over* L'*, then* L *is axiomatizable over* L' *by finitely many stable canonical formulas.*

Proof. This is an immediate consequence of the proof of the previous theorem. ☐

We will now focus on a particular class of stable canonical formulas: the Jankov formulas [35, 36, 37]. For each $\mathbf{A} \in \mathsf{bi\text{-}GA}_{SI}^{<\omega}$, we call $\mathcal{J}(\mathbf{A}) := \gamma(\mathbf{A}, A^2)$ the *Jankov formula* of $\mathbf{A}$. We compile the defining properties of these formulas in the following lemma, and subsequently use them to characterize the splitting logics of the lattice $\Lambda(\mathsf{bi\text{-}GD})$, as well as finding its cardinality.

Lemma 4.9 (Jankov Lemma)**.** *If* $\mathbf{B} \in \mathsf{bi\text{-}GA}$ *and* $\mathbf{A} \in \mathsf{bi\text{-}GA}_{SI}^{<\omega}$*, then the following conditions are equivalent:*

(i) $\mathbf{B} \not\models \mathcal{J}(\mathbf{A})$*;*
(ii) *there exists a bi-Heyting algebra embedding* $h\colon \mathbf{A} \hookrightarrow \mathbf{C}$*, for some* $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$*;*
(iii) $\mathbf{A} \in \mathbb{SH}(\mathbf{B})$*;*
(iv) $\mathbf{A} \in \mathbb{HS}(\mathbf{B})$*.*

Proof. Firstly, let us note that the equivalence (i) $\iff$ (ii) is just a particular instance of the Stable Jankov Lemma 4.3, and that (ii) clearly implies (iii). The equivalence (iii) $\iff$ (iv) follows from Proposition 2.2.(iii) and the fact that bi-GA has EDPC (see Corollary 3.8). Finally, (iv) $\implies$ (i) follows from the easily checked fact that $\mathbf{A} \not\models \mathcal{J}(\mathbf{A})$, and by noting that the operators $\mathbb{H}$ and $\mathbb{S}$ preserve the validity of formulas. ☐

Corollary 4.10. *If* $\mathbf{B} \in \mathsf{bi\text{-}GA}_{SI}$*, then* $\mathbb{V}(\mathbf{B})_{SI}^{<\omega} = \mathbb{IS}(\mathbf{B})^{<\omega}$*.*

Proof. We start by noting that $\mathbb{IS}(\mathbf{B})^{<\omega} \subseteq \mathbb{V}(\mathbf{B})_{SI}^{<\omega}$ follows directly from Corollary 3.9. To prove the other inclusion, we use the Jankov Lemma and the fact that the product of algebras preserves the validity of formulas to deduce that if $\mathbf{A} \in \mathbb{V}(\mathbf{B})_{SI}^{<\omega}$, then $\mathbf{B} \not\models \mathcal{J}(\mathbf{A})$, i.e., $\mathbf{A} \in \mathbb{SH}(\mathbf{B})$. As bi-GA is a semi-simple variety (see Corolary 3.8) and simple algebras have no nontrivial homomorphic images, $\mathbf{B} \in \mathsf{bi\text{-}GA}_{SI}$ now implies that $\mathbf{A} \in \mathbb{IS}(\mathbf{B})^{<\omega}$, as desired. ☐

Given a lattice $\mathbf{L}$ and elements $a, b \in L$, we call (a, b) a *splitting pair* for $\mathbf{L}$ if $L = {\uparrow}a \uplus {\downarrow}b$ (we use the symbol $\uplus$ to denote the union of sets which are pairwise disjoint). In particular, if $\mathbf{L} = \Lambda(\mathsf{bi\text{-}GD})$ then a is said to be a *splitting logic*.

Theorem 4.11 (Splitting Theorem)**.** *If* $L \in \Lambda(\mathsf{bi\text{-}GD})$*, then:*

(i) *L is a splitting logic iff L is axiomatized by a single Jankov formula,*
(ii) *L is a join of splitting logics iff L is axiomatized by Jankov formulas.*

Proof. We start by noting that condition (i) clearly implies (ii), hence we only prove the former equivalence. Suppose that (L, L') is a splitting pair for $\Lambda(\mathsf{bi\text{-}GD})$, for some $L' \in \Lambda(\mathsf{bi\text{-}GD})$. As bi-GA is a congruence distributive variety (by Proposition 2.2 together with Corollary 3.8) with the FMP (see Corollary 4.5), it follows from a result by McKenzie [49] that $L' = Log(\mathbf{A})$, for some $\mathbf{A} \in \mathsf{bi\text{-}GA}_{SI}^{<\omega}$. Using the definition of a splitting pair together with the fact $\mathbf{A} \not\models \mathcal{J}(\mathbf{A})$, it is easy to see that the equivalence $\mathbf{B} \models \mathcal{J}(\mathbf{A})$ iff $\mathbf{B} \models L$ holds for all $\mathbf{B} \in \mathsf{bi\text{-}GA}$. Thus, $L = \mathsf{bi\text{-}GD} + \mathcal{J}(\mathbf{A})$.

Conversely, assume $L = \mathsf{bi\text{-}GD} + \mathcal{J}(\mathbf{A})$ for some $\mathbf{A} \in \mathsf{bi\text{-}GA}_{SI}^{<\omega}$. Set $L' := Log(\mathbf{A})$ and notice that $\mathbf{A} \not\models \mathcal{J}(\mathbf{A})$ implies $L \not\subseteq L'$. Now, take $E \in \Lambda(\mathsf{bi\text{-}GD})$ and suppose $L \not\subseteq E$, i.e., $\mathcal{J}(\mathbf{A}) \notin E$. By a simple application of the Jankov Lemma 4.9, this implies $\mathbf{A} \in \mathsf{V}_E = \{\mathbf{B} \in \mathsf{bi\text{-}HA} \colon \mathbf{B} \models E\}$. Equivalently, $E \subseteq Log(\mathbf{A}) = L'$. We just proved that for $E \in \Lambda(\mathsf{bi\text{-}GD})$, $E \notin {\uparrow}L$ entails $E \in {\downarrow}L'$, i.e., that $\Lambda(\mathsf{bi\text{-}GD}) = {\uparrow}L \uplus {\downarrow}L'$. Therefore, (L, L') is a splitting pair for $\Lambda(\mathsf{bi\text{-}GD})$. ☐

A *splitting algebra* of a variety V is an SI member $\mathbf{A}$ for which there exists the largest subvariety $\mathsf{V}' \subseteq \mathsf{V}$ omitting $\mathbf{A}$. In this case, $(\mathbb{V}(\mathbf{A}), \mathsf{V}')$ is a splitting pair for the lattice of subvarieties of V. Translating the Splitting Theorem 4.11 into algebraic terms characterizes the splitting algebras of the variety bi-GA as the finite SI bi-Gödel algebras (we note that the equality between splitting algebras and finite SI algebras holds more in general for every variety of finite type with EDPC and the FMP, as shown in [12, Cor. 3.2] and [49]).

Theorem 4.12. *The splitting algebras of* bi-GA *are exactly the finite SI bi-Gödel algebras.*

It is well known that the analogue of the previous theorem holds for the variety of Heyting algebras (see, e.g., [17]): the splitting algebras of HA are exactly its finite SI elements. However, this is far from the case for bi-Heyting algebras; a result by Wolter [59] shows that the only splitting algebras in bi-HA are the two-element and three-element chains. This is the main reason why the theories of stable canonical formulas cannot be developed for bi-IPC.

4.2. **The cardinality of $\Lambda(\text{bi-GD})$.** The goal of this section is to prove that the cardinality of the lattice $\Lambda(\text{bi-GD})$ is $2^{\aleph_0}$. Accordingly, let us define a partial order $\leqslant$ on the class $\text{bi-GA}_{SI}^{<\omega}$ by $\mathbf{A} \leqslant \mathbf{B}$ iff $\mathbf{A} \in \mathbb{HS}(\mathbf{B})$. Note that, by Corollary 4.10, we have

$$\mathbf{A} \leqslant \mathbf{B} \iff \mathbf{A} \in \mathbb{IS}(\mathbf{B}).$$

We will show $|\Lambda(\text{bi-GD})| = 2^{\aleph_0}$ by proving that there exists a countably infinite $\leqslant$-antichain $\Omega \subseteq \text{bi-GA}_{SI}^{<\omega}$ (that is, the elements of Ω are pairwise $\leqslant$-incomparable). That the existence of Ω suffices to establish the desired equality follows easily from the next proposition, as we shall see in a moment.

Proposition 4.13. *Let $\Omega \subseteq \text{bi-GA}_{SI}^{<\omega}$ be a countably infinite $\leqslant$-antichain. If $\Omega_1, \Omega_2 \in \mathcal{P}(\Omega)$ are distinct, then*

$$\text{bi-GD} + \mathcal{J}(\Omega_1) \neq \text{bi-GD} + \mathcal{J}(\Omega_2),$$

where $\mathcal{J}(\Omega_i) := \{\mathcal{J}(\mathbf{A}) \colon \mathbf{A} \in \Omega_i\}$.

Proof. Without loss of generality, suppose that there exists $\mathbf{B} \in \Omega_1 \smallsetminus \Omega_2$. Since $\mathbf{B} \not\models \mathcal{J}(\mathbf{B})$, it is clear that $\mathbf{B} \not\models \text{bi-GD} + \mathcal{J}(\Omega_1)$. On the other hand, if $\mathbf{B} \not\models \text{bi-GD} + \mathcal{J}(\Omega_2)$ then there is $\mathbf{A} \in \Omega_2$ such that $\mathbf{B} \not\models \mathcal{J}(\mathbf{A})$. By the Jankov Lemma, it follows $\mathbf{A} \leqslant \mathbf{B}$. But this is a contradiction, since $\mathbf{A}$ and $\mathbf{B}$ are in Ω, an $\leqslant$-antichain. Therefore, $\mathbf{B} \models \text{bi-GD} + \mathcal{J}(\Omega_2)$, and we conclude

$$\text{bi-GD} + \mathcal{J}(\Omega_1) \neq \text{bi-GD} + \mathcal{J}(\Omega_2).$$ □

Suppose that we have Ω satisfying the conditions of the previous proposition. As our language is countable, we know that there are at most continuum many extensions of bi-GD, that is, $|\Lambda(\text{bi-GD})| \leqslant 2^{\aleph_0}$. But we just proved that distinct subsets of the countably infinite $\leqslant$-antichain Ω give rise to distinct extensions of bi-GD, hence it follows $|\mathcal{P}(\Omega)| = 2^{\aleph_0} \leqslant |\Lambda(\text{bi-GD})|$. Therefore, we get the desired equality.

We end this discussion by noting that we can use the bi-Esakia duality and Example 2.13 to translate the partial order $\leqslant$ defined above into one on the class of finite co-trees: $\mathcal{X} \leqslant \mathcal{Y}$ iff $\mathcal{X}$ is a bi-p-morphic image of $\mathcal{Y}$. It is now clear that to find our desired $\leqslant$-antichain of finite SI bi-Gödel algebras, it suffices to find a countably infinite $\leqslant$-antichain of finite co-trees. In order to do this, we rely on the following observation. Recall that $x \prec y$ denotes that y is an immediate successor of x.

Lemma 4.14. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a bi-p-morphism between co-trees. If $x \prec y \in \mathcal{X}$, then either $f(x) = f(y)$ or $f(x) \prec f(y)$.*

Proof. Assume $x \prec y$. As bi-p-morphisms are order preserving, $x \prec y$ entails $f(x) \leqslant f(y)$. If $f(x) = f(y)$ we are done, so let us suppose that $f(x) < f(y)$. Suppose as well that $f(x) \leqslant u \leqslant f(y)$, for some $u \in \mathcal{Y}$. By the up condition (see the Definition 2.9), there exists $z \in {\uparrow}x$ satisfying $f(z) = u$. Notice that ${\uparrow}x = \{x\} \uplus {\uparrow}y$, since $x \prec y$ and the principal upsets of $\mathcal{X}$ are chains. If $z = x$, then $f(x) = f(z) = u$. If $z \in {\uparrow}y$, then $f(y) \leqslant f(z) = u \leqslant f(y)$, and thus $f(y) = u$. We conclude $f(x) \prec f(y)$. □

Let $\mathcal{T} := \{T_n \colon n \in \omega\}$ be the family of finite co-trees depicted in Figure 3. The next result proves that this is an $\leqslant$-antichain of finite co-trees.[‡] Its proof makes extensive use, often without reference, of Proposition 2.10 (in particular, of the fact that if $f \colon \mathcal{X} \to \mathcal{Y}$ is a bi-p-morphism between co-trees, then minimal points must be mapped to minimal points, and the co-root of $\mathcal{X}$ must be mapped to the co-root of $\mathcal{Y}$) and of Lemma 4.14.

Proposition 4.15 (Hodkinson)**.** *The set $\mathcal{T} := \{T_n \colon n \in \omega\}$ is a countably infinite $\leqslant$-antichain of finite co-trees.*

Proof. Firstly, let us note that for $n \in \mathbb{Z}^+$, if $m < n$ then $|T_m| < |T_n|$, hence clearly $T_n \nleqslant T_m$. Moreover, if $f \colon T_n \to T_0$ is a bi-p-morphism, then the co-root u_1 of T_n must be mapped to the co-root of T_0. By Lemma 4.14, this entails that both nontrivial predecessors of v_1, which are

[‡]This $\leqslant$-antichain was constructed by Ian Hodkinson (personal communication). We use a different method to show that this is in fact an $\leqslant$-antichain

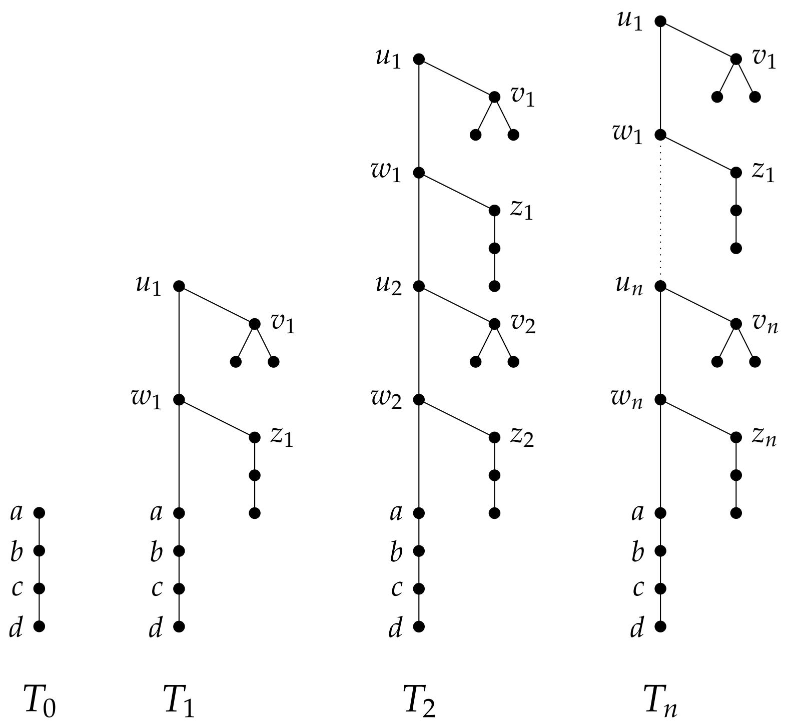

FIGURE 3. The co-trees T_0, T_1, T_2, and T_n.

minimal, must be mapped to nonminimal points, contradicting Proposition 2.10. So f cannot exist, and thus $T_0 \not\leqslant T_n$.

It remains to show $T_m \not\leqslant T_n$, for all $m \in \mathbb{Z}^+$ such that $m < n$. We denote the elements of T_m by u'_1, v'_1, a', b', etc., and those of T_n by u_1, v_1, a, b, etc. Set $U'_m := \{u'_j \in T_m : j \leqslant m\}$ and $U_n := \{u_i \in T_n : i \leqslant n\}$. In a similar way, define the sets V'_m and V_n, W'_m and W_n, Z'_m and Z_n. Suppose, with a view to contradiction, that there exists a surjective bi-p-morphism $f : T_n \twoheadrightarrow T_m$, and take $i \leqslant n$ and $j \leqslant m$.

Before we begin a proof by induction, let us establish a series of useful equalities involving these newly defined sets. First we show that

$$f[V_n] \cap (U'_m \cup W'_m \cup Z'_m \cup \{a'\}) = \varnothing. \tag{2}$$

Since we have $|{\downarrow} v_i| = 3 < Min\{|{\downarrow} u'_j|, |{\downarrow} w'_j|, |{\downarrow} a'|\}$, the down condition of bi-p-morphisms (see Definition 2.9) already implies $f(v_i) \notin U'_m \cup W'_m \cup \{a'\}$. To see that $f(v_i) \notin Z'_m$, suppose otherwise. By Lemma 4.14, this implies that both nontrivial predecessors of v_i, which are minimal, must be mapped to nonminimal points, a contradiction. This establishes the desired equality. We now prove

$$f[Z_n] \cap (U'_m \cup W'_m \cup V'_m \cup \{a'\}) = \varnothing. \tag{3}$$

Notice that $|{\downarrow} z_i| = 3 < Min\{|{\downarrow} u'_j|, |{\downarrow} w'_j|, |{\downarrow} a'|\}$ holds, so the down condition forces $f(z_i) \notin U'_m \cup W'_m \cup \{a'\}$. Moreover, $f(z_i) = v'_j$ would imply that $f[{\downarrow} z_i] = {\downarrow} v'_j$, again by the down condition, which clearly contradicts the fact that f is order preserving. Finally, to see that

$$f[U_n] \cap W'_m = \varnothing = f[W_n] \cap U'_m, \tag{4}$$

suppose $f(u_i) = w'_j$. It follows from Lemma 4.14 that $f(v_i) \in \{w'_j, z'_j, \alpha(j)\}$, where $\alpha(j) := u'_{j+1}$ if $j < m$, and $\alpha(j) := a'$ otherwise. Since this contradicts (2), we conclude $f[U_n] \cap W'_m = \varnothing$. Suppose now that $f(w_i) = u'_j$. By Lemma 4.14, $f(z_i) \in \{u'_j, v'_j, w'_j\}$, so (3) yields a contradiction. Thus, we have $f[W_n] \cap U'_m = \varnothing$.

We now prove by strong induction that for all $0 < i \leqslant m$,

$$f(u_i) = u'_i \qquad f(v_i) = v'_i \qquad f(w_i) = w'_i \qquad f(z_i) = z'_i.$$

Let $i = 1$. By Proposition 2.10, the co-root u_1 of T_n must be mapped to the co-root u'_1 of T_m. In other words, $f(u_1) = u'_1$. By Lemma 4.14, $f(v_1) \in \{u'_1, v'_1, w'_1\}$ follows. So (2) now forces $f(v_1) = v'_1$. To see that $f(w_1) = w'_1$, notice that Lemma 4.14 and (4) already imply $f(w_1) \in \{w'_1, v'_1\}$, so it suffices to show that $f(w_1) \neq v'_1$. Suppose otherwise, i.e., that $f(w_1) = v'_1$. By the definition of a bi-p-morphism, we must have $f[{\downarrow} w_1] = {\downarrow} v'_1$. Since the aforementioned equalities $f(u_1) = u_1$ and $f(v_1) = v'_1$ entail $f[\{u_1\} \cup {\downarrow} v_1] = \{u'_1\} \cup {\downarrow} v'_1$, it now follows from the structures of both T_n and T_m that

$$f[T_n] = f[\{u_1\} \cup {\downarrow} v_1 \cup {\downarrow} w_1] = f[\{u_1\} \cup {\downarrow} v_1] \cup f[{\downarrow} w_1] = \{u'_1\} \cup {\downarrow} v'_1 \subsetneq T_m.$$

But this contradicts our assumption that f is surjective, hence we conclude $f(w_1) = w'_1$. Finally, Lemma 4.14 now implies $f(z_1) \in \{w'_1, z'_1, \alpha(1)\}$ (where $\alpha(1) = u'_2$ if $1 < m$, and $\alpha(1) = a'$ otherwise), so (3) yields $f(z_1) = z'_1$, as desired.

If $m = 1$, we are done with our proof by induction, so let us assume otherwise. Then consider $1 < i \leqslant m$ and suppose that for all $j \in \mathbb{Z}^+$ such that $j < i \leqslant m$, the induction hypothesis holds true.

Since this entails $f(w_{i-1}) = w'_{i-1}$, Lemma 4.14, together with $f[U_n] \cap W'_m = \varnothing$, forces $f(u_i) \in \{u'_i, z'_{i-1}\}$. Let us show that u_i must be mapped to u'_i, by proving that $f(u_i) = z'_{i-1}$ cannot happen. For suppose otherwise. Then $f[{\downarrow} u_i] = {\downarrow} z'_{i-1}$. By simply looking at the poset structure of T_n and T_m, we see that this equality implies $a' \notin f[{\downarrow} u_i]$. But our induction hypothesis and the definition of a bi-p-morphism ensure that no element in $T_n \smallsetminus {\downarrow} u_i$ is mapped to a'. It follows $a' \notin f[{\downarrow} u_i] \cup f[T_n \smallsetminus {\downarrow} u_i] = f[T_n]$, contradicting our assumption that f is surjective. Thus, we must have $f(u_i) = u'_i$.

That $f(w_i) = w'_i$ is proved in a very similar way: $f(u_i) = u'_i$, Lemma 4.14, and (4) imply $f(w_i) \in \{v'_i, w'_i\}$, and $f(w_i) = v'_i$ cannot happen, since this would force $a' \notin f[{\downarrow} w_i] = {\downarrow} v'_i$, contradicting the surjectivity of f.

With the equalities $f(u_i) = u'_i$ and $f(w_i) = w'_i$ now established, we can use Lemma 4.14 together with condition (2) to show that $f(v_i) = v'_i$, and with condition (3) to show that $f(z_i) = z'_i$. This finishes our proof by induction.

As a consequence, we now know $f(w_m) = w'_m$. Since $f[U_n] \cap W'_m = \varnothing$ by (4), Lemma 4.14 entails $f(u_{m+1}) = \{a', z'_m\}$. The same argument used above (to show that $f(u_i) \neq z'_{i-1}$, for all $i \leqslant m$) ensures $f(u_{m+1}) \neq z'_m$. Thus, we must have $f(u_{m+1}) = a'$. But, by Lemma 4.14, this implies that both nontrivial predecessors of v_{m+1}, which are minimal, must be mapped to nonminimal points, a contradiction. Therefore, f cannot exist, and we showed $T_m \nleqslant T_n$, as desired. □

By our previous discussion, the following theorem follows immediately.

Theorem 4.16. *The cardinality of the lattice* $\Lambda(\mathsf{bi\text{-}GD})$ *is* $2^{\aleph_0}$.

4.3. **Subframe formulas.** Let $\mathbf{A} \in \mathsf{bi\text{-}GA}^{<\omega}_{SI}$ and introduce, for each $a \in A$, a fresh propositional variable $p_a \in Prop$. Let Γ be the formula describing the algebraic structure of the $(\vee, \leftarrow)$-reduct of $\mathbf{A}$, that is,

$$\Gamma := \bigwedge \{p_{a \vee b} \leftrightarrow (p_a \vee p_b) \colon (a,b) \in A^2\} \wedge \bigwedge \{p_{a \leftarrow b} \leftrightarrow (p_a \leftarrow p_b) \colon (a,b) \in A^2\}.$$

We define the *subframe formula* of $\mathbf{A}$ by

$$\beta(\mathbf{A}) := \neg \sim \Gamma \to \neg \bigwedge \{p_a \leftarrow p_b \colon (a,b) \in A^2 \text{ and } a \nleqslant b\}.$$

In order to state the analogue of the Stable Jankov Lemma for subframe formulas, we need to introduce the notion of a $(\vee, \leftarrow)$-*homomorphism* between co-Heyting algebras, i.e., a map $f \colon \mathbf{A} \to \mathbf{B}$ that preserves both $\vee$ and $\leftarrow$. Notice that any such map must always preserve 0, since the equation $x \leftarrow x \approx 0$ is valid on all co-Heyting algebras, and therefore

$$f(0) = f(a \leftarrow a) = f(a) \leftarrow f(a) = 0.$$

If f is moreover injective, then it is called a $(\vee, \leftarrow)$-*embedding*, denoted by $f \colon \mathbf{A} \hookrightarrow \mathbf{B}$.

As before, we use the facts stated in Lemma 4.2 without further reference in the following proof.

Lemma 4.17 (Subframe Jankov Lemma)**.** *(Subframe Jankov Lemma) Let* $\mathbf{B} \in \mathsf{bi\text{-}GA}$*. If* $\mathbf{A} \in \mathsf{bi\text{-}GA}^{<\omega}_{SI}$*, then* $\mathbf{B} \not\models \beta(\mathbf{A})$ *iff there exists a* $(\vee, \leftarrow)$*-embedding* $h\colon \mathbf{A} \hookrightarrow \mathbf{C}$*, for some* $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$*.*

Proof. To show that the left to right implication holds, we can apply the argument used to prove the same direction of the Stable Jankov Lemma 4.3, but since now the formula Γ only describes the algebraic structure of the $(\vee, \leftarrow)$-reduct of $\mathbf{A}$, the injective map $h\colon A \to C$ is simply a $(\vee, \leftarrow)$-embedding, as desired.

Conversely, suppose there exists a $(\vee, \leftarrow)$-embedding $h\colon \mathbf{A} \hookrightarrow \mathbf{C}$, for some $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$. Let $v\colon Prop \to C$ be a valuation on $\mathbf{C}$ satisfying $v(p_a) = h(a)$, for all $a \in A$. We show that $\mathbf{C}$ refutes $\beta(\mathbf{A})$ via v, which is a sufficient condition for $\mathbf{B} \not\models \beta(\mathbf{A})$. Firstly, note that since h is a $(\vee, \leftarrow)$-homomorphism, we have for that all $a, b \in A$,

$$v(p_{a\vee b}) = h(a \vee b) = h(a) \vee h(b) = v(p_a) \vee v(p_b),$$

hence $v(p_{a\vee b} \leftrightarrow p_a \vee p_b) = 1$. Similarly, we have

$$v(p_{a\leftarrow b}) = h(a \leftarrow b) = h(a) \leftarrow h(b) = v(p_a) \leftarrow v(p_b),$$

and thus $v(p_{a\leftarrow b} \leftrightarrow p_a \leftarrow p_b) = 1$. By the equalities above, we see that $v(\Gamma) = 1$, and therefore $v(\neg\sim\Gamma) = \neg\sim v(\Gamma) = \neg\sim 1 = 1$. Now, if $a, b \in A$ are such that $a \nleq b$, i.e., $a \leftarrow b \neq 0$, then it follows that $0 \neq h(a \leftarrow b) = v(p_{a_\leftarrow b})$, since h is an injective map that preserves 0. This proves that $0 \notin \{v(p_a \leftarrow p_b) \colon a, b \in A \text{ and } a \nleq b\}$. As $\mathbf{C}$ is SI by assumption, $0_{\mathbf{C}}$ is $\wedge$-irreducible (see Theorem 3.7), and thus we obtain $\bigwedge\{v(p_a \leftarrow p_b) \colon a, b \in A \text{ and } a \nleq b\} \neq 0$. Equivalently,

$$\neg \bigwedge\{v(p_a \leftarrow p_b) \colon a, b \in A \text{ and } a \nleq b\} \neq 1,$$

and we conclude

$$v(\beta(\mathbf{A})) = 1 \to \neg \bigwedge\{v(p_a \leftarrow p_b) \colon a, b \in A \text{ and } a \nleq b\} \neq 1.$$ ☐

Next we introduce partial co-Esakia morphisms, which enable us to translate the Subframe Jankov Lemma into terms of bi-Esakia spaces.

Definition 4.18. Let $\mathcal{X}$ and $\mathcal{Y}$ be co-Esakia spaces. A partial map $f\colon \mathcal{X} \to \mathcal{Y}$ is a *partial co-Esakia morphism* if it satisfies the following conditions:

(i) $\forall x, z \in dom(f)\ \big(x \leqslant z \implies f(x) \leqslant f(z)\big)$;
(ii) $\forall x \in dom(f), \forall y \in Y\ \big(y \leqslant f(x) \implies \exists z \in {\downarrow} x\ (f(z) = y)\big)$;
(iii) $\forall x \in X\ \big(x \in dom(f) \iff \exists y \in Y\ (f[{\downarrow} x] = {\downarrow} y)\big)$;
(iv) $\forall x \in X\ \big(f[{\downarrow} x] \in Cl(\mathcal{Y})\big)$;
(v) $\forall U \in ClopUp(\mathcal{Y})\ \big({\uparrow} f^{-1}U \in ClopUp(\mathcal{X})\big)$.

Proposition 4.19. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be co-Heyting algebras, while* $\mathcal{X}$ *and* $\mathcal{Y}$ *are co-Esakia spaces.*

(i) *If* $f\colon \mathcal{X} \to \mathcal{Y}$ *is a partial co-Esakia morphism, then setting*

$$f^*(U) := {\uparrow} f^{-1}U,$$

for every $U \in ClopUp(\mathcal{Y})$*, yields a* $(\vee, \leftarrow)$*-homomorphism* $f^*\colon \mathcal{Y}^* \to \mathcal{X}^*$ *between co-Heyting algebras. If* f *is moreover surjective, then* f^* *is a* $(\vee, \leftarrow)$*-embedding.*

(ii) *If* $h\colon \mathbf{A} \to \mathbf{B}$ *is a* $(\vee, \leftarrow)$*-homomorphism between co-Heyting algebras, then setting*

$$dom(h_*) := \{x \in B_* \colon h^{-1}x \in A_*\} \text{ and } h_*(x) := h^{-1}x,$$

for everyx $\in dom(h_*)$*, yields a partial co-Esakia morphism* $h_*\colon \mathbf{B}_* \to \mathbf{A}_*$ *between co-Esakia spaces. If* h *is moreover a* $(\vee, \leftarrow)$*-embedding, then* h_* *is surjective.*

Proof. Both (i) and (ii) can be proven simply by order dualizing the proofs of the analogous results for partial Esakia morphisms and $(\wedge, \to)$-homomorphisms between Heyting algebras (see, e.g., [6]). ☐

Before we present the Dual Subframe Lemma and some of its equivalent conditions, we need one more definition and a subsequent lemma.

Definition 4.20. A map $f\colon \mathcal{X} \to \mathcal{Y}$ between posets is called an *order-embedding* if it is *order-invariant*, that is, if

$$w \leqslant v \iff f(w) \leqslant f(v),$$

for all $w, v \in \mathcal{X}$. In this case, we say that $\mathcal{X}$ *order-embeds into* $\mathcal{Y}$ (via f), and denote this by $f\colon \mathcal{X} \hookrightarrow \mathcal{Y}$.

Remark 4.21. It is clear by definition that $\mathcal{X}$ order-embeds into $\mathcal{Y}$ iff $\mathcal{X}$ can be viewed as a subposet of $\mathcal{Y}$.

Lemma 4.22. *If $\mathcal{Y}$ is a finite co-tree and $\mathcal{X}$ a co-Esakia space, then $\mathcal{Y}$ order-embeds into $\mathcal{X}$ iff there exists a surjective partial co-Esakia morphism $f\colon \mathcal{X} \twoheadrightarrow \mathcal{Y}$.*

Proof. This is exactly the order-dual version of [5, Thm. 3.6] (and thus we omit the proof). □

We are finally ready to translate the Subframe Jankov Lemma into the language of bi-Esakia co-forests.

Lemma 4.23 (Dual Subframe Jankov Lemma)**.** *(Dual Subframe Jankov Lemma) If* $\mathbf{B} \in \mathsf{bi\text{-}GA}$ *and* $\mathbf{A} \in \mathsf{bi\text{-}GA}^{<\omega}_{SI}$*, then the following conditions are equivalent:*
1. $\mathbf{B} \not\models \beta(\mathbf{A})$*;*
2. *there exists a* $(\vee, \leftarrow)$*-embedding* $h\colon \mathbf{A} \hookrightarrow \mathbf{C}$*, for some* $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$*;*
3. *there exists a surjective partial co-Esakia morphism* $f\colon \mathbf{C}_* \twoheadrightarrow \mathbf{A}_*$*, for some* $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$*;*
4. $\mathbf{A}_*$ *order-embeds into* $\mathbf{C}_*$*, for some* $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$*;*
5. $\mathbf{A}_*$ *order-embeds into* $\mathbf{B}_*$*;*
6. *there exists a surjective partial co-Esakia morphism* $f\colon \mathbf{B}_* \to \mathbf{A}_*$*;*
7. *there exists a* $(\vee, \leftarrow)$*-embedding* $h\colon \mathbf{A} \to \mathbf{B}$*.*

Proof. The equivalence (1) $\iff$ (2) is just the Subframe Jankov Lemma 4.17, while (2) $\iff$ (3) follows immediately from the duality between $(\vee, \leftarrow)$-homomorphisms of co-Heyting algebras and partial co-Esakia morphisms stated in Proposition 4.19. Notice that this result also yields (6) $\iff$ (7). As an immediate consequence of Lemma 4.22, we have that both (3) $\iff$ (4) and (5) $\iff$ (6) hold true.

Finally, to see that (4) $\implies$ (5), let $\mathbf{C} \in \mathbb{H}(\mathbf{B})_{SI}$ and note that if $\mathbf{A}_*$ order-embeds into a bi-generated subframe of $\mathbf{B}_*$, such as $\mathbf{C}_*$, then clearly $\mathbf{A}_*$ order-embeds into $\mathbf{B}_*$.

Conversely, suppose that $\mathbf{A}_*$ order-embeds into $\mathbf{B}_*$. Since $\mathbf{A}$ is nontrivial, $\mathbf{A}_*$ is nonempty, hence so is $\mathbf{B}_*$. Therefore, we can write $B_* = \biguplus_{i \in I} T_i$ as a nonempty disjoint union of maximal (with respect to inclusion) co-trees. By the definition of an order-embedding, it is clear that the co-tree $\mathbf{A}_*$ is mapped to a single co-tree T_i. Since we can view T_i as a bi-generated subframe of $\mathbf{B}_*$, by equipping T_i with the subspace topology, we conclude by duality that (4) holds. Thus, we proved (5) $\implies$ (4). □

Before we present some applications of subframe formulas, we need a few definitions. Let $\mathcal{X}$ be a co-tree and $n \in \mathbb{Z}^+$. If $\mathcal{X}$ has a chain with n elements, and all chains of $\mathcal{X}$ have at most n elements, we say that $\mathcal{X}$ has *depth* n, and write $dp(\mathcal{X}) = n$. Otherwise, we say that $\mathcal{X}$ has *infinite depth*.

Furthermore, if $\mathcal{X}$ has an antichain (i.e., a subposet whose elements are pairwise incomparable) with n elements, and all antichains in $\mathcal{X}$ have at most n elements, we say that $\mathcal{X}$ has *width* n, and write $wd(\mathcal{X}) = n$. Otherwise, we say that $\mathcal{X}$ has *infinite width*.

We prove that if $n \in \mathbb{Z}^+$, then the bi-intermediate logic of co-trees of depth (respectively, width) less than n can be axiomatized by a single subframe formula.

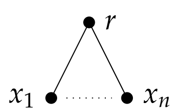

FIGURE 4. The n-co-fork $\mathfrak{F}_n$.

Proposition 4.24. *Let $n \in \mathbb{Z}^+$, $\mathfrak{L}_n$ be the n-chain, and $\mathfrak{F}_n$ be the n-co-fork (see Figure 4). If $\mathcal{X}$ is bi-Esakia co-tree, then:*

(i) $\mathcal{X} \models \beta(\mathfrak{L}_n^*) \iff dp(\mathcal{X}) < n$. *Equivalently,* bi-GD $+\beta(\mathfrak{L}_n^*)$ *is the bi-intermediate logic of co-trees of depth less that n;*
(ii) $\mathcal{X} \models \beta(\mathfrak{F}_n^*) \iff wd(\mathcal{X}) < n$. *Equivalently,* bi-GD $+\beta(\mathfrak{F}_n^*)$ *is the bi-intermediate logic of co-trees of width less that n.*

Proof. The desired equivalences follow immediately from Lemma 4.23, noting that by the definition of an order-embedding, we clearly have that $\mathfrak{L}_n$ does not order-embed into $\mathcal{X}$ iff $dp(\mathcal{X}) < n$, while $\mathfrak{F}_n$ does not order-embed into $\mathcal{X}$ iff $wd(\mathcal{X}) < n$.

The last part of both statements are now an immediate consequence of the algebraic completeness of bi-GD and the bi-Esakia duality. □

As a corollary, we obtain a different axiomatization of the bi-intuitionistic linear calculus

$$\text{bi-LC} = \text{bi-IPC} + (p \to q) \vee (q \to p) + \neg[(q \leftarrow p) \wedge (p \leftarrow q)].$$

Corollary 4.25. *The bi-intermediate logic* bi-LC *of chains coincides with* bi-GD $+\beta(\mathfrak{F}_2^*)$.

Proof. Let **A** be a bi-Heyting algebra. By Theorem 3.10, we know that $\mathbf{A} \in (\mathsf{V}_{\text{bi-LC}})_{SI}$ iff $\mathbf{A}_*$ is a nonempty bi-Esakia chain. The latter condition is clearly equivalent to $\mathbf{A}_*$ being a bi-Esakia co-tree of width 1, which in turn is equivalent to **A** being an SI bi-Heyting algebra that validates bi-GD $+\beta(\mathfrak{F}_2^*)$, by Proposition 4.24.(ii). □

5. Locally tabular extensions of bi-GD

A bi-intermediate logic L is said to be *locally tabular* if for every positive integer n, there are only finitely many non-L-equivalent formulas built from the propositional variables $p_1, \dots, p_n$. Equivalently, when V_L is *locally finite* (i.e., every finitely generated algebra in V_L is finite).

In this section we present a characterization of locally tabular extensions of bi-GD: they are exactly those which contain at least one of the Jankov formulas associated with the *finite combs* (a particular class of bi-Esakia co-trees defined below). If is an immediate consequence of this criterion that the logic of the finite combs is the only pre-locally tabular extension of bi-GD (recall that a logic is said to be *pre-locally tabular* if it is not locally tabular, but all of its proper extensions are so).

For each positive integer n, we define the *n-comb* $\mathfrak{C}_n := (C_n, \leqslant)$ as the finite bi-Esakia co-tree depicted in Figure 5. Our aim in this section is to prove the following criterion:

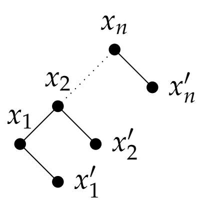

FIGURE 5. The n-comb $\mathfrak{C}_n$.

Theorem 5.1. *If $L \in \Lambda$(bi-GD), then L is locally tabular iff $L \vdash \mathcal{J}(\mathfrak{C}_n^*)$, for some $n \in \mathbb{Z}^+$.*

The above theorem is proved in three steps:

- Step 1: we prove that for all $n \in \mathbb{Z}^+$ and all $\mathbf{A} \in$ bi-GA, $\mathbf{A} \not\models \mathcal{J}(\mathfrak{C}_n^*)$ iff $\mathbf{A} \not\models \beta(\mathfrak{C}_n^*)$;
- Step 2: we show that a variety $\mathsf{V} \subseteq$ bi-GA containing all the algebraic duals of the finite combs cannot be locally finite;
- Step 3: we establish the existence of a natural bound for the size of m-generated SI bi-Gödel algebras whose bi-Esakia duals do not admit the n-comb $\mathfrak{C}_n$ as a subposet.

The left to right implication of Step 1 is a straightforward consequence of the defining properties of the Jankov and subframe formulas established in the previous section. As for the reverse implication, we prove a sufficient condition for it to hold, namely, "if $\mathfrak{C}_n$ order-embeds into a bi-Esakia co-tree $\mathcal{X}$, then there exists a surjective bi-Esakia morphism $f\colon \mathcal{X} \twoheadrightarrow \mathfrak{C}_n$". The construction of the map f is very lengthy and technical, requiring a heavily combinatorial and careful manipulation of the clopens of $\mathcal{X}$ that will be the f-pre-images of the points in $\mathfrak{C}_n$.

Step 2 is arguably the easiest step. It can be shown by proving that $\mathfrak{C}_n^*$ is 1-generated as a bi-Heyting algebra, for every $n \in \mathbb{Z}^+$. Since the finite combs are arbitrarily large bi-Esakia co-trees, the assumption that their algebraic duals are all contained in V now yields that there are arbitrarily large 1-generated algebras in V. Therefore, the 1-generated free V-algebra must be infinite, and thus V cannot be locally finite.

Step 3 makes extensive use of the Coloring Theorem 2.19. We first show that in a bi-Esakia co-forest, two particular equivalence relations must always be bi-E-partitions. By assuming that an SI bi-Gödel algebra $\mathbf{A}$ is m-generated and $\mathbf{A}_*$ does not admit $\mathfrak{C}_n$ as a subposet, we can use the definitions of the aforementioned bi-E-partitions, together with the Coloring Theorem, to not only derive a bound for the depth of $\mathbf{A}_*$, but also to establish the existence of a bound for the cardinality of $\mathbf{A}$.

5.1. **Step 1.** Recall our convention that the "arrow operators" of generating upsets and downsets bind stronger than other set theoretic operations (e.g., expressions of the form ${\uparrow}U \smallsetminus V$ and ${\downarrow}U \cap V$ are to be read as $({\uparrow}U) \smallsetminus V$ and $({\downarrow}U) \cap V$, respectively). We will start by proving two elementary lemmas about posets. A subset W of a poset is said to be *convex* if $W = {\downarrow}W \cap {\uparrow}W$. Since we always have $W \subseteq {\downarrow}W \cap {\uparrow}W$, this is equivalent to demanding that ${\downarrow}W \cap {\uparrow}W \subseteq W$.

Lemma 5.2. *Let $\mathcal{X} = (X, \leqslant)$ be a poset and $W, V \subseteq X$. If W is convex, then:*
(i) *$V \subseteq {\downarrow}W \smallsetminus W$ implies $V \cap {\uparrow}W = \varnothing$,*
(ii) *$V \subseteq {\uparrow}W \smallsetminus W$ implies $V \cap {\downarrow}W = \varnothing$.*

Proof. (i) Suppose, with a view towards contradiction, that $x \in V \cap {\uparrow}W$. As $V \subseteq {\downarrow}W$, it follows that $x \in {\downarrow}W \cap {\uparrow}W$, i.e., that $x \in W$, since W is convex. But then $x \in V \cap W$, contradicting the definition of V. Condition (ii) is proved analogously. ☐

Lemma 5.3. *Let $\mathcal{X} = (X, \leqslant)$ be a poset and $W, V \subseteq X$. The following equivalences hold true:*
$$ {\downarrow}W \cap {\uparrow}{\downarrow}V = \varnothing \iff {\downarrow}W \cap {\downarrow}V = \varnothing \iff {\uparrow}{\downarrow}W \cap {\downarrow}V = \varnothing. $$

Proof. By symmetry, it suffices to show the first equivalence, whose right to left implication follows immediately from the inclusion ${\downarrow}V \subseteq {\uparrow}{\downarrow}V$. We prove the reverse implication by contraposition. Suppose $x \in {\downarrow}W \cap {\uparrow}{\downarrow}V$, i.e., that $x \in {\downarrow}W$ and $y \leqslant x$, for some $y \in {\downarrow}V$. Then we clearly have $y \in {\downarrow}W \cap {\downarrow}V$, as desired. ☐

For the remainder of this subsection, we fix an arbitrary positive integer n and a bi-Esakia co-tree $\mathcal{X}$ that admits $\mathfrak{C}_n$ as a subposet. Without loss of generality, we suppose that the co-root of $\mathfrak{C}_n$, x_n, is also the co-root of $\mathcal{X}$. The bulk of this subsection will be dedicated to proving the next result. Notice the use of the symbol $\uplus$ to denote the union of sets which are pairwise disjoint.

Proposition 5.4. *There are clopens $X_1, \ldots, X_n, X_1', \ldots, X_n' \subseteq \mathcal{X}$ satisfying:*
- (IH.i) *For all $i \leqslant n$, we have that X_i is a convex set containing x_i, that X_i' is a downset containing x_i', and that $X_i \cap {\uparrow}{\downarrow}x_j' = \varnothing$ for all j such that $i < j \leqslant n$;*
- (IH.ii) ${\downarrow}X_1 = X_1 \uplus X_1'$ *and* ${\downarrow}X_i = X_i \uplus {\downarrow}X_{i-1} \uplus X_i'$*, for all $1 < i \leqslant n$;*
- (IH.iii) ${\downarrow}X_{i-1} \cap {\uparrow}X_i' = \varnothing$*, for all $1 < i \leqslant n$;*
- (IH.iv) ${\uparrow}X_1' = {\uparrow}X_1 \uplus X_1'$ *and* ${\uparrow}X_i = {\uparrow}X_{i-1} \cap {\uparrow}X_i'$*, for all $1 < i \leqslant n$.*

We will prove this by induction on $m \leqslant n$. Let us assume $m = 1$. By the properties of bi-Esakia spaces,
$$ Y := {\uparrow}{\downarrow}\{x_2', \ldots, x_n'\} \cup {\downarrow}x_1' $$

is a closed set. Moreover, the structure of the n-comb and that of co-trees forces $x_1 \notin Y$. To see this, notice that if $n = 1$ then clearly $x_1 \notin Y = {\downarrow}x_1'$, and if $1 < i \leqslant n$, then since $\mathcal{X}$ is a co-tree, $x_1 \in {\uparrow}{\downarrow}x_i'$ would imply either $x_1 \leqslant x_i'$ or $x_i' \leqslant x_1$, contradicting the order of $\mathfrak{C}_n$. It follows that x_1 is contained in the open set Y^c, and as bi-Esakia spaces are 0-dimensional, there exists a clopen V_1 satisfying $x_1 \in V_1 \subseteq Y^c$. Again by the properties of bi-Esakia spaces, the following sets are all clopens:

$$U_1 := {\downarrow}V_1 \cap {\uparrow}V_1 \qquad U_1' := {\downarrow}U_1 \smallsetminus U_1 \qquad W_1 := {\uparrow}U_1' \cap U_1 \qquad W_1' := {\downarrow}W_1 \smallsetminus W_1.$$

By the above definitions, it is clear that ${\downarrow}W_1 = W_1 \uplus {\downarrow}W_1'$, $x_1 \in U_1$ and that $x_1' \notin U_1$. As $x_1' \leqslant x_1$, it follows $x_1' \in U_1' = {\downarrow}U_1 \smallsetminus U_1$, hence $x_1 \in {\uparrow}U_1' \cap U_1 = W_1$. Consequently, we have $x_1' \in W_1'$. Now, to see that ${\downarrow}W_1 \cap {\uparrow}{\downarrow}x_j' = \emptyset$, for all j such that $1 < j \leqslant n$, just note that if this intersection was nonempty, then by our construction of W_1 it would follow that $V_1 \cap {\uparrow}{\downarrow}x_j' \neq \emptyset$, contradicting the definition of V_1.

We now prove that ${\downarrow}W_1 \cap {\uparrow}W_1 \subseteq W_1$ i.e., that W_1 is convex. Suppose $x \in {\downarrow}W_1 \cap {\uparrow}W_1$, so there are $y, z \in W_1 = {\uparrow}U_1' \cap U_1$ satisfying $y \leqslant x \leqslant z$. As $y \in {\uparrow}U_1'$, $y \leqslant x$ entails $x \in {\uparrow}U_1'$, so to establish $x \in W_1$ we only need to prove $x \in U_1$. By definition, $U_1 = {\downarrow}V_1 \cap {\uparrow}V_1$, so $y, z \in U_1$ and $y \leqslant x \leqslant z$ yield $x \in {\downarrow}V_1 \cap {\uparrow}V_1 = U_1$, as desired.

Using $W_1 = {\downarrow}W_1 \cap {\uparrow}W_1$ and the definition of W_1', it is easy to see that W_1' is a downset, i.e., that $W_1' = {\downarrow}W_1'$. Consequently, we can write

$${\downarrow}W_1 = W_1 \uplus {\downarrow}W_1' = W_1 \uplus W_1',$$

by above.

Finally, we prove that ${\uparrow}W_1' = {\uparrow}W_1 \uplus W_1'$. To see that the sets on the right side of the equality are disjoint, just notice that $W_1 \cap W_1' = \emptyset$ by definition of these clopens, hence the fact that W_1 is convex entails ${\uparrow}W_1 \cap W_1' = \emptyset$, by Lemma 5.2. To establish the inclusion ${\uparrow}W_1' \subseteq {\uparrow}W_1 \uplus W_1'$, suppose $x \in {\uparrow}W_1' \smallsetminus W_1'$, i.e., that there is $y \in W_1'$ such that $y \leqslant x$ and $x \notin W_1'$. Since $W_1' \subseteq {\downarrow}W_1$ by definition of W_1', there exists $z \in W_1$ satisfying $y \leqslant z$. As $\mathcal{X}$ is a co-tree, we have $y \leqslant x \leqslant z$ or $y \leqslant z \leqslant x$. If $y \leqslant x \leqslant z \in W_1$, then $x \in {\downarrow}W_1 = W_1 \uplus W_1'$ by above, so $x \notin W_1'$ yields $x \in W_1 \subseteq {\uparrow}W_1$. If $y \leqslant z \leqslant x$, then $x \in {\uparrow}W_1$ since $z \in W_1$. We conclude ${\uparrow}W_1' \subseteq {\uparrow}W_1 \uplus W_1'$.

To prove that the reverse inclusion holds, it suffices to show that ${\uparrow}W_1 \subseteq {\uparrow}W_1'$, since $W_1' \subseteq {\uparrow}W_1'$. To this end, suppose $x \in {\uparrow}W_1$, i.e., that x lies above some $y \in W_1 = {\uparrow}U_1' \cap U_1$. It follows that there is $z \in U_1'$ such that $z \leqslant y \leqslant x$. Notice that $U_1' = {\downarrow}U_1 \smallsetminus U_1$ entails $z \notin U_1$, hence $z \notin W_1 \subseteq U_1$. From $z \leqslant y \in W_1$, we can now infer $z \in {\downarrow}W_1 \smallsetminus W_1$, i.e., that $z \in W_1'$. As $z \leqslant x$, it now follows that $x \in {\uparrow}W_1'$, as desired.

If $n = 1$, just set $X_1 := W_1$ and $X_1' := W_1'$, and we are done with the proof of Proposition 5.4. So, let us now assume that $n > 1$. Before continuing, we derive some consequences of the conditions in our induction hypothesis. They will not only help us in our proof by induction, but they are essential for proving the main result of this subsection.

Lemma 5.5. *Clopens $X_1, \dots, X_m, X_1', \dots, X_m' \subseteq \mathcal{X}$ that satisfy conditions* (IH.i-iv) *also satisfy the following conditions:*

1. ${\downarrow}X_1 \subseteq {\downarrow}X_2 \subseteq \cdots \subseteq {\downarrow}X_m$ *and* $X_i' \subseteq {\downarrow}X_i$*, for all* $i \leqslant m$*;*
2. ${\uparrow}X_m \subseteq \cdots \subseteq {\uparrow}X_2 \subseteq {\uparrow}X_1$ *and* ${\uparrow}X_i \subseteq {\uparrow}X_i'$*, for all* $i \leqslant m$*;*
3. ${\downarrow}X_i \cap {\uparrow}X_j' = {\uparrow}{\downarrow}X_i \cap X_j' = {\downarrow}X_i \cap {\uparrow}X_j = \emptyset$*, for all* $i < j \leqslant m$*;*
4. ${\uparrow}X_{i-1} \smallsetminus X_{i-1} = {\uparrow}X_i = {\uparrow}X_i' \smallsetminus X_i'$*, for all* $i \leqslant m$*;*
5. *The sets* $X_1, \dots, X_m, X_1', \dots, X_m'$ *are pairwise disjoint.*

Proof. Condition (1) is immediate from (IH.ii), while (2) clearly follows from (IH.iv).

(3) Suppose $i < j \leqslant m$. Since ${\downarrow}X_i \subseteq {\downarrow}X_{j-1}$ by condition (1), then (IH.iii) implies

$$\emptyset = {\downarrow}X_i \cap {\uparrow}X_j' = {\downarrow}X_i \cap {\uparrow}{\downarrow}X_j',$$

where the last equality follows from the fact that X_j' is a downset, by (IH.i). That the above intersections are empty already entails $\emptyset = {\uparrow}{\downarrow}X_i \cap {\downarrow}X_j' = {\uparrow}{\downarrow}X_i \cap X_j'$ by Lemma 5.3, and that ${\downarrow}X_i \cap {\uparrow}X_j = \emptyset$, since ${\uparrow}X_j \subseteq {\uparrow}X_j'$ by (IH.iv).

(4) Let us start by proving the inclusion ${\uparrow}X_{i-1} \smallsetminus X_{i-1} \subseteq {\uparrow}X_i$. Suppose that $x \in {\uparrow}X_{i-1} \smallsetminus X_{i-1}$, i.e., $x \notin X_{i-1}$ but there is $y \in X_{i-1}$ such that $y \leqslant x$. As $X_{i-1} \subseteq {\downarrow}X_i$ by (IH.ii), there must be a $z \in X_i$ above y. As $\mathcal{X}$ is a co-tree, we have $y \leqslant z \leqslant x$ or $y \leqslant x \leqslant z$, with both cases yielding $x \in {\uparrow}X_i$. The former is clear since $z \in X_i$, and if $y \leqslant x \leqslant z$, then $z \in X_i$ and (IH.ii) imply

$$x \in {\downarrow}X_i = X_i \uplus {\downarrow}X_{i-1} \uplus X_i'.$$

By assumption, $x \notin X_{i-1}$ and $x \in {\uparrow}X_{i-1}$, so (IH.i) (in particular, that X_{i-1} is convex) entails $x \notin {\downarrow}X_{i-1}$. It follows that either $x \in X_i$ or $x \in X_i'$. But $x \in X_i'$ cannot happen, since ${\uparrow}{\downarrow}X_{i-1} \cap X_i' = \emptyset$ by condition (3) proved above. Thus, we must have $x \in X_i \subseteq {\uparrow}X_i$, as desired.

To prove ${\uparrow}X_i \subseteq {\uparrow}X_{i-1} \smallsetminus X_{i-1}$, notice that ${\uparrow}X_i \subseteq {\uparrow}X_{i-1}$ by (2), so it suffices to show ${\uparrow}X_i \cap X_{i-1} = \emptyset$, which follows immediately from condition (3) proved above. The equality ${\uparrow}X_i = {\uparrow}X_i' \smallsetminus X_i'$ is proved analogously, hence we omit it.

Finally, by combining (IH.ii-iii) with conditions (3) and (4), it is clear that (5) is verified. □

We now resume our proof by induction. Suppose that we have clopens $X_1, \dots, X_{m-1}$, $X_1', \dots, X_{m-1}' \subseteq \mathcal{X}$ satisfying our induction hypothesis, that is, conditions (IH.i-iv). Using a similar argument as for the base case, we can easily prove $x_m', x_m \notin {\uparrow}{\downarrow}\{x_{m+1}', \dots, x_n'\}$. Furthermore, by (IH.i), we know that $X_{m-1} \cap {\uparrow}{\downarrow}x_m' = \emptyset$. Notice that this equality clearly implies ${\downarrow}X_{m-1} \cap {\uparrow}{\downarrow}x_m' = \emptyset$, hence it follows from Lemma 5.3 that ${\uparrow}{\downarrow}X_{m-1} \cap {\downarrow}x_m' = \emptyset$. As $x_m' \in {\downarrow}x_m'$, we established $x_m' \notin {\uparrow}{\downarrow}X_{m-1}$. Since $\mathcal{X}$ is a bi-Esakia space, there must be a clopen U_m' satisfying

$$x_m' \in U_m' \subseteq ({\uparrow}{\downarrow}\{x_{m+1}', \dots, x_n'\} \cup {\uparrow}{\downarrow}X_{m-1})^c.$$

We know that $x_m \in {\uparrow}{\downarrow}x_m'$ by the order of $\mathfrak{C}_n$, so the fact ${\downarrow}X_{m-1} \cap {\uparrow}{\downarrow}x_m' = \emptyset$ proved above yields $x_m \notin {\downarrow}X_{m-1}$. But, since $x_{m-1} \leqslant x_m$ and $x_{m-1} \in X_{m-1}$, by the structure of $\mathfrak{C}_n$ and (IH.i), respectively, it follows $x_m \in {\uparrow}X_{m-1}$. As $U_m' \cap {\uparrow}{\downarrow}X_{m-1} = \emptyset$ by definition of U_m', we must have $x_m \notin {\downarrow}U_m'$, since otherwise there would exist $y \in U_m'$ satisfying $x_m \leqslant y$, and therefore $y \in U_m' \cap {\uparrow}X_{m-1} \subseteq U_m' \cap {\uparrow}{\downarrow}X_{m-1}$, a contradiction. We conclude that there exists a clopen U_m such that

$$x_m \in U_m \subseteq ({\uparrow}{\downarrow}\{x_{m+1}', \dots, x_n'\} \cup {\downarrow}X_{m-1} \cup {\downarrow}U_m')^c.$$

As $\mathcal{X}$ is a bi-Esakia space, the following sets are all clopens:

- $W_m := {\downarrow}U_m \cap {\uparrow}X_{m-1} \cap {\uparrow}{\downarrow}U_m'$;
- $W_m' := {\downarrow}U_m' \cap {\downarrow}W_m$;
- $W_i := X_i \cap {\downarrow}W_m$, for $i \leqslant m-1$;
- $W_i' := X_i' \cap {\downarrow}W_m$, for $i \leqslant m-1$.

The following lemma establishes some crucial properties of our newly defined clopens.

Lemma 5.6. *The following conditions hold:*

1. $W_m = {\downarrow}U_m \cap {\uparrow}W_{m-1} \cap {\uparrow}W_m'$;
2. *W_i is convex, for all $i \leqslant m$;*
3. ${\downarrow}W_{m-1} \cap {\uparrow}W_m' = {\downarrow}W_{m-1} \cap W_m' = {\uparrow}{\downarrow}W_{m-1} \cap W_m' = \emptyset$;
4. *For all $i < m$,*
$${\downarrow}W_i = {\downarrow}X_i \cap {\downarrow}W_m \quad \textit{and} \quad {\downarrow}W_i' = {\downarrow}X_i' \cap {\downarrow}W_m;$$
5. *For all $i < m$,*
$${\uparrow}W_i = ({\uparrow}X_i \cap {\downarrow}W_m) \cup {\uparrow}W_m \quad \textit{and} \quad {\uparrow}W_i' = ({\uparrow}X_i' \cap {\downarrow}W_m) \cup {\uparrow}W_m.$$

Proof. (1) The right to left inclusion is clear, since by the definitions of W_{m-1} and W_m', we have ${\uparrow}W_{m-1} \subseteq {\uparrow}X_{m-1}$ and ${\uparrow}W_m' \subseteq {\uparrow}{\downarrow}U_m'$, hence

$${\downarrow}U_m \cap {\uparrow}W_{m-1} \cap {\uparrow}W_m' \subseteq {\downarrow}U_m \cap {\uparrow}X_{m-1} \cap {\uparrow}{\downarrow}U_m' = W_m.$$

To prove the reverse inclusion, let $x \in W_m = {\downarrow}U_m \cap {\uparrow}X_{m-1} \cap {\uparrow}{\downarrow}U'_m$, so there are $y \in X_{m-1}$ and $z \in {\downarrow}U'_m$ such that $y, z \leqslant x$. As $x \in W_m$, we have both

$$y \in X_{m-1} \cap {\downarrow}W_m = W_{m-1} \text{ and } z \in {\downarrow}U'_m \cap {\downarrow}W_m = W'_m.$$

Since $x \in {\downarrow}U_m$ by definition of W_m, it is now clear that $x \in {\downarrow}U_m \cap {\uparrow}W_{m-1} \cap {\uparrow}W'_m$, as desired.

(2) That W_m is convex follows immediately from the easily checked fact that the intersection of a downset and an upset is always convex.

Now, for $i < m$, notice that if $x \in {\downarrow}W_i \cap {\uparrow}W_i$, there are $y, z \in W_i = X_i \cap {\downarrow}W_m$ satisfying $y \leqslant x \leqslant z$. As $X_i = {\downarrow}X_i \cap {\uparrow}X_i$ by (IH.i), it follows that $x \in X_i$, and since $z \in {\downarrow}W_m$, we conclude $x \in X_i \cap {\downarrow}W_m = W_i$. This shows ${\downarrow}W_i \cap {\uparrow}W_i \subseteq W_i$, i.e., that W_i is convex, as desired.

(3) We prove that ${\downarrow}W_{m-1} \cap {\uparrow}{\downarrow}W'_m = \varnothing$, and therefore that

$${\downarrow}W_{m-1} \cap {\downarrow}W'_m = \varnothing = {\uparrow}{\downarrow}W_{m-1} \cap {\downarrow}W'_m,$$

by Lemma 5.3. As W'_m is the intersection of two downsets, and thus itself a downset, the above three intersections are exactly those we want to prove to be empty. We suppose, with a view towards contradiction, that $x \in {\downarrow}W_{m-1} \cap {\uparrow}{\downarrow}W'_m = {\downarrow}W_{m-1} \cap {\uparrow}W'_m$, i.e., that there are $y \in W'_m$ and $z \in W_{m-1}$ satisfying $y \leqslant x \leqslant z$. As $W'_m \subseteq {\downarrow}U'_m$ and $W_{m-1} \subseteq X_{m-1}$, $y \leqslant z$ entails $y \in {\downarrow}U'_m \cap {\downarrow}X_{m-1}$. It follows that $U'_m \cap {\uparrow}{\downarrow}X_{m-1} \neq \varnothing$, contradicting the definition of $U'_m \subseteq ({\uparrow}{\downarrow}X_{m-1})^c$, as desired.

(4) Let us first prove ${\downarrow}W_i = {\downarrow}X_i \cap {\downarrow}W_m$, for $i < m$. The left to right inclusion is clear, since $W_i = X_i \cap W_m$. For the reverse inclusion, suppose that $x \in {\downarrow}X_i \cap {\downarrow}W_m$, i.e., that there are $z \in X_i$ and $y \in W_m$ such that $x \leqslant y, z$. As $\mathcal{X}$ is a co-tree, we have $x \leqslant z \leqslant y$ or $x \leqslant y \leqslant z$. The former clearly yields $x \in {\downarrow}W_i$, since in this case we have $z \in X_i \cap {\downarrow}W_m = W_i$. We now prove that $x \leqslant y \leqslant z$ cannot happen. For if this was the case, we would have $z \in X_i \cap {\uparrow}W_m \subseteq {\downarrow}X_i \cap {\uparrow}W_m$, hence Lemma 5.5.(1) (in particular, that ${\downarrow}X_i \subseteq {\downarrow}X_{m-1}$) yields

$${\downarrow}X_{m-1} \cap {\uparrow}W_m \neq \varnothing.$$

Since $W_m \subseteq {\uparrow}{\downarrow}U'_m$ by definition of W_m, it now follows that ${\downarrow}X_{m-1} \cap {\uparrow}{\downarrow}U'_m \neq \varnothing$. By Lemma 5.3, this is equivalent to ${\uparrow}{\downarrow}X_{m-1} \cap {\downarrow}U'_m \neq \varnothing$, which clearly implies ${\uparrow}{\downarrow}X_{m-1} \cap U'_m \neq \varnothing$, contradicting the definition of $U'_m \subseteq ({\uparrow}{\downarrow}X_{m-1})^c$.

To prove the nontrivial inclusion of ${\downarrow}W'_i = {\downarrow}X'_i \cap {\downarrow}W_m$, i.e., that ${\downarrow}X'_i \cap {\downarrow}W_m \subseteq {\downarrow}W'_i$, suppose that $x \leqslant z, y$, for some $z \in X'_i$ and $y \in W_m$. As $\mathcal{X}$ is a co-tree, we have either $x \leqslant z \leqslant y$ or $x \leqslant y \leqslant z$. The former case immediately yields $x \in {\downarrow}W'_i = {\downarrow}(X'_i \cap {\downarrow}W_m)$, and the latter again yields a contradiction. To see this, notice that we would then have $z \in {\downarrow}X_i \cap {\uparrow}W_m$, since $X'_i \subseteq {\downarrow}X_i$ by (IH.ii), but we just proved above that ${\downarrow}X_i \cap {\uparrow}W_m \neq \varnothing$ cannot happen.

(5) We start by proving

$${\uparrow}W_i \subseteq ({\uparrow}X_i \cap {\downarrow}W_m) \cup {\uparrow}W_m.$$

Suppose that $x \in {\uparrow}W_i = {\uparrow}(X_i \cap {\downarrow}W_m)$, so there is $y \in X_i \cap {\downarrow}W_m$ such that $y \leqslant x$. As $y \in {\downarrow}W_m$, we have $y \leqslant z$ for some $z \in W_m$. Since $\mathcal{X}$ is a co-tree, it follows that $y \leqslant x \leqslant z$ or $y \leqslant z \leqslant x$. If $y \leqslant x \leqslant z$, then $x \in {\uparrow}X_i \cap {\downarrow}W_m$, and if $y \leqslant z \leqslant x$, then $x \in {\uparrow}W_m$. Since both cases yield $x \in ({\uparrow}X_i \cap {\downarrow}W_m) \cup {\uparrow}W_m$, we are done.

Now, for the other inclusion, suppose that $x \in {\uparrow}W_m$, i.e, that $z \leqslant x$ for some $z \in W_m$. By definition of this set, we know $W_m \subseteq {\uparrow}X_{m-1}$. Furthermore, by applying Lemma 5.5.(2) we get the inclusion $W_m \subseteq {\uparrow}X_{m-1} \subseteq {\uparrow}X_i$. Hence, there is $y \in X_i \cap {\downarrow}W_m = W_i$ satisfying $y \leqslant z \leqslant x$, and therefore $x \in {\uparrow}W_i$. It is an easy to see that ${\uparrow}X_i \cap {\downarrow}W_m \subseteq {\uparrow}W_i$, thus establishing the desired inclusion.

That ${\uparrow}W'_i = ({\uparrow}X'_i \cap {\downarrow}W_m) \cup {\uparrow}W_m$ is proved analogously. □

In order to satisfy (IH.ii), we would like to have the following equality:

$${\downarrow}W_m = W_m \uplus {\downarrow}W_{m-1} \uplus W'_m.$$

Notice that W'_m is a downset by definition, and that $W_m \subseteq {\uparrow}W_{m-1} \cap {\uparrow}W'_m$ by Lemma 5.6.(1). Consequently, from condition (3) of the same lemma we can infer that the 3 clopens on the right side of the above equality are in fact pairwise disjoint. To see this, notice that ${\downarrow}W_{m-1} \cap {\uparrow}W'_m = \varnothing$

yields that ${\downarrow}W_{m-1}$ is disjoint from W'_m and from $W_m \subseteq {\uparrow}W'_m$, and that ${\uparrow}{\downarrow}W_{m-1} \cap W'_m = \varnothing$ yields that W'_m is disjoint from $W_m \subseteq {\uparrow}W_{m-1}$.

Although $W_m \uplus {\downarrow}W_{m-1} \uplus W'_m \subseteq {\downarrow}W_m$ follows immediately from the definitions of W_{m-1} and W'_m, nothing in our construction ensures that the reverse inclusion holds true. In other words, there can be points lying strictly below W_m which are not contained in ${\downarrow}W_{m-1} \cup W'_m$, that is, the set ${\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$ can be nonempty. We will now characterize these *inconvenient* points.

Definition 5.7. A point $x \in {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$ is a *2-point* if

$$ {\downarrow}x \cap {\downarrow}W_{m-1} \neq \varnothing \neq {\downarrow}x \cap W'_m. $$

We will prove that 2-points (they are called as such because their downsets intersect with both ${\downarrow}W_{m-1}$ and W'_m) do not exist. But first, let us show a helpful equivalence.

Lemma 5.8. *If* $x \in {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$*, then*

$$ {\downarrow}x \cap {\downarrow}W_{m-1} \neq \varnothing \iff {\downarrow}x \cap W_{m-1} \neq \varnothing $$

Proof. The right to left implication is clear since $W_{m-1} \subseteq {\downarrow}W_{m-1}$. For the other direction, suppose that $y \in {\downarrow}x \cap {\downarrow}W_{m-1}$. As $y \in {\downarrow}W_{m-1}$, there exists $z \in W_{m-1}$ such that $y \leqslant z$, and since $\mathcal{X}$ is a co-tree, we have $y \leqslant x \leqslant z$ or $y \leqslant z \leqslant x$. But $y \leqslant x \leqslant z$ cannot happen, since $z \in W_{m-1}$ and $x \notin {\downarrow}W_{m-1}$ by assumption. Thus, we must have $y \leqslant z \leqslant x$, and it now follows that $z \in {\downarrow}x \cap W_{m-1}$, as desired. ☐

Now, let us suppose that x is a 2-point, so, in particular, we have $x \in {\downarrow}W_m \smallsetminus W_m$. Using the previous lemma, our assumption on x yields

$$ {\downarrow}x \cap W_{m-1} \neq \varnothing \neq {\downarrow}x \cap W'_m, $$

i.e., that $x \in {\uparrow}W_{m-1} \cap {\uparrow}W'_m$. As $W_m \subseteq {\downarrow}U_m$ by definition of W_m, $x \in {\downarrow}W_m$ now entails

$$ x \in {\downarrow}U_m \cap {\uparrow}W_{m-1} \cap {\uparrow}W'_m = W_m, $$

where the equality above follows from Lemma 5.6.(1). But this contradicts our assumption $x \notin W_m$. We conclude that 2-points do not exist, as desired.

Definition 5.9. A point $x \in {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$ is a *1-point* if

$$ {\downarrow}x \cap {\downarrow}W_{m-1} \neq \varnothing = {\downarrow}x \cap W'_m. $$

Equivalently, when

$$ {\downarrow}x \cap W_{m-1} \neq \varnothing = {\downarrow}x \cap W'_m. $$

The next lemma provides an equivalent characterization of 1-points, which are called as such because their downsets intersect with only one of ${\downarrow}W_{m-1}$ and W'_m (namely, with ${\downarrow}W_{m-1}$).

Lemma 5.10 (1-point Lemma)**.** *If* $x \in \mathcal{X}$*, then* x *is a 1-point iff* $x \in {\uparrow}W_{m-1} \smallsetminus (W_{m-1} \cup {\uparrow}W_m)$.

Proof. Let us first prove the left to right implication. Suppose that x is a 1-point, so by definition we have $x \in {\downarrow}W_m \smallsetminus (W_m \cup W_{m-1})$ and ${\downarrow}x \cap W_{m-1} \neq \varnothing$, i.e., that $x \in {\uparrow}W_{m-1}$. This already shows that $x \in {\uparrow}W_{m-1} \smallsetminus W_{m-1}$, so it remains to prove that $x \notin {\uparrow}W_m$. But we know $x \in {\downarrow}W_m \smallsetminus W_m$ by above, so the fact that W_m is convex proved in Lemma 5.6.(2) forces $x \notin {\uparrow}W_m$, by Lemma 5.2.

To prove the reverse implication, let us assume $x \in {\uparrow}W_{m-1} \smallsetminus (W_{m-1} \cup {\uparrow}W_m)$. Since this already entails $x \notin W_{m-1} \cup W_m$ and ${\downarrow}x \cap W_{m-1} \neq \varnothing$, to establish x as a 1-point it remains to show $x \in {\downarrow}W_m \smallsetminus W'_m$ and ${\downarrow}x \cap W'_m = \varnothing$. Equivalently, that $x \in {\downarrow}W_m \smallsetminus {\uparrow}W'_m$. Since $x \in {\uparrow}W_{m-1}$, there exists $y \in W_{m-1}$ such that $y \leqslant x$. By the definition of $W_{m-1} = X_{m-1} \cap {\downarrow}W_m$, we have $y \leqslant z$ for some $z \in W_m$. As $\mathcal{X}$ is a co-tree, it follows that $y \leqslant z \leqslant x$ or $y \leqslant x \leqslant z$. The former cannot happen, since $z \in W_m$ and by hypothesis $x \notin {\uparrow}W_m$. Hence, we have $y \leqslant x \leqslant z$, and we proved $x \in {\downarrow}W_m$. To see that $x \notin {\uparrow}W'_m$, just notice that $W_m \subseteq {\downarrow}U_m$ by definition, so $x \leqslant z \in W_m$ entails $x \in {\downarrow}U_m$. As $x \in {\uparrow}W_{m-1}$ and $x \notin W_m$ by assumption, the fact $W_m = {\downarrow}U_m \cap {\uparrow}W_{m-1} \cap {\uparrow}W'_m$ proved in Lemma 5.6.(1) clearly implies $x \notin {\uparrow}W'_m$. ☐

In analogy with the previous definition, we define the $1'$-points and provide an equivalent characterization for them, whose proof we skip since it uses a very similar argument to the one detailed above. Notice that $1'$-points are called as such because they intersect with only one of ${\downarrow}W_{m-1}$ and W'_m (namely, with W'_m).

Definition 5.11. A point $x' \in {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$ is a *$1'$-point* if

$$ {\downarrow}x' \cap {\downarrow}W_{m-1} = \varnothing \neq {\downarrow}x' \cap W'_m. $$

Lemma 5.12 ($1'$-point Lemma)**.** *If $x' \in \mathcal{X}$, then x' is a $1'$-point iff $x' \in {\uparrow}W'_m \smallsetminus (W'_m \cup {\uparrow}W_m)$.*

The following auxiliary lemma provides some sufficient conditions for a point in $\mathcal{X}$ to be a 1-point or a $1'$point. They will not only help us in the characterization of the inconvenient points, that is, the points in ${\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$, but we can also use these conditions to deduce easily that 1-points and $1'$-points are incomparable.

Lemma 5.13. *The following conditions hold, for any $y \in \mathcal{X}$:*
1. *If $y \in ({\downarrow}W_m \smallsetminus W_m) \cap {\uparrow}x$ or $y \in ({\uparrow}W_{m-1} \smallsetminus W_{m-1}) \cap {\downarrow}x$ for some 1-point x, then y is also a 1-point;*
2. *If $y \in ({\downarrow}W_m \smallsetminus W_m) \cap {\uparrow}x'$ or $y \in ({\uparrow}W'_m \smallsetminus W'_m) \cap {\downarrow}x'$ for some $1'$-point x', then y is also a $1'$-point;*
3. *1-points and $1'$-points are incomparable.*

Proof. (1) Suppose that $y \in ({\downarrow}W_m \smallsetminus W_m) \cap {\uparrow}x$, for some 1-point x. We prove that y is contained in ${\uparrow}W_{m-1} \smallsetminus (W_{m-1} \cup {\uparrow}W_m)$, i.e., that y is a 1-point, by Lemma 5.10. As W_m is convex by Lemma 5.6.(2), our assumption $y \in {\downarrow}W_m \smallsetminus W_m$ already entails $y \notin {\uparrow}W_m$, by Lemma 5.2. Furthermore, as 1-points are contained in ${\uparrow}W_{m-1}$ by definition, $x \leqslant y$ yields $y \in {\uparrow}W_{m-1}$. Finally, to see that $y \notin W_{m-1}$, notice that otherwise we would have $x \in {\downarrow}W_{m-1}$, contradicting the assumption that x is a 1-point. We conclude that y is indeed a 1-point, as desired.

Suppose now that $y \in ({\uparrow}W_{m-1} \smallsetminus W_{m-1}) \cap {\downarrow}x$ for some 1-point x. By the 1-point Lemma 5.10, we know that $x \notin {\uparrow}W_m$. Consequently, we must have $y \notin {\uparrow}W_m$, since we assumed $y \leqslant x$. It follows that $y \in {\uparrow}W_{m-1} \smallsetminus (W_{m-1} \cup {\uparrow}W_m)$, i.e., that y is a 1-point, by the aforementioned lemma.

(2) This is proved analogously to condition (1) above, but in place of the fact that W_{m-1} is convex, we use that W'_m is a downset.

(3) Let x be a 1-point and x' a $1'$-point. Recall that, by definition, both points lie in ${\downarrow}W_m \smallsetminus W_m$. If $x \leqslant x'$, then $x' \in ({\downarrow}W_m \smallsetminus W_m) \cap {\uparrow}x$, so x' is a 1-point by condition (1) proved above. If $x' \leqslant x$, then $x \in ({\downarrow}W_m \smallsetminus W_m) \cap {\uparrow}x'$, hence x is a $1'$-point by condition (2) proved above. Both cases yield a contradiction, since the definitions of 1-points and $1'$-points are clearly mutually exclusive. ☐

We will now define the last class of points that we will need to fully characterize the set of inconvenient points ${\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$. They are called 0-points because, unsurprisingly, their downsets do not intersect with ${\downarrow}W_{m-1}$ nor with W'_m.

Definition 5.14. A point $x \in {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$ is a *0-point* if

$$ {\downarrow}x \cap {\downarrow}W_{m-1} = \varnothing = {\downarrow}x \cap W'_m, $$

and x does not lie below a 1-point, nor below a $1'$-point.

Remark 5.15. It is clear by the definition above that points lying below 0-points must also be 0-points.

We can now use the mutually exclusive nature of the definitions above (just notice which intersections are empty and which ones are not in their respective definitions) to fully characterize the points in ${\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$. They are, and this will be proved shortly, of five distinct forms:

- 1-points;
- points which are not 1-points but lie below a 1-point;
- $1'$-points;
- points which are not $1'$-points but lie below a $1'$-point;
- 0-points.

Our solution to "deal" with these inconvenient points is the following: 1-points will be added to W_{m-1}; points which are not 1-points but lie below one will be added to W'_{m-1}; 1′-points, their downsets, and 0-points will all be added to W'_m.

Note that by the 1-point Lemma 5.10, the set of 1-points can be written as

$$Z := {\uparrow}W_{m-1} \smallsetminus (W_{m-1} \cup {\uparrow}W_m),$$

and is therefore a clopen. Note as well that the set of inconvenient points which are not 1-points but lie below a 1-point takes the form of

$${\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1}),$$

which is clearly a clopen as well. Finally, let us denote the set of 1′-points by Z' and the set of 0-points by Z_0. As $\mathcal{X}$ is a bi-Esakia space, the following sets are all clopens:

- $V_{m-1} := W_{m-1} \cup Z$;
- $V'_{m-1} := W'_{m-1} \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]$;
- $V'_m := {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}V_{m-1})$.

We will now prove that the definition of the set V'_m complies with our "solution" stated above, that is, V'_m is simply W'_m together with all the 0-points and all the points in the downsets of 1′-points.

Lemma 5.16. *The following equality holds:*

$$V'_m = W'_m \cup {\downarrow}Z' \cup Z_0.$$

Proof. We start by proving the left to right inclusion. Suppose that $x \in V'_m$ i.e., that $x \in {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}V_{m-1})$. If $x \in W'_m$ we are done, so let us assume otherwise. It now follows from our assumption on x and the definition of $V_{m-1} = W_{m-1} \cup Z$ that

$$x \in {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m).$$

Since we proved that 2-points do not exist, ${\downarrow}x \cap {\downarrow}W_{m-1} \neq \emptyset \neq {\downarrow}x \cap W'_m$ cannot happen. If

$${\downarrow}x \cap {\downarrow}W_{m-1} \neq \emptyset = {\downarrow}x \cap W'_m,$$

then x would satisfy the definition of a 1-point, i.e., $x \in Z$. But this contradicts our assumption $x \notin {\downarrow}V_{m-1}$, since $Z \subseteq V_{m-1}$. If

$${\downarrow}x \cap {\downarrow}W_{m-1} = \emptyset \neq {\downarrow}x \cap W'_m,$$

then x satisfies the definition of a 1′-point, i.e., $x \in Z'$, and we are done. Finally, suppose

$${\downarrow}x \cap {\downarrow}W_{m-1} = \emptyset = {\downarrow}x \cap W'_m.$$

Since the case $x \in {\downarrow}Z'$ is immediate, let us assume otherwise, i.e., that x does not lie below a 1′-point. Since we also assumed $x \notin {\downarrow}V_{m-1} = {\downarrow}(W_{m-1} \cup Z)$, in particular, that x cannot lie below a 1-point, it now follows that x satisfies the definition of a 0-point, i.e., $x \in Z_0$, and we are done. We conclude $V'_m \subseteq W'_m \cup {\downarrow}Z' \cup Z_0$, as desired.

We now prove that $W'_m \cup {\downarrow}Z' \cup Z_0 \subseteq V'_m = {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}V_{m-1})$. Let us start by showing

$$W'_m \cup {\downarrow}Z' \cup Z_0 \subseteq {\downarrow}W_m \smallsetminus W_m.$$

That $W'_m \subseteq {\downarrow}W_m \smallsetminus W_m$ was already proved above (see the comment before the definition of 2-points), and that ${\downarrow}Z' \cup Z_0 \subseteq {\downarrow}W_m \smallsetminus W_m$ follows immediately from the definitions of 1′-points and 0-points.

To establish the desired inclusion, it remains to show $W'_m \cup {\downarrow}Z' \cup Z_0 \subseteq {\downarrow}W_m \smallsetminus {\downarrow}V_{m-1}$. As we already know $W'_m \cup {\downarrow}Z' \cup Z_0 \subseteq {\downarrow}W_m$, it suffices to show

$$(W'_m \cup {\downarrow}Z' \cup Z_0) \cap {\downarrow}V_{m-1} = \emptyset$$

We will prove this by noting that

$$(W'_m \cup {\downarrow}Z' \cup Z_0) \cap {\downarrow}V_{m-1} = (W'_m \cap {\downarrow}V_{m-1}) \cup ({\downarrow}Z' \cap {\downarrow}V_{m-1}) \cup (Z_0 \cap {\downarrow}V_{m-1}),$$

and showing that the three intersections on the right side of the above equality are all empty. Recall that ${\downarrow}V_{m-1} = {\downarrow}W_{m-1} \cup {\downarrow}Z$, by definition. That $W'_m \cap ({\downarrow}W_{m-1} \cup {\downarrow}Z) = \emptyset$ follows from

Lemma 5.6.(3) and the definition of 1-points. To see that $\downarrow Z' \cap (\downarrow W_{m-1} \cup \downarrow Z) = \emptyset$, recall that $\downarrow Z \cap \downarrow W_{m-1}$ follows immediately from the definition of 1′-points, and note that as 1-points and 1′-points are incomparable (see Lemma 5.13.(3)), the fact that $\mathcal{X}$ is a co-tree forces $\downarrow Z \cap \downarrow Z' = \emptyset$. Finally, that $Z_0 \cap (\downarrow W_{m-1} \cup \downarrow Z) = \emptyset$ is immediate from the definition of 0-points. □

We are finally ready to finish our proof by induction. To improve the readability of what follows and for ease of reference, below we re-label some of our clopens, restating their definitions as well as some useful equalities:

- $V_i := W_i = X_i \cap \downarrow W_m$, for all $i < m - 1$;
- $V_i' := W_i' = X_i' \cap \downarrow W_m$, for all $i < m - 1$;
- $W_{m-1} = X_{m-1} \cap \downarrow W_m$;
- $W_{m-1}' = X_{m-1}' \cap \downarrow W_m$;
- $W_m' = \downarrow U_m' \cap \downarrow W_m$;
- $V_{m-1} = W_{m-1} \cup Z$;
- $V_{m-1}' = W_{m-1}' \cup [\downarrow Z \smallsetminus (Z \cup \downarrow W_{m-1})]$;
- $V_m' = \downarrow W_m \smallsetminus (W_m \cup \downarrow V_{m-1}) = W_m' \cup \downarrow Z' \cup Z_0$;
- $V_m = W_m = \downarrow U_m \cap \uparrow X_{m-1} \cap \uparrow\downarrow U_m' = \downarrow U_m \cap \uparrow W_{m-1} \cap \uparrow W_m'$.

Recall that $x_i \in U_i$, $x_i' \in U_i'$ and that $U_i' \subseteq (\uparrow\downarrow x_j')^c$, for all $i < j \leqslant n$. These facts will be used repeatedly in the next proof.

Proposition 5.17. *The clopens $V_1, \dots, V_m, V_1', \dots, V_m' \subseteq \mathcal{X}$ satisfy condition (IH.i), that is, for all $i \leqslant m$, we have that V_i is a convex set containing x_i, that V_i' is a downset containing x_i', and that $V_i \cap \uparrow\downarrow x_j' = \emptyset$ for all j such that $i < j \leqslant n$.*

Proof. Firstly, we show that the statement holds when $i = m$. As $x_{m-1} \in X_{m-1}$ by (IH.i) and $x_m' \in U_m'$ by definition of U_m', it follows $x_m \in \uparrow X_{m-1} \cap \uparrow\downarrow U_m'$, since $x_m' \leqslant x_m$ by the order of $\mathfrak{C}_n$. As we also know $x_m \in U_m$ by the definition of this set, we indeed have

$$x_m \in \downarrow U_m \cap \uparrow X_{m-1} \cap \uparrow\downarrow U_m' = W_m = V_m.$$

Furthermore, that this set is convex was already established in Lemma 5.6.(2).

We stated above that $x_m' \in U_m'$, so $x_m' \leqslant x_m \in W_m$ now yields $x_m' \in \downarrow U_m' \cap \downarrow W_m = W_m'$. As $W_m' \subseteq V_m'$ by Lemma 5.16, it follows $x_m' \in V_m'$. Note as well that this lemma also implies that V_m' is a downset, since it is characterized as the union of three downsets (recall that $W_m' = \downarrow U_m' \cap \downarrow W_m$ and $\downarrow Z'$ are downsets by definition, and that Z_0 being a downset is an immediate consequence of the definitions of 0-points, since any point lying below a 0-point must also be a 0-point).

Finally, that $\downarrow V_m \cap \uparrow\downarrow x_j' = \emptyset$ for all j such that $m < j \leqslant n$, follows easily from the definitions of V_m and U_m, since $V_m = W_m \subseteq \downarrow U_m$ and $U_m \subseteq (\uparrow\downarrow x_j')^c$.

Suppose now that $i < m - 1$. By (IH.i), we know that $x_i \in X_i$ and $x_i' \in X_i'$. As $x_m \in W_m$ by above, the order of $\mathfrak{C}_n$ yields both $x_i \in X_i \cap \downarrow W_m = V_i$ and $x_i' \in X_i' \cap \downarrow W_m = V_i$. Moreover, that $V_i = W_i$ is convex was already established in Lemma 5.6.(2), and since X_i' is a downset by (IH.i), it is clear that $V_i' = X_i' \cap \downarrow W_m$ is also a downset. It remains to show $V_i \cap \uparrow\downarrow x_j' = \emptyset$ for all j such that $i < j \leqslant n$, which follows easily from the definition $V_i = X_i \cap \downarrow W_m \subseteq X_i$ and from $X_i \cap \uparrow\downarrow x_j' = \emptyset$ by (IH.i).

It remains to consider the case $i = m - 1$. By an argument similar to the one detailed for the previous case, we can easily show that $x_{m-1} \in W_{m-1}$ and $x_{m-1}' \in W_{m-1}'$. As $W_{m-1} \subseteq V_{m-1}$ and $W_{m-1}' \subseteq V_{m-1}'$ by the definitions of V_{m-1} and V_{m-1}', we have $x_{m-1} \in V_{m-1}$ and $x_{m-1}' \in V_{m-1}'$. To see that $V_{m-1} = W_{m-1} \cup Z$ is convex, i.e., that $\downarrow V_{m-1} \cap \uparrow V_{m-1} \subseteq V_{m-1}$, recall that W_{m-1} is convex by Lemma 5.6.(2) and that $Z \subseteq \uparrow W_{m-1}$ by the definition of 1-points. It follows that

$$\begin{aligned}\downarrow V_{m-1} \cap \uparrow V_{m-1} &= (\downarrow W_{m-1} \cup \downarrow Z) \cap (\uparrow W_{m-1} \cup \uparrow Z) = (\downarrow W_{m-1} \cup \downarrow Z) \cap \uparrow W_{m-1} \\ &= (\downarrow W_{m-1} \cap \uparrow W_{m-1}) \cup (\downarrow Z \cap \uparrow W_{m-1}) = W_{m-1} \cup (\downarrow Z \cap \uparrow W_{m-1}).\end{aligned}$$

Thus, to show that V_{m-1} is convex, it suffices to prove the inclusion $\downarrow Z \cap \uparrow W_{m-1} \subseteq W_{m-1} \cup Z = V_{m-1}$. As Z is the set of 1-points, this inclusion is an immediate consequence of Lemma 5.13.(1), and we are done.

We now prove that $V'_{m-1} = W'_{m-1} \cup [\downarrow Z \smallsetminus (Z \cup \downarrow W_{m-1})]$ is a downset. Recall that X'_{m-1} is a downset by (IH.i), hence $W'_{m-1} = X'_{m-1} \cap \downarrow W_m$ is also a downset. To see that $\downarrow Z \smallsetminus (Z \cup \downarrow W_{m-1})$ is a downset as well, we suppose otherwise and arrive at a contradiction. Assume that there are x and y satisfying $x \in \downarrow Z \smallsetminus (Z \cup \downarrow W_{m-1})$, $y \notin \downarrow Z \smallsetminus (Z \cup \downarrow W_{m-1})$, and $y \leqslant x$. Notice that there must exist $z \in Z$ such that $x \leqslant z$, hence $y \leqslant x \leqslant z$ forces $y \in \downarrow Z$. This, together with our assumption on y, yields $y \in Z \cup \downarrow W_{m-1}$. If $y \in Z$, then we have $x \in \downarrow Z \cap \uparrow Z$. But we know that Z is convex (this is an immediate consequence of Lemma 5.13.(1)), so it follows $x \in Z$, a contradiction. If $y \in \downarrow W_{m-1}$, then there is $w \in W_{m-1}$ such that $y \leqslant w$. Since $\mathcal{X}$ is a co-tree, $y \leqslant w, x$ entails $y \leqslant x \leqslant w$ or $y \leqslant x \leqslant w$. The former cannot happen, since $x \notin \downarrow W_{m-1}$ by assumption, so we must have $y \leqslant w \leqslant x$. It follows $x \in (\uparrow W_{m-1} \smallsetminus W_{m-1}) \cap \downarrow z$, and since z is a 1-point, Lemma 5.13.(1) now yields that x is a 1-point, i.e., that $x \in Z$, another contradiction. We conclude that no such y can exist, that is, that $\downarrow Z \smallsetminus (Z \cup \downarrow W_{m-1})$ is a downset. Thus, V'_{m-1} is the union of two downsets, and therefore a downset, as desired.

Finally, we prove that $\downarrow V_{m-1} \cap \uparrow\downarrow x'_j = \varnothing$, for all j such that $m-1 < j \leqslant n$. By the definition of V_{m-1}, we have

$$\downarrow V_{m-1} \cap \uparrow\downarrow x'_j = \downarrow(W_{m-1} \cup Z) \cap \uparrow\downarrow x'_j = (\downarrow W_{m-1} \cup \downarrow Z) \cap \uparrow\downarrow x'_j.$$

Since $W_{m-1} = X_{m-1} \cap \downarrow W_m \subseteq X_{m-1}$ and we know $X_{m-1} \cap \uparrow\downarrow x'_j = \varnothing$ by (IH.i), it suffices to show

$$\downarrow Z \cap \uparrow\downarrow x'_j = \varnothing.$$

If $j > m$, then this already follows from the equality $V_m \cap \uparrow\downarrow x'_j = \varnothing$ proved above, since $\downarrow Z \subseteq \downarrow W_m = \downarrow V_m$ by the definition of 1-points. If $j = m$, then as x'_m is contained in the downset W'_m by above, we can use the definition of 1-points, in particular, that $\downarrow Z \cap W'_m = \varnothing$, to conclude $\downarrow Z \cap \uparrow\downarrow x'_m = \varnothing$. ☐

In the next result, we will use the conventions $W_0, X_0, W'_0, X'_0 \in \{\varnothing\}$.

Proposition 5.18. *The clopens $V_1, \dots, V_m, V'_1, \dots, V'_m \subseteq \mathcal{X}$ satisfy condition (IH.ii), that is, for all $i \leqslant m$, we have $\downarrow V_i = V_i \uplus \downarrow V_{i-1} \uplus V'_i$.*

Proof. We first show that $\downarrow W_i = W_i \uplus \downarrow W_{i-1} \uplus W'_i$, for all $i \leqslant m-1$. Recall that in this case, we have $W_i = X_i \cap \downarrow W_m$ and $W'_i = X'_i \cap \downarrow W_m$, by the definitions of W_i and W'_i, respectively. Recall as well that $\downarrow W_i = \downarrow X_i \cap \downarrow W_m$ and $\downarrow W_{i-1} = \downarrow X_{i-1} \cap \downarrow W_m$ follow from Lemma 5.6.(4), and that we assumed $\downarrow X_i = X_i \uplus \downarrow X_{i-1} \uplus X'_i$ in (IH.ii). Compiling all of these equalities yields

$$\begin{aligned} W_i \cup \downarrow W_{i-1} \cup W'_i &= (X_i \cap \downarrow W_m) \cup (\downarrow X_{i-1} \cap \downarrow W_m) \cup (X'_i \cap \downarrow W_m) \\ &= (X_i \cup \downarrow X_{i-1} \cup X'_i) \cap \downarrow W_m = \downarrow X_i \cap \downarrow W_m \\ &= \downarrow W_i. \end{aligned}$$

As the clopens $X_i, \downarrow X_{i-1}, X'_i$ are pairwise disjoint by (IH.ii), and since they contain $W_i, \downarrow W_{i-1}, W'_i$, respectively, we conclude

$$\downarrow W_i = W_i \uplus \downarrow W_{i-1} \uplus W'_i \tag{5}$$

for all $i \leqslant m-1$, as desired. Since, by definition, we have $V_j = W_j$ and $V'_j = W'_j$ for all $j < m-1$, we just proved that the statement holds when $i < m-1$.

Next we prove the case where $i = m-1$, that is, we establish the equality

$$\downarrow V_{m-1} = V_{m-1} \uplus \downarrow V_{m-2} \uplus V'_{m-1}.$$

Equivalently (using our previous notation), we prove

$$\downarrow(W_{m-1} \cup Z) = (W_{m-1} \cup Z) \uplus \downarrow W_{m-2} \uplus (W'_{m-1} \cup [\downarrow Z \smallsetminus (Z \cup \downarrow W_{m-1})]).$$

Using the equality ${\downarrow}W_{m-1} = W_{m-1} \uplus {\downarrow}W_{m-2} \uplus W'_{m-1}$, a particular instance of (5) proved above, we can re-write

$$(W_{m-1} \cup Z) \cup {\downarrow}W_{m-2} \cup (W'_{m-1} \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]) =$$
$$(W_{m-1} \cup {\downarrow}W_{m-2} \cup W'_{m-1}) \cup (Z \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]) =$$
$${\downarrow}W_{m-1} \cup (Z \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]).$$

Since we clearly have ${\downarrow}(W_{m-1} \cup Z) = {\downarrow}W_{m-1} \cup {\downarrow}Z = {\downarrow}W_{m-1} \cup Z \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]$, the above display now entails

$${\downarrow}(W_{m-1} \cup Z) = (W_{m-1} \cup Z) \cup {\downarrow}W_{m-2} \cup (W'_{m-1} \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]).$$

To finish the case $i = m-1$, it remains to show that the three sets on the right side of the above equality are pairwise disjoint. We again rely on

$${\downarrow}W_{m-1} = W_{m-1} \uplus {\downarrow}W_{m-2} \uplus W'_{m-1}.$$

Not only this equality yields that $W_{m-1}, {\downarrow}W_{m-2}$, and W'_{m-1} are pairwise disjoint, but also that ${\downarrow}W_{m-2} \subseteq {\downarrow}W_{m-1}$. Since 1-points (i.e., the elements of Z) are by definition inconvenient points (i.e., the elements of ${\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup {\downarrow}W'_m)$), the previous inclusion ensures $Z \cap {\downarrow}W_{m-2} = \varnothing$. This proves $(W_{m-1} \cup Z) \cap {\downarrow}W_{m-2} = \varnothing$. Again using the inclusion ${\downarrow}W_{m-2} \subseteq {\downarrow}W_{m-1}$, we infer that

$${\downarrow}W_{m-2} \cap [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})] = \varnothing,$$

and since ${\downarrow}W_{m-2} \cap W'_{m-1} = \varnothing$ was already established above, we conclude

$${\downarrow}W_{m-2} \cap (W'_{m-1} \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]) = \varnothing.$$

To see that $(W_{m-1} \cup Z) \cap (W'_{m-1} \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]) = \varnothing$, notice that: W_{m-1} and W'_{m-1} are disjoint by above; W_{m-1} and Z are clearly disjoint from ${\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})$; and that $Z \cap {\downarrow}W_{m-1} = \varnothing$ by the definition of 1-points, so the fact $W'_{m-1} \subseteq {\downarrow}W_{m-1}$ (an immediate consequence of (5)) now yields $Z \cap W'_{m-1} = \varnothing$. Concluding, and returning to our current notation, we proved

$${\downarrow}V_{m-1} = V_{m-1} \uplus {\downarrow}V_{m-2} \uplus V'_{m-1}.$$

Finally, the case where $i = m$ follows immediately form the definitions. Recall that $V_m = W_m$, $V_{m-1} = W_{m-1} \cup Z$, $W_{m-1} = X_{m-1} \cap {\downarrow}W_m$, $V'_m = {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}V_{m-1})$, and that $Z \subseteq {\downarrow}W_m$ by the definition of 1-points. This ensures

$${\downarrow}W_m = W_m \cup {\downarrow}(W_{m-1} \cup Z) \cup ({\downarrow}W_m \smallsetminus [W_m \cup {\downarrow}(W_{m-1} \cup Z)]),$$

that is,

$${\downarrow}V_m = V_m \cup {\downarrow}V_{m-1} \cup V'_m.$$

That $V'_m = {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}V_{m-1})$ is disjoint from both W_m (i.e., from V_m) and V_{m-1} is clear. To see that ${\downarrow}V_{m-1} \cap V_m = \varnothing$, i.e., that ${\downarrow}(W_{m-1} \cup Z) \cap W_m = \varnothing$, recall that ${\downarrow}W_{m-1} \cap W_m = \varnothing$ is an immediate consequence of Lemma 5.6.(3) (since $W_m \subseteq {\uparrow}W'_m$), and that ${\downarrow}Z \cap W_m = \varnothing$ by the definition of 1-points. Thus, we have

$${\downarrow}V_m = V_m \uplus {\downarrow}V_{m-1} \uplus V'_m. \qquad \square$$

Proposition 5.19. *The clopens $V_1, \dots, V_m, V'_1, \dots, V'_m \subseteq \mathcal{X}$ satisfy condition (IH.iii), that is, for all $1 < i \leqslant m$, we have ${\downarrow}V_{i-1} \cap {\uparrow}V'_i = \varnothing$.*

Proof. The case $1 < i < m-1$ follows immediately from the definitions and (IH.iii), since $V_{i-1} = W_{i-1} = X_{i-1} \cap {\downarrow}W_m$, $V'_i = W'_i = X'_i \cap {\downarrow}W_m$, and ${\downarrow}X_{i-1} \cap {\uparrow}X'_i = \varnothing$.

Suppose that $i = m-1$. As $V_{m-2} = W_{m-2}$ and $V'_{m-1} = W'_{m-1} \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]$ by their definitions, what we want to prove is

$${\downarrow}W_{m-2} \cap {\uparrow}(W'_{m-1} \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]) = \varnothing.$$

Using the same argument as for the previous case, we can easily show that ${\downarrow}W_{m-2}$ is disjoint form ${\uparrow}W'_{m-1}$. Therefore, it only remains to prove

$${\downarrow}W_{m-2} \cap {\uparrow}[{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})] = \varnothing.$$

Let us assume otherwise, so there are $w \in W_{m-2}$ and $z \in {\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})$ satisfying $z \leqslant x \leqslant w$, for some x. In particular, we have $z \leqslant w$. Using the inclusion ${\downarrow}W_{m-2} \subseteq {\downarrow}W_{m-1}$ established in the proof of the previous result, $z \leqslant w$ and $w \in W_{m-2}$ yield $z \in {\downarrow}W_{m-1}$, contradicting our assumption $z \notin {\downarrow}W_{m-1}$. We conclude that there is no such x, i.e., that

$$ {\downarrow}V_{m-2} \cap {\uparrow}V'_{m-1} = {\downarrow}W_{m-2} \cap {\uparrow}(W'_{m-1} \cup [{\downarrow}Z \smallsetminus (Z \cup {\downarrow}W_{m-1})]) = \varnothing, $$

as desired.

Finally, we show that ${\downarrow}V_{m-1} \cap {\uparrow}V'_m = \varnothing$. Using the definition of V_{m-1} and the equality proved in Lemma 5.16, that the previous intersection is empty can be written as

$$ {\downarrow}(W_{m-1} \cup Z) \cap {\uparrow}(W'_m \cup {\downarrow}Z' \cup Z_0) = \varnothing. $$

Equivalently,

$$ ({\downarrow}W_{m-1} \cup {\downarrow}Z) \cap ({\uparrow}W'_m \cup {\uparrow}{\downarrow}Z' \cup {\uparrow}Z_0) = \varnothing. $$

That ${\downarrow}W_{m-1}$ is disjoint from ${\uparrow}W'_m$ was already established in Lemma 5.6.(3). To see that ${\downarrow}W_{m-1} \cap {\uparrow}Z_0 = \varnothing$, suppose otherwise, so there are $w \in W_{m-1}$ and $z \in Z_0$ satisfying $z \leqslant x \leqslant w$, for some x. This implies $z \in {\downarrow}W_{m-1}$, a contradiction, since 0-points (i.e., the elements of Z_0) are defined to be in ${\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m)$. By Lemma 5.3, ${\downarrow}W_{m-1} \cap {\uparrow}{\downarrow}Z' = \varnothing$ iff ${\downarrow}W_{m-1} \cap {\downarrow}Z' = \varnothing$, and the latter equality is immediate from the definition of 1′-points, since the downsets generated by the points in Z' are disjoint from ${\downarrow}W_{m-1}$. All of this establishes

$$ {\downarrow}W_{m-1} \cap ({\uparrow}W'_m \cup {\uparrow}{\downarrow}Z' \cup {\uparrow}Z_0) = \varnothing, $$

so it remains to show

$$ {\downarrow}Z \cap ({\uparrow}W'_m \cup {\uparrow}{\downarrow}Z' \cup {\uparrow}Z_0) = \varnothing. $$

Again using Lemma 5.3, and the fact that W'_m is a downset, we see that ${\downarrow}Z \cap {\uparrow}W'_m = \varnothing$ iff ${\downarrow}Z \cap W'_m = \varnothing$. Just note that the latter equality is clear, since the downsets generated by 1-points are defined to be disjoint from W'_m. The aforementioned lemma also yields the equivalence ${\downarrow}Z \cap {\uparrow}{\downarrow}Z' = \varnothing$ iff ${\downarrow}Z \cap {\downarrow}Z' = \varnothing$, whose right side condition can be easily deduced using the fact that 1-points and 1′-points are incomparable (see Lemma 5.13) together with the co-tree structure of $\mathcal{X}$. Finally, to establish that ${\downarrow}Z$ and ${\uparrow}Z_0$ are disjoint, simply recall that by definition, 0-points do not lie below 1-points. Therefore, we have proved

$$ {\downarrow}Z \cap ({\uparrow}W'_m \cup {\uparrow}{\downarrow}Z' \cup {\uparrow}Z_0) = \varnothing, $$

and we are done. □

Proposition 5.20. *The clopens $V_1, \dots, V_m, V'_1, \dots, V'_m \subseteq \mathcal{X}$ satisfy condition (IH.iv), that is, we have*

$$ {\uparrow}V'_1 = {\uparrow}V_1 \uplus V'_1 \text{ and } {\uparrow}V_i = {\uparrow}V_{i-1} \cap {\uparrow}V'_i, $$

for all $1 < i \leqslant m$.

Proof. Let us start by noting that for all $1 < i \leqslant m-1$, we have

$$ \begin{aligned} {\uparrow}W_{i-1} \cap {\uparrow}W'_i &= (({\uparrow}X_{i-1} \cap {\downarrow}W_m) \cup {\uparrow}W_m) \cap (({\uparrow}X'_i \cap {\downarrow}W_m) \cup {\uparrow}W_m) \\ &= (({\uparrow}X_{i-1} \cap {\downarrow}W_m) \cap ({\uparrow}X'_i \cap {\downarrow}W_m)) \cup {\uparrow}W_m \\ &= (({\uparrow}X_{i-1} \cap {\uparrow}X'_i) \cap {\downarrow}W_m) \cup {\uparrow}W_m \\ &= ({\uparrow}X_i \cap {\downarrow}W_m) \cup {\uparrow}W_m \\ &= {\uparrow}W_i, \end{aligned} \tag{6} $$

where the first and last equalities follow from Lemma 5.6.(5), the second and third follow from the distributivity of the set theoretic operations, while the fourth uses our assumption (IH.iv), namely, that ${\uparrow}X_{i-1} \cap {\uparrow}X'_i = {\uparrow}X_i$. As $V_j = W_j$ and $V'_j = V'_j$ for all $j < m-1$, we just proved ${\uparrow}V_i = {\uparrow}V_{i-1} \cap {\uparrow}V'_i$ for all $1 < i < m-1$.

We now prove that ${\uparrow}V_{m-1} = {\uparrow}V_{m-2} \cap {\uparrow}V'_{m-1}$. The left to right inclusion is straightforward. Just note that $Z \subseteq {\uparrow}W_{m-1}$ by the 1-point Lemma 5.10, so using the definition of $V_{m-1} = W_{m-1} \cup Z$ yields

$$ {\uparrow}V_{m-1} = {\uparrow}(W_{m-1} \cup Z) = {\uparrow}W_{m-1}. $$

Consequently, to show that the desired inclusion holds it suffices to prove ${\uparrow}W_{m-1} \subseteq {\uparrow}V_{m-2} \cap {\uparrow}V'_{m-1}$. Just recall that, by their respective definitions, we have $V_{m-2} = W_{m-2}$ and $W'_{m-1} \subseteq V'_{m-1}$, hence the previous inclusion is now immediate from the equality

$${\uparrow}W_{m-1} = {\uparrow}W_{m-2} \cap {\uparrow}W'_{m-1},$$

a particular instance of (6) proved above.

To see that ${\uparrow}V_{m-2} \cap {\uparrow}V'_{m-1} \subseteq {\uparrow}V_{m-1}$, let us suppose that $x \in {\uparrow}V_{m-2} \cap {\uparrow}V'_{m-1}$. Note that by Proposition 5.19, we know that ${\downarrow}V_{m-2} \cap {\uparrow}V'_{m-1} = \varnothing$. As V'_{m-1} is a downset by Proposition 5.17, the previous equality is equivalent to ${\downarrow}V_{m-2} \cap {\uparrow}{\downarrow}V'_{m-1} = \varnothing$, which in turn is equivalent to ${\uparrow}{\downarrow}V_{m-2} \cap {\downarrow}V'_{m-1} = \varnothing$ by Lemma 5.3. In particular, this implies that if $x \in {\uparrow}V_{m-2} \cap {\uparrow}V'_{m-1}$, then $x \notin V'_{m-1}$, so we must have $x \in {\uparrow}V'_{m-1} \smallsetminus V'_{m-1}$.

We now prove that ${\uparrow}V'_{m-1} \smallsetminus V'_{m-1} \subseteq {\uparrow}V_{m-1}$, thus establishing our desired inclusion ${\uparrow}V_{m-2} \cap {\uparrow}V'_{m-1} \subseteq {\uparrow}V_{m-1}$. If $x \in {\uparrow}V'_{m-1} \smallsetminus V'_{m-1}$, then there exists $y \in V'_{m-1}$ such that $y \leqslant x$. As $V'_{m-1} \subseteq {\downarrow}V_{m-1}$ by Proposition 5.18, there must be a $z \in V_{m-1}$ such that $y \leqslant z$. Since $\mathcal{X}$ is a co-tree, we have $y \leqslant z \leqslant x$ or $y \leqslant x \leqslant z$, and both possibilities yield $x \in {\uparrow}V_{m-1}$. The former is clear, since $z \in V_{m-1}$. Assuming $y \leqslant x \leqslant z$ entails $x \in {\downarrow}V_{m-1}$. Equivalently,

$$x \in V_{m-1} \uplus {\downarrow}V_{m-2} \uplus V'_{m-1}$$

by Proposition 5.18. By hypothesis, $x \in {\uparrow}V'_{m-1} \smallsetminus V'_{m-1}$, so we not only know $x \notin V'_{m-1}$, but we can also infer $x \notin {\downarrow}V_{m-2}$, since ${\downarrow}V_{m-2} \cap {\uparrow}V'_{m-1} = \varnothing$ by Proposition 5.19. Thus, we must have $x \in V_{m-1} \subseteq {\uparrow}V_{m-1}$, as desired. We conclude

$${\uparrow}V_{m-1} = {\uparrow}V_{m-2} \cap {\uparrow}V'_{m-1}.$$

To finish the proof of the second part of the statement, it remains to show that

$${\uparrow}V_m = {\uparrow}V_{m-1} \cap {\uparrow}V'_m.$$

Let us recall the characterization of $V_m = W_m = {\downarrow}U_m \cap {\uparrow}W_{m-1} \cap {\uparrow}W'_m$ given in Lemma 5.6.(1), the definition of $V_{m-1} = W_{m-1} \cup Z$ and the equality $V'_m = W'_m \cup {\downarrow}Z' \cup Z_0$ established in Lemma 5.16. Notice that from the three previous equalities it is clear that the desired left to right inclusion holds true, so let us prove the reverse inclusion.

By the 1-point Lemma 5.10, we know that $Z \subseteq {\uparrow}W_{m-1}$. Similarly, the 1′-point Lemma 5.12 yields $Z' \subseteq {\uparrow}W'_m$. Consequently, we have both ${\uparrow}Z \subseteq {\uparrow}W_{m-1}$ and ${\uparrow}Z' \subseteq {\uparrow}W'_m$, hence what we need to prove can be written as

$$\begin{aligned} {\uparrow}V_{m-1} \cap {\uparrow}V'_m &= {\uparrow}(W_{m-1} \cup Z) \cap {\uparrow}(W'_m \cup {\downarrow}Z' \cup Z_0) \\ &= {\uparrow}W_{m-1} \cap ({\uparrow}W'_m \cup {\uparrow}{\downarrow}Z' \cup {\uparrow}Z_0) \\ &\subseteq {\uparrow}W_m = {\uparrow}V_m. \end{aligned}$$

To see that ${\uparrow}W_{m-1} \cap {\uparrow}W'_m \subseteq {\uparrow}W_m$, suppose that $x \in {\uparrow}W_{m-1} \cap {\uparrow}W'_m$, so there exists $w \in W_{m-1}$ such that $w \leqslant x$. As $W_{m-1} = X_{m-1} \cap {\downarrow}W_m$ by the definition of W_{m-1}, there must be a $y \in W_m$ satisfying $w \leqslant y$. So $w \leqslant x, y$ yields $x \leqslant y$ or $y \leqslant x$, since $\mathcal{X}$ is a co-tree. If $y \leqslant x$ we are done, as this implies $x \in {\uparrow}W_m$. If $x \leqslant y$, then notice that $y \in W_m = {\downarrow}U_m \cap {\uparrow}W_{m-1} \cap {\uparrow}W'_m$ entails $x \in {\downarrow}U_m$. This, together with our assumption $x \in {\uparrow}W_{m-1} \cap {\uparrow}W'_m$, yields $x \in {\downarrow}U_m \cap {\uparrow}W_{m-1} \cap {\uparrow}W'_m = W_m \subseteq {\uparrow}W_m$, as desired.

Let us now prove that ${\uparrow}W_{m-1} \cap {\uparrow}{\downarrow}Z' \subseteq {\uparrow}W_m$. Suppose $x \in {\uparrow}W_{m-1} \cap {\uparrow}{\downarrow}Z'$, so, in particular, there are $z' \in Z'$ and $y \in {\downarrow}z'$ satisfying $y \leqslant z', x$. Since $\mathcal{X}$ is a co-tree, this entails $x \leqslant z'$ or $z' \leqslant x$. But $x \leqslant z'$ cannot happen, as our assumption $x \in {\uparrow}W_{m-1}$ would then force $z' \in {\uparrow}W_{m-1}$, contradicting the definition of 1′-points (more specifically, that their downsets are disjoint from W_{m-1}). Thus, we must have $z' \leqslant x$. As mentioned above, the 1′-point Lemma 5.12 ensures $z' \in {\uparrow}W'_m$, so $z' \leqslant x$ now implies $x \in {\uparrow}W'_m$. This, together with our assumption $x \in {\uparrow}W_{m-1}$, yields $x \in {\uparrow}W_{m-1} \cap {\uparrow}W'_m$. Since we proved in the previous paragraph that ${\uparrow}W_{m-1} \cap {\uparrow}W'_m \subseteq {\uparrow}W_m$, we are done.

It remains to show ${\uparrow}W_{m-1} \cap {\uparrow}Z_0 \subseteq {\uparrow}W_m$. Take $x \in {\uparrow}W_{m-1} \cap {\uparrow}Z_0$ and suppose, with a view towards contradiction, that $x \notin {\uparrow}W_m$. Recall that $Z_0 \subseteq {\downarrow}W_m$ by the definition of 0-points, so $x \in {\uparrow}Z_0$ implies that there are $z \in Z_0$ and $w \in W_m$ such that $z \leqslant x, w$. As $\mathcal{X}$ is a co-tree, we have

$x \leqslant w$ or $w \leqslant x$. But $w \in W_m$ and we assumed $x \notin {\uparrow}W_m$, so we must have $x \leqslant w$. This shows $x \in {\downarrow}W_m \smallsetminus W_m$. Furthermore, that $z \leqslant x$ and $z \in Z_0$ forces $x \notin {\downarrow}W_{m-1} \cup W'_m$, since 0-points are defined to lie outside of this union (recall that $W'_m = {\downarrow}U'_m \cup {\downarrow}W_m$ is a downset by definition).

We have proved

$$x \in {\downarrow}W_m \smallsetminus (W_m \cup {\downarrow}W_{m-1} \cup W'_m),$$

i.e., that x is an inconvenient point. This, together with our assumption $x \in {\uparrow}W_{m-1}$, implies ${\downarrow}x \cap {\downarrow}W_{m-1} \neq \emptyset$, by Lemma 5.8. Recall that 2-points, that is, inconvenient points satisfying

$${\downarrow}u \cap {\downarrow}W_{m-1} \neq \emptyset \neq {\downarrow}u \cap W'_m,$$

do not exist. Therefore, it follows that x is an inconvenient point satisfying

$${\downarrow}x \cap {\downarrow}W_{m-1} \neq \emptyset = {\downarrow}x \cap W'_m,$$

i.e., x satisfies the definition of a 1-point. But now our assumption $x \in {\uparrow}Z_0$ yields a contradiction, since by definition, no 0-point lies below a 1-point. We conclude that $x \in {\uparrow}W_m = {\uparrow}V_m$, as desired. Therefore, we have established

$${\uparrow}V_m = {\uparrow}V_{m-1} \cap {\uparrow}V'_m,$$

and finished the proof that the second part of the statement holds.

Finally, let us show that ${\uparrow}V'_1 = {\uparrow}V_1 \uplus V'_1$. Notice that the definitions of V_1 and V'_1 depend on whether $m > 2$ or $m = 2$. If $m > 2$, we need to prove that ${\uparrow}V'_1 = {\uparrow}W'_1 = {\uparrow}W_1 \uplus W'_1$. Using the definitions of these clopens, together with Lemma 5.6.(5) and (IH.iv), we have

$$\begin{aligned}{\uparrow}W'_1 &= ({\uparrow}X'_1 \cap {\downarrow}W_m) \cup {\uparrow}W_m = (({\uparrow}X_1 \cup X'_1) \cap {\downarrow}W_m) \cup {\uparrow}W_m \\ &= (({\uparrow}X_1 \cap {\downarrow}W_m) \cup (X'_1 \cap {\downarrow}W_m)) \cup {\uparrow}W_m \\ &= (({\uparrow}X_1 \cap {\downarrow}W_m) \cup {\uparrow}W_m) \cup (X'_1 \cap {\downarrow}W_m) = {\uparrow}W_1 \cup W'_1.\end{aligned}$$

To see that ${\uparrow}W_1$ and W'_1 are in fact disjoint, notice that $W'_1 \subseteq {\downarrow}W_1 \smallsetminus W_1$ by Proposition 5.18 and that W_1 is convex by Proposition 5.17, so ${\uparrow}W_1 \cap W'_1 = \emptyset$ follows from Lemma 5.2. We have proved ${\uparrow}V_1 = {\uparrow}V_1 \uplus V'_1$ if $m > 2$, as desired.

Suppose now that $m = 2$. Recall that we proved above ${\uparrow}V'_{m-1} \smallsetminus V'_{m-1} \subseteq {\uparrow}V_{m-1}$. Since we are assuming that $m = 2$, this means ${\uparrow}V'_1 \smallsetminus V'_1 \subseteq {\uparrow}V_1$, and it is now clear that ${\uparrow}V'_1 \subseteq {\uparrow}V_1 \cup V'_1$. To prove the reverse inclusion, noting that clearly $V'_1 \subseteq {\uparrow}V'_1$, it suffices to show ${\uparrow}V_1 \subseteq {\uparrow}V'_1$, i.e., that ${\uparrow}(W_1 \cup Z) \subseteq {\uparrow}V'_1$. Equivalently, that ${\uparrow}W_1 \subseteq {\uparrow}V'_1$, since $Z \subseteq {\uparrow}W_1$ by the 1-point Lemma 5.10. By using the same argument as above, we can show that ${\uparrow}W'_1 = {\uparrow}W_1 \cup W'_1$, and thus infer ${\uparrow}W_1 \subseteq {\uparrow}W'_1 \subseteq {\uparrow}V'_1$, as desired. We have proved that the equality ${\uparrow}V'_1 = {\uparrow}V_1 \cup V'_1$ holds, so it remains to show that the sets on the right side are disjoint. Just notice that as $V'_1 \subseteq {\downarrow}V_1 \smallsetminus V_1$ by Proposition 5.18 and V_1 is convex by Proposition 5.17, ${\uparrow}V_1 \cap V'_1 = \emptyset$ follows from Lemma 5.2. ☐

With the four previous results now proven, we have finished the induction step of our proof by induction, thus showing that Proposition 5.4 holds true. We are finally ready to prove the main result of this subsection:

Theorem 5.21. *Let $n \in \mathbb{Z}^+$. If $\mathcal{X}$ is a bi-Esakia co-tree, then $\mathcal{X}$ admits the n-comb $\mathfrak{C}_n$ as a subposet iff $\mathfrak{C}_n$ is a bi-Esakia morphic image of $\mathcal{X}$.*

Proof. We first prove the left to right implication. Suppose that $\mathcal{X}$ admits $\mathfrak{C}_n$ as a subposet. Without loss of generality, we can assume that the co-root x_n of $\mathfrak{C}_n$ is identified with the co-root of $\mathcal{X}$. By Proposition 5.4, there are clopens $X_1, \dots, X_n, X'_1, \dots, X'_n \subseteq \mathcal{X}$ satisfying:

(i) For all $i \leqslant n$, we have that X_i is a convex set containing x_i, that X'_i is a downset containing x'_i, and that $X_i \cap {\uparrow}{\downarrow}x'_j = \emptyset$ for all j such that $i < j \leqslant n$;

(ii) ${\downarrow}X_1 = X_1 \uplus X'_1$ and ${\downarrow}X_i = X_i \uplus {\downarrow}X_{i-1} \uplus X'_i$, for all $1 < i \leqslant n$;

(iii) ${\downarrow}X_{i-1} \cap {\uparrow}X'_i = \emptyset$, for all $1 < i \leqslant n$;

(iv) ${\uparrow}X'_1 = {\uparrow}X_1 \uplus X'_1$ and ${\uparrow}X_i = {\uparrow}X_{i-1} \cap {\uparrow}X'_i$, for all $1 < i \leqslant n$.

Notice that since x_n is the co-root of $\mathcal{X}$, then $x_n \in X_n$ entails ${\downarrow}X_n = \mathcal{X}$. By successive applications of condition (ii) to this equality, we have

$$\begin{aligned}\mathcal{X} &= {\downarrow}X_n = X_n \uplus {\downarrow}X_{n-1} \uplus X'_n = (X_n \uplus X'_n) \uplus {\downarrow}X_{n-1}\\ &= (X_n \uplus X'_n) \uplus (X_{n-1} \uplus {\downarrow}X_{n-2} \uplus X'_{n-1})\\ &= (X_n \uplus X'_n) \uplus (X_{n-1} \uplus X'_{n-1}) \uplus {\downarrow}X_{n-2}\\ &= (X_n \uplus X_{n-1} \uplus X'_n \uplus X'_{n-1}) \uplus {\downarrow}X_{n-2}\\ &= \cdots = \biguplus_{i=1}^{n} X_i \uplus \biguplus_{i=1}^{n} X'_i.\end{aligned}$$

We define the map $f\colon \mathcal{X} \to \mathfrak{C}_n$ by

$$f(z) := \begin{cases} x_i & \text{if } z \in X_i,\\ x'_i & \text{if } z \in X'_i,\end{cases}$$

and prove that it is a surjective bi-Esakia morphism. That f is well-defined follows immediately from $\mathcal{X} = \biguplus_{i=1}^{n} X_i \uplus \biguplus_{i=1}^{n} X'_i$. Moreover, since for each $i \leqslant n$, X_i and X'_i are clopens containing x_i and x'_i, respectively, it is clear by definition that f is both continuous and surjective.

We now prove that f is order preserving. As X_n contains the co-root x_n of $\mathcal{X}$, then for any $w \in {\uparrow}X_n$, since we always have $w \leqslant x_n$, it follows $w \in {\uparrow}X_n \cap {\downarrow}X_n$. But X_n is convex by condition (i), hence we proved ${\uparrow}X_n \subseteq X_n$, and it is now clear that $X_n = {\uparrow}X_n$. This fact will be used to establish the last equality of the display below, whose other (nontrivial) equalities follow from successive applications of Lemma 5.5.(4), in particular, of the fact that ${\uparrow}X_j \smallsetminus X_j = {\uparrow}X_{j+1}$, for all $j < n$. For each $i \leqslant n$, we have

$$\begin{aligned}{\uparrow}X_i &= X_i \uplus ({\uparrow}X_i \smallsetminus X_i) = X_i \uplus {\uparrow}X_{i+1}\\ &= X_i \uplus (X_{i+1} \uplus ({\uparrow}X_{i+1} \smallsetminus X_{i+1}))\\ &= X_i \uplus X_{i+1} \uplus {\uparrow}X_{i+2}\\ &= X_i \uplus X_{i+1} \uplus (X_{i+2} \uplus ({\uparrow}X_{i+2} \smallsetminus X_{i+2}))\\ &= X_i \uplus X_{i+1} \uplus X_{i+2} \uplus {\uparrow}X_{i+3}\\ &= \dots\\ &= X_i \uplus X_{i+1} \uplus \cdots \uplus {\uparrow}X_n\\ &= X_i \uplus X_{i+1} \uplus \cdots \uplus X_n.\end{aligned}$$

Recall that the aforementioned lemma also ensures ${\uparrow}X'_i \smallsetminus X'_i = {\uparrow}X_i$, so the above display immediately yields

$${\uparrow}X'_i = X'_i \uplus ({\uparrow}X'_i \smallsetminus X'_i) = X'_i \uplus {\uparrow}X_i = X'_i \uplus X_i \uplus X_{i+1} \uplus \cdots \uplus X_n.$$

Using the descriptions of $\mathcal{X}$, ${\uparrow}X_i$, and ${\uparrow}X'_i$ proved above, it is now easy to see that f is indeed order preserving. For suppose $z \leqslant y \in \mathcal{X}$. As $\mathcal{X} = \biguplus_{i=1}^{n} X_i \uplus \biguplus_{i=1}^{n} X'_i$, either $z \in X_i$ or $z \in X'_i$, for some $i \leqslant n$. If $z \in X_i$, then

$$y \in {\uparrow}X_i = X_i \uplus X_{i+1} \uplus \cdots \uplus X_n$$

implies $y \in X_j$, for some $j \geqslant i$, and therefore that $f(z) = x_i \leqslant x_j = f(y)$. If $z \in X'_i$, then

$$y \in {\uparrow}X'_i = X'_i \uplus X_i \uplus X_{i+1} \uplus \cdots \uplus X_n$$

entails either $y \in X'_i$ or $y \in X_j$, for some $j \geqslant i$. Thus, either $f(z) = x'_i = f(y)$, or $f(z) = x'_i \leqslant x_j = f(y)$. We conclude that f is order preserving.

Next we show that f satisfies the up condition (see the definition of a bi-p-morphism 2.9). To this end, suppose that $f(z) \leqslant x$, for some $z \in \mathcal{X}$ and $x \in \mathfrak{C}_n$. If x is of the form x'_i, for some $i \leqslant n$, then x is a minimal point of $\mathfrak{C}_n$ (recall our definition of the n-comb in Figure 5). So $f(z) \leqslant x$ forces $f(z) = x'_i$ and we are done.

If instead we have $x = x_i$, for some $i \leqslant n$, then $f(z) \leqslant x_i$ and the order of $\mathfrak{C}_n$ entail $f(z) \in \{x_j, x_j'\}$, for some $j \leqslant i$. By Lemma 5.5.(1), we know

$$X_j' \subseteq {\downarrow} X_j \subseteq {\downarrow} X_{j+1} \subseteq \cdots \subseteq {\downarrow} X_i.$$

It is now clear that both possibilities for $f(z) \in \{x_j, x_j'\}$ (equivalently, by the definition of f, $z \in X_j \cup X_j'$), yield $z \in {\downarrow} X_i$, i.e., that there exists a $y \in X_i$ satisfying $z \leqslant y$ and $f(y) = x_i$, as desired.

It remains to prove that f satisfies the down condition (see Definition 2.9). Let $z \in \mathcal{X}$ and $x \in \mathfrak{C}_n$, and suppose $x \leqslant f(z)$. As the case $x = f(z)$ is trivial, we can assume without loss of generality that $x < f(z)$. In particular, we are assuming that $f(z)$ is not a minimal point of $\mathfrak{C}_n$, which forces $f(z) = x_i$, for some $i \leqslant n$. This has two immediate consequences: $z \in X_i$, by our definition of the map f; and $x = x_i'$ or $x \in \{x_j, x_j'\}$ for some $j < i$, by the structure of $\mathfrak{C}_n$.

If $x = x_i'$, then the inclusion $X_i \subseteq {\uparrow} X_i'$ (recall that ${\uparrow} X_i' \smallsetminus X_i' = {\uparrow} X_i$, by Lemma 5.5.(4)) ensures the existence of a point $y \in X_i'$ satisfying $y \leqslant z \in X_i$ and $f(y) = x_i'$, as required.

On the other hand, if $x \in \{x_j, x_j'\}$ for some $j < i$, then as

$$X_i \subseteq {\uparrow} X_i \subseteq {\uparrow} X_{i-1} \subseteq \cdots \subseteq {\uparrow} X_j \subseteq {\uparrow} X_j'$$

follows from Lemma 5.5.(2), and thus $z \in X_i \subseteq {\uparrow} X_j \subseteq {\uparrow} X_j'$, we can easily find, for both possibilities on $x \in \{x_j, x_j'\}$, a point $y \in {\uparrow} X_j'$ such that $y \leqslant z$ and $f(y) = x$, as desired. This finishes the proof that f satisfies the down condition, and we conclude that f is a surjective bi-Esakia morphism.

Therefore, $\mathfrak{C}_n$ is a bi-Esakia morphic image of $\mathcal{X}$, and we proved the left to right implication of the desired equivalence.

The reverse implication is straightforward. Suppose that $\mathfrak{C}_n$ is a bi-Esakia morphic image of $\mathcal{X}$. By the Dual Subframe Jankov Lemma 4.23, it is clear that $\mathfrak{C}_n \nvDash \beta(\mathfrak{C}_n^*)$, since $\mathfrak{C}_n$ trivially order-embeds into itself. As the validity of formulas is preserved under taking bi-Esakia morphic images, it follows $\mathcal{X} \nvDash \beta(\mathfrak{C}_n^*)$, i.e., that $\mathfrak{C}_n$ order-embeds into $\mathcal{X}$, again by the aforementioned lemma. Therefore, $\mathfrak{C}_n$ can be regarded as a subposet of $\mathcal{X}$, as desired. □

We can now derive the following corollary, thus finishing the first step in the proof of our criterion.

Corollary 5.22. *Let $n \in \mathbb{Z}^+$. If $\mathcal{X}$ is a bi-Esakia co-forest, then $\mathcal{X} \nvDash \beta(\mathfrak{C}_n^*)$ iff $\mathcal{X} \nvDash \mathcal{J}(\mathfrak{C}_n^*)$. Equivalently, if $\mathbf{B} \in$ bi-GA, then $\mathbf{B} \nvDash \beta(\mathfrak{C}_n^*)$ iff $\mathbf{B} \nvDash \mathcal{J}(\mathfrak{C}_n^*)$.*

Proof. We prove the second part of the statement, which is equivalent to the first part by duality. Let n be a positive integer and $\mathbf{B} \in$ bi-GA. Note the following equivalences:

$$\begin{aligned} \mathbf{B} \nvDash \beta(\mathfrak{C}_n^*) &\iff \mathfrak{C}_n \text{ order-embeds into } \mathbf{D}_*, \text{ for some } \mathbf{D} \in \mathbb{H}(\mathbf{B})_{SI} \\ &\iff \mathbf{D} \nvDash \mathcal{J}(\mathfrak{C}_n^*), \text{ for some } \mathbf{D} \in \mathbb{H}(\mathbf{B})_{SI} \\ &\iff \mathbf{B} \nvDash \mathcal{J}(\mathfrak{C}_n^*). \end{aligned}$$

The first equivalence follows from (1) $\iff$ (4) of the Dual Subframe Jankov Lemma 4.23. To see that the second equivalence holds, notice that since $\mathbf{D}$ is an SI bi-Gödel algebra, it follows from Theorem 3.7 that $\mathbf{D}$ has no nontrivial homomorphic images and that $\mathbf{D}_*$ is a co-tree. By the previous theorem, $\mathfrak{C}_n$ order-embeds into $\mathbf{D}_*$ iff $\mathfrak{C}_n$ is a bi-Esakia morphic image of $\mathbf{D}_*$, which in turn, by duality and by our previous comment, is equivalent to $\mathfrak{C}_n^* \in \mathbb{S}(\mathbf{D}) = \mathbb{SH}(\mathbf{D})$. Equivalently, $\mathbf{D} \nvDash \mathcal{J}(\mathfrak{C}_n^*)$, by the Jankov Lemma 4.9, and we established the second equivalence.

Finally, the last equivalence is an immediate consequence of the aforementioned Jankov Lemma. □

5.2. **Step 2.** The nontrivial part of this step consists in showing that the algebraic duals of the finite combs are all 1-generated as bi-Heyting algebras. Before we prove this, we need a short lemma about bi-E-partitions on bi-Esakia spaces (see Definition 2.16).

Lemma 5.23. *Let $\mathcal{X}$ be a bi-Esakia space and E a bi-E-partition on $\mathcal{X}$. If $x \leqslant y \leqslant z \in \mathcal{X}$, then $(x,z) \in E$ implies $(x,y) \in E$.*

In particular, if $\mathcal{X}$ is the n-comb $\mathfrak{C}_n$ and $(x_i, x_j) \in E$ for some $i < j \leqslant n$, then $(x_i, x_{i+1}) \in E$.

Proof. Suppose $x \leqslant y \leqslant z$ and $(x,z) \in E$. With a view towards contradiction, we assume $(x,y) \notin E$. By the refined condition of E, there exists an E-saturated clopen upset V that separates x and y. As $x \leqslant y$, y must be the point contained in V, since V is an upset. But then $y \leqslant z$ entails $z \in V$. As $(x,z) \in E$ and V is E-saturated, we now have that $x \in V$, contradicting the definition of V.

The second part of the statement is clearly a particular instance of what we just proved, since if $i < j \leqslant n$, then the order of $\mathfrak{C}_n$ entails $x_i < x_{i+1} \leqslant x_j$. □

Proposition 5.24. *$\mathfrak{C}_n^*$ is a 1-generated bi-Heyting algebra, for every $n \in \mathbb{Z}^+$.*

Proof. Firstly, the algebraic dual of the 1-comb is the three element chain, which is generated as a bi-Heyting algebra by its only element distinct from 0 and 1. Let then $n \geqslant 2$ and recall that by the Coloring Theorem 2.19, to show that $\mathfrak{C}_n^*$ is 1-generated, i.e., that there exists $U \in Up(\mathfrak{C}_n)$ such that $\mathfrak{C}_n^* = \langle U \rangle$, it suffices to show that every proper bi-E-partition on $\mathfrak{C}_n$ identifies points of different colors, where the coloring of $\mathfrak{C}_n = (C_n, \leqslant)$ is defined by

$$col(w) = \begin{cases} 1 & \text{if } w \in U, \\ 0 & \text{if } w \notin U, \end{cases}$$

for all $w \in C_n$. To this end, let $U := \{x_1\} \cup {\uparrow}\{x'_i \in C_n \colon i \text{ is even}\}$ and E be a proper bi-E-partition on $\mathfrak{C}_n$.

We now prove by complete induction on $i < n$ that if $(x_i, x_{i+1}) \in E$, then E must identify points of different colors. Suppose $(x_i, x_{i+1}) \in E$ and that our induction hypothesis holds true for all $j < i$. By the definition of E, in particular, since it satisfies the down condition, the fact $x'_{i+1} \leqslant x_{i+1}$ together with our assumption $(x_i, x_{i+1}) \in E$ entails $(w, x'_{i+1}) \in E$, for some $w \leqslant x_i$. By the structure of $\mathfrak{C}_n$, the condition $w \leqslant x_i$ can be split into three cases: $w = x'_i$, or $w = x'_j$ for some $j < i$, or $w = x_j$ for some $j \leqslant i$.

If $w = x'_i$, then $(x'_i, x'_{i+1}) \in E$ and we are done, since $col(x'_i) \neq col(x'_{i+1})$, by our definition of the coloring of $\mathfrak{C}_n$.

If $w = x'_j$ for some $j < i$, then applying the up condition of E to $x'_j \leqslant x_j$ and $(x'_j, x'_{i+1}) \in E$ yields $(z, x_j) \in E$, for some $z \in {\uparrow}x'_{i+1}$. If $z = x'_{i+1}$, then by the down condition, and noting that x'_{i+1} is minimal in $\mathfrak{C}_n$, it follows that ${\downarrow}x_j \subseteq [\![x'_{i+1}]\!]_E$, where $[\![x'_{i+1}]\!]_E$ is the E-equivalence class of x'_{i+1}. Since $x_1, x'_1 \in {\downarrow}x_j$, the inclusion ${\downarrow}x_j \subseteq [\![x'_{i+1}]\!]_E$ forces $(x_1, x'_1) \in E$ and we are done, since $col(x'_1) = 0 \neq 1 = col(x_1)$. On the other hand, if $z \neq x'_{i+1}$, then we must have $z = x_t$, for some $t \geqslant i+1$. As $(x_j, z) \in E$, i.e., $(x_j, x_t) \in E$, and since $j < i < t$, Lemma 5.23 entails $(x_j, x_{j+1}) \in E$, which falls under our induction hypothesis. Thus, E identifies points of different colors, as desired.

It remains to consider the case $w = x_j$, for $j \leqslant i$. Just note that in this case, $(x_j, x'_{i+1}) \in E$ and the fact that x'_{i+1} is a minimal point entail ${\downarrow}x_j \subseteq [\![x'_{i+1}]\!]_E$, by the down condition of E. By the same reasoning as above, this implies that x_1 and x'_1, two points of different colors, are E-equivalent, as desired.

We are finally ready to prove that if E is a proper bi-E-partition on $\mathfrak{C}_n$, then E identifies points of different colors, and therefore the Coloring Theorem ensures that $\mathfrak{C}_n^* = \langle U \rangle$ is a 1-generated bi-Heyting algebra. Let $i < j \leqslant n$. If $(x_i, x_j) \in E$, then Lemma 5.23 entails $(x_i, x_{i+1}) \in E$, and the result now follows by our previous discussion. If $(x'_i, x'_j) \in E$ or $(x_i, x'_j) \in E$, using the same argument as the one detailed above (namely, when dealing with the case "$w = x'_j$" in the proof by induction) yields E-equivalent points of different colors, as desired. The only cases that remain are either $(x'_i, x_i) \in E$ or $(x'_i, x_j) \in E$, which are clear, since the minimality of x'_i implies either ${\downarrow}x_i \subseteq [\![x'_i]\!]_E$ or ${\downarrow}x_j \subseteq [\![x'_i]\!]_E$, respectively, hence $x'_1 E x_i E x_1$. Since we have $col(x'_1) \neq col(x_1)$, the result follows. □

Corollary 5.25. *Let* V *be a variety of bi-Gödel algebras. If* V *contains all the algebraic duals of the finite combs, then* V *is not locally finite.*

Proof. Let us note that, by definition, the finite combs are arbitrarily large finite co-trees. Hence, their bi-Heyting duals are arbitrarily large finite bi-Gödel algebras. If we assume that these duals are all contained in V, then it follows from the previous proposition that there are arbitrarily large finite 1-generated algebras in V. Therefore, the 1-generated free V-algebra must be infinite, and thus V cannot be locally finite. ☐

5.3. **Step 3.** The third and last result we will need to prove Theorem 5.1 consists in establishing, for arbitrary $n, m \in \mathbb{Z}^+$, the existence of a natural bound $k(n,m)$ for the size of m-generated SI bi-Gödel algebras whose bi-Esakia duals do not admit the n-comb as a subposet. We need a few auxiliary lemmas.

Definition 5.26. Given a poset $\mathcal{X}$ and a chain $H \subseteq \mathcal{X}$ with a least element m_0 and a greatest element m_1, we say that H is an *isolated chain* (in $\mathcal{X}$) if

$$\downarrow m_1 \smallsetminus H = \downarrow m_0 \smallsetminus \{m_0\} \text{ and } \uparrow m_0 \smallsetminus H = \uparrow m_1 \smallsetminus \{m_1\}.$$

Example 5.27. Consider the poset $\mathcal{X}$ depicted in Figure 6. The set $H := \{m_0, d, m_1\}$ forms an isolated chain in $\mathcal{X}$, since $\downarrow m_1 \smallsetminus H = \{e, f\} = \downarrow m_0 \smallsetminus \{m_0\}$ and $\uparrow m_0 \smallsetminus H = \{b, c, a\} = \uparrow m_1 \smallsetminus \{m_1\}$. On the other hand, the chain $G := \{m_1, b, a\}$ is not isolated in $\mathcal{X}$, since, for example, $c \in \downarrow a \smallsetminus G$ but $c \notin \downarrow m_1 \smallsetminus \{m_1\}$.

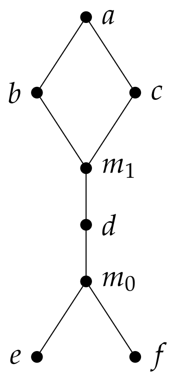

FIGURE 6. The poset $\mathcal{X}$

Lemma 5.28. *If $\mathcal{X}$ is a bi-Esakia space and $H \subseteq \mathcal{X}$ is an isolated chain, then $E := H^2 \cup Id_X$, the least equivalence relation identifying the points in H, is a bi-E-partition on $\mathcal{X}$.*

Proof. That E is an equivalence relation that satisfies the up and down conditions follows immediately from the definition of E and that of an isolated chain. It remains to show that E is refined, i.e., that every two non-E-equivalent points are separated by an E-saturated clopen upset of $\mathcal{X}$ (notice that, by the definition of E, a clopen upset U is E-saturated iff $H \subseteq U$ or $H \cap U = \emptyset$). To this end, let $w, v \in X$ and suppose $(w, v) \notin E$. There are only two possible cases: either $w, v \notin H$, or, without loss of generality, $w \in H$ and $v \notin H$.

We first suppose that $w, v \notin H$. Since $(w, v) \notin E$, we have $w \neq v$, and we can suppose without loss of generality that $w \nleq v$. By the PSA, there exists $U \in ClopUp(\mathcal{X})$ satisfying $w \in U$ and $v \notin U$. If $w < m_1 := MAX(H)$, then H being an isolated chain in $\mathcal{X}$ and $w \notin H$ imply $w < m_0 := Min(H)$, hence we have $H \subseteq U$, since U is an upset containing w. Thus, U is an E-saturated clopen upset that separates w from v. On the other hand, if $w \nless m_1$, then by the PSA there exists $V \in ClopUp(\mathcal{X})$ such that $w \in V$ and $m_1 \notin V$. Since V is an upset not containing m_1, it follows $H \cap V = \emptyset$, and it is easy to see that $U \cap V$ is an E-saturated clopen upset that separates w from v, as desired.

Suppose now that $w \in H$ and $v \notin H$. If $m_1 \nleq v$, then we also have $m_0 \nleq v$, by the definition of an isolated chain. The PSA now yields $U \in ClopUp(\mathcal{X})$ satisfying $m_0 \in U$ and $v \notin U$. Since

U is an upset containing m_0, we have $H \subseteq U$ and clearly U satisfies our desired conditions. On the other hand, if $m_1 \leqslant v$, we must have $m_1 < v$ because $v \notin H$. Consequently, in this case, we have $v \nleqslant m_1$. Therefore, we can apply the PSA obtaining some $V \in ClopUp(\mathcal{X})$ such that $v \in V$ and $m_1 \notin V$. Since V is an upset, it follows $H \cap V = \emptyset$, and we conclude that V is an E-saturated clopen upset that separates v from w, as desired. □

Recall that an *order-isomorphism* is an order-invariant bijection between posets (in other words, a surjective order-embedding), and that given two points w and v in a poset $\mathcal{X} = (X, \leqslant)$, we denote $[w, v] := \{x \in X : w \leqslant x \leqslant v\}$. Notice that if $\mathcal{X}$ is a co-forest, then $[w, v]$ is a chain.

Lemma 5.29. *Let $\mathcal{X}$ be a bi-Esakia co-forest and $w, v \in X$ two distinct points with a common immediate successor. If both $\downarrow w$ and $\downarrow v$ are finite, and there exists an order-isomorphism $f : \downarrow w \to \downarrow v$, then*

$$E := \big\{(x, y) \in X^2 : \big(x \in \downarrow w \text{ and } f(x) = y\big) \text{ or } \big(x \in \downarrow v \text{ and } f(y) = x\big)\big\} \cup Id_X$$

is a bi-E-partition on $\mathcal{X}$.

Proof. We start by noting that, by its definition, E is clearly an equivalence relation. Furthermore, that E satisfies the down condition is immediate from the definition of E and that of an order-isomorphism. Since we assumed that w and v share an immediate successor, and since in a co-forest points have at most one immediate successor, it follows that E satisfies the up condition. To see this, let us denote the unique immediate successor of both w and v by u, and note that, for $x \in \downarrow w$ (or $x \in \downarrow v$), we have a description $\uparrow x = [x, w] \uplus \uparrow u$ (respectively, $\uparrow x = [x, v] \uplus \uparrow u$), since the principal upsets of $\mathcal{X}$ are chains. Using this description of $\uparrow x$, the definition of E, and that of an order-isomorphism, it is now clear that E satisfies the up condition.

We now show that E is refined, thus ensuring that E is a bi-E-partition on $\mathcal{X}$. Let $x, y \in X$ and suppose that $(x, y) \notin E$. So $x \neq y$, and we can suppose without loss of generality that $x \nleqslant y$. We proceed by cases:

- <u>**Case 1:** $\{x, y\} \cap (\downarrow w \cup \downarrow v) = \emptyset$;</u>
  In this case, we have $x \nleqslant w$ and $x \nleqslant v$. Since we also assumed $x \nleqslant y$, by the PSA there are $U_y, U_w, U_v \in ClopUp(\mathcal{X})$ all containing x, and such that $y \notin U_y$, $w \notin U_w$, and $v \notin U_v$. As U_w is an upset not containing w, we have $\downarrow w \cap U_w = \emptyset$. Similarly, it follows $\downarrow v \cap U_v = \emptyset$. Thus, $U := U_y \cap U_w \cap U_v$ is an E-saturated (since $U \cap (\downarrow w \cup \downarrow v) = \emptyset$) clopen upset separating x from y, as desired.
- <u>**Case 2:** $x \notin \downarrow w \cup \downarrow v$ and $y \in \downarrow w \cup \downarrow v$;</u>
  By assumption, we have $x \nleqslant w$ and $x \nleqslant v$, so by the PSA there are $U_w, U_v \in ClopUp(\mathcal{X})$, both containing x, satisfying $w \notin U_w$ and $v \notin U_v$. As U_w is an upset not containing w, we have $\downarrow w \cap U_w = \emptyset$. Similarly, it follows $\downarrow v \cap U_v = \emptyset$. Thus, $U := U_w \cap U_v$ is an E-saturated (since $U \cap (\downarrow w \cup \downarrow v) = \emptyset$) clopen upset separating x from y, since we assumed $y \in \downarrow w \cup \downarrow v$. We note that the previous argument can also be used when $y \notin \downarrow w \cup \downarrow v$ and $x \in \downarrow w \cup \downarrow v$, by replacing x with y, and vice-versa.
- <u>**Case 3:** $x, y \in \downarrow w \cup \downarrow v$.</u>
  Without loss of generality, we suppose that $x \in \downarrow w$ and $y \in \downarrow w$ (if $y \in \downarrow v$ or $x \in \downarrow v$, we can replace y or x in the following argument by $f^{-1}(y)$ or $f^{-1}(x)$, respectively, where f^{-1} is the inverse of the order-isomorphism f). As $\downarrow w$ is finite by hypothesis, we can enumerate $\downarrow w \smallsetminus \downarrow y := \{x_1, \dots, x_n\}$. Notice that for all $i \leqslant n$, we have $x_i \nleqslant y$ by the definition of x_i, and $x_i \nleqslant f(y)$, since $f(y) \in \downarrow v$ and $x_i \notin \downarrow v$ (recall that w and v are distinct points in a co-forest with a common immediate successor, hence we have $\downarrow w \cap \downarrow v = \emptyset$). Using the same argument as in the previous cases, $x_i \nleqslant y$ and $x_i \nleqslant f(y)$ imply, by the PSA, that there exists $U_i \in ClopUp(\mathcal{X})$ satisfying $x_i \in U_i$ and $y, f(y) \notin U_i$. As U_i is an upset, it follows that $U_i \cap \big(\downarrow y \cup \downarrow f(y)\big) = \emptyset$. Furthermore, by the definition of an order-isomorphism, $x_i \nleqslant y$ entails $f(x_i) \nleqslant f(y)$, and since we have $f(x_i) \in \downarrow v$ and $y \in \downarrow w$, it follows $f(x_i) \nleqslant y$. Again, the PSA yields some $V_i \in ClopUp(\mathcal{X})$ satisfying $f(x_i) \in V_i$ and $V_i \cap \big(\downarrow y \cup \downarrow f(y)\big) = \emptyset$. Let $U := \bigcup_{i=1}^n U_i \cup \bigcup_{i=1}^n V_i$, and note that this is a clopen upset satisfying

$$\{x_1, \dots, x_n, f(x_1), \dots, f(x_n)\} \subseteq U \text{ and } U \cap \big(\downarrow y \cup \downarrow f(y)\big) = \emptyset.$$

As we assumed $x \in {\downarrow}w$ and $x \nleqslant y$, it now follows $x \in \{x_1, \ldots, x_n\} = {\downarrow}w \smallsetminus {\downarrow}y$, and thus $x \in U$. By the way we defined U and E, we conclude that U is an E-saturated clopen upset separating x from y.

Therefore, E is indeed a bi-E-partition on $\mathcal{X}$. □

We now have all the necessary tools to obtain the desired bound.

Proposition 5.30. *If n and m are positive integers, then there is a natural bound $k(n, m)$ (only dependent on n and m) on the size of m-generated SI bi-Gödel algebras whose bi-Esakia duals do not admit the n-comb $\mathfrak{C}_n$ as a subposet.*

Proof. Let n and m be positive integers. Take a bi-Esakia co-tree $\mathcal{X}$ which does not admit the n-comb as a subposet, and suppose that $\mathcal{X}^*$ is m-generated as a bi-Heyting algebra, so there are $U_1, \ldots, U_m \in ClopUp(\mathcal{X})$ such that $\mathcal{X}^* = \langle U_1, \ldots, U_m \rangle$. By the Coloring Theorem 2.19, every proper bi-E-partition on $\mathcal{X}$ must identify points of different colors, where the coloring of $\mathcal{X}$ is defined by $V(p_i) = U_i$, for $i \leqslant m$.

First we prove that if $w \in min(\mathcal{X})$ then $|{\uparrow}w| \leqslant (m+1) \cdot n$. Take $w \in min(\mathcal{X})$ and notice that, since the U_i are upsets, we can re-enumerate them in such a way as to satisfy

$${\uparrow}w \cap U_1 \subseteq \cdots \subseteq {\uparrow}w \cap U_m.$$

Set $H_1 := {\uparrow}w \cap U_1$, $H_i := {\uparrow}w \cap (U_i \smallsetminus U_{i-1})$ for every $i \in \{2, \ldots, m\}$, and $H_{m+1} := {\uparrow}w \smallsetminus U_m$. It is clear that ${\uparrow}w = \biguplus_{i=1}^{m+1} H_i$. We now show that $|H_i| \leqslant n$, for all $i \leqslant m+1$. For suppose this is not the case, i.e., that $|H_i| > n$ for some $i \leqslant m+1$. As H_i is, by definition, contained in the chain ${\uparrow}w$, H_i must also be a chain. Hence, $|H_i| > n$ implies that there exists a strictly ascending sequence $a_1 < \cdots < a_n < a_{n+1}$ contained in H_i.

Let $j \leqslant n$ and suppose that $[a_j, a_{j+1}]$ is an isolated chain in $\mathcal{X}$. By the definitions of H_i and of our coloring of $\mathcal{X}$, the inclusion $[a_j, a_{j+1}] \subseteq H_i$ forces all the points in this isolated chain to have the same color. But now Lemma 5.28 yields a proper (since $a_j < a_{j+1}$) bi-E-partition on $\mathcal{X}$ which does not identify points of different colors, contradicting the Coloring Theorem 2.19. Thus, the chain $[a_j, a_{j+1}]$ cannot be isolated.

Since $\mathcal{X}$ is a co-tree, it is clear that both

$${\uparrow}a_j \smallsetminus [a_j, a_{j+1}] = {\uparrow}a_{j+1} \smallsetminus \{a_{j+1}\} \text{ and } {\downarrow}a_j \smallsetminus \{a_j\} \subseteq {\downarrow}a_{j+1} \smallsetminus [a_j, a_{j+1}],$$

hold true. Therefore, the fact that the chain $[a_j, a_{j+1}]$ is not isolated in $\mathcal{X}$ (see Definition 5.26) implies

$${\downarrow}a_{j+1} \smallsetminus [a_j, a_{j+1}] \nsubseteq {\downarrow}a_j \smallsetminus \{a_j\}.$$

Equivalently, there must exist $x_j \in [a_j, a_{j+1}] \smallsetminus \{a_j\}$ such that ${\downarrow}x_j \smallsetminus ([a_j, a_{j+1}] \cup {\downarrow}a_j) \neq \emptyset$.

As j was arbitrary in the above discussion, we can now fix, for each $j \leqslant n$, an element $x'_j \in {\downarrow}x_j \smallsetminus ([a_j, a_{j+1}] \cup {\downarrow}a_j)$. Thus, we have found a subposet of $\mathcal{X}$,

$$(\{x_j \colon j \leqslant n\} \cup \{x'_j \colon j \leqslant n\}, \leqslant),$$

which is clearly a copy of the n-comb $\mathfrak{C}_n$, contradicting our hypothesis. Therefore, there can be no chain $a_1 < \cdots < a_n < a_{n+1}$ contained in H_i, and it follows $|H_i| \leqslant n$ for all $i \leqslant m+1$.

Consequently, we conclude that ${\uparrow}w = \biguplus_{i=1}^{m+1} H_i$ consists of at most $m+1$ pieces, each of size at most n, that is, $|{\uparrow}w| \leqslant (m+1) \cdot n$ as desired.

Since every point in a bi-Esakia space lies above a minimal one (see Proposition 2.14), it now follows from the definition of the depth of a co-tree that $dp(\mathcal{X}) \leqslant (m+1) \cdot n$. Notice that $\mathcal{X}$ being a co-tree of finite depth entails that every point distinct from its co-root r has a unique immediate successor. Let $\{w_i\}_{i \in I} \subseteq min(\mathcal{X})$, and suppose they all share their unique immediate successor, v. Note that there are only 2^m distinct colors, and that $i \neq j \in I$ implies $col(w_i) \neq col(w_j)$, otherwise Lemma 5.29 would contradict the Coloring Theorem. Thus, we have $|I| \leqslant 2^m$ and $|{\downarrow}v| \leqslant 2^m + 1$.

Now, let $u \in X$ be such that all of its strict predecessors are either minimal, or are immediate successors of minimal points. Set $\{v_i\}_{i \in I} := \{y \in X \colon y \prec u\}$, and notice that for all $i \in I$, we have $|{\downarrow}v_i| \leqslant 2^m + 1$ by above. Moreover, since there are only 2^m distinct colors, there exists a natural bound $b(m)$ for the number of possible distinct colored configurations (by which we mean poset structure together with a coloring) of the posets $\downarrow v_i$. As the v_i all share their unique immediate successor, we cannot have that for $i \neq j \in I$, $\downarrow v_i$ and $\downarrow v_j$ have both the same poset structure (i.e., there exists an order-isomorphism from $\downarrow v_i$ to $\downarrow v_j$) and coloring, otherwise Lemma 5.29 would contradict the Coloring Theorem. Hence $|I| \leqslant b(m)$, and we now have $|\downarrow u| \leqslant (2^m + 1) \cdot b(m) + 1$.

Since we have a natural bound for the depth of $\mathcal{X}$, we can now iterate the above argument a finite number of times (namely, at most $(m+1) \cdot n$ times) to find a bound $k_0(n,m) \in \omega$ for the size of X, i.e., $|X| = |\downarrow r| \leqslant k_0(n,m)$. By the nature of the argument that led to this bound, $k_0(n,m)$ depends only on n and m, and not on $\mathcal{X}$.

As there are only finitely many co-trees of size less than or equal to $k_0(n,m)$, it follows that there are only finitely many bi-Esakia co-trees which do not admit $\mathfrak{C}_n$ as a subposet and whose algebraic dual is m-generated. Therefore, we can now find a natural bound $k(n,m)$ (only dependent on $k_0(n,m)$) for the size of the bi-Heyting duals of these bi-Esakia co-trees. □

5.4. **The proof of Theorem 5.1 and a criterion for local tabularity.** We are finally ready to prove Theorem 5.1.

Proof. Let L be an extension of bi-GD. We start by proving the contrapositive of the left to right implication. Accordingly, let us suppose that for all $n \in \mathbb{Z}^+$, we have $\mathcal{J}(\mathfrak{C}_n^*) \notin L$. Equivalently, that $\mathsf{V}_L \not\models \mathcal{J}(\mathfrak{C}_n^*)$, by duality. It is now an immediate consequence of the Jankov Lemma 4.9 that V_L contains all the algebraic duals of the finite combs. By Corollary 5.25, V_L is not locally finite, and thus L is not locally tabular, as desired.

To prove the reverse implication, suppose that $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some $n \in \mathbb{Z}^+$. By duality, this is equivalent to $\mathsf{V}_L \models \mathcal{J}(\mathfrak{C}_n^*)$, which in turn is equivalent to $\mathsf{V}_L \models \beta(\mathfrak{C}_n^*)$ by Corollary 5.22. In particular, it now follows that for each positive $m \in \omega$, if $\mathbf{A}$ is an SI m-generated algebra contained in V_L, then $\mathbf{A} \models \beta(\mathfrak{C}_n^*)$. By the Dual Subframe Jankov Lemma 4.23, this is equivalent to $\mathbf{A}_*$ not admitting $\mathfrak{C}_n$ as a subposet. Since $\mathbf{A}$ satisfies all the conditions in the statement of Proposition 5.30, we now have that $|A| \leqslant k(n,m)$, and we can use Theorem 2.3 to conclude that V_L is locally finite, i.e., that L is locally tabular. □

We can now derive the following criterion for local tabularity. Let us denote the logic of the finite combs by $Log(FC) := \{\varphi \in Fm \colon \forall n \in \mathbb{Z}^+ \ (\mathfrak{C}_n \models \varphi)\}$.

Corollary 5.31. *If $L \in \Lambda(\text{bi-GD})$, then the following conditions are equivalent:*

(i) *L is locally tabular;*
(ii) *$\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some $n \in \mathbb{Z}^+$;*
(iii) *$\beta(\mathfrak{C}_n^*) \in L$, for some $n \in \mathbb{Z}^+$;*
(iv) *$L \nsubseteq Log(FC)$.*

Consequently, $Log(FC)$ is the only pre-locally tabular extension of bi-GD.

Proof. The equivalence (i) $\iff$ (ii) is just Theorem 5.1, while (ii) $\iff$ (iii) follows immediately by duality and Corollary 5.22.

We now prove (ii) $\iff$ (iv). Suppose $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some $n \in \mathbb{Z}^+$. Since $\mathfrak{C}_n^* \not\models \mathcal{J}(\mathfrak{C}_n^*)$ by the Jankov Lemma 4.9, it is clear that $L \nsubseteq Log(FC) \subseteq Log(\mathfrak{C}_n^*)$. Conversely, if $L \nsubseteq Log(FC)$, i.e., if there exists a finite comb satisfying $\mathfrak{C}_n \not\models L$, then the Jankov Lemma yields $\mathcal{J}(\mathfrak{C}_n^*) \in L$, as desired.

The last part of the statement is an immediate consequence of the equivalence (i) $\iff$ (iv). □

We close the paper by comparing some properties of the logic bi-GD (algebraized by bi-GA) with those of the thoroughly investigated linear calculus LC which is algebraized by the variety GA of Gödel algebras, i.e., the class of Heyting algebras satisfying the Gödel-Dummett axiom. In the table below, SREC is a short hand for strongly rooted Esakia chain (i.e., an Esakia chain with an isolated least element). The fact that $\Lambda(\text{bi-GD})$ is not a chain is an immediate consequence of the proof of Theorem 4.16, while the previous result clearly ensures that bi-GD is not locally tabular.

$\mathsf{LC} = \mathsf{IPC} + (p \to q) \vee (q \to p)$	$\mathsf{bi\text{-}GD} = \mathsf{bi\text{-}IPC} + (p \to q) \vee (q \to p)$
$\mathbf{A} \in \mathsf{GA} \iff \mathbf{A}_*$ is an Esakia co-forest	$\mathbf{A} \in \mathsf{bi\text{-}GA} \iff \mathbf{A}_*$ is a bi-Esakia co-forest
$\mathbf{A} \in \mathsf{GA}_{SI} \iff \mathbf{A}_*$ is a SREC	$\mathbf{A} \in \mathsf{bi\text{-}GA}_{SI} \iff \mathbf{A}_*$ is a bi-Esakia co-tree
LC has the FMP	bi-GD has the FMP
LC is locally tabular	bi-GD is not locally tabular
$\Lambda(\mathsf{LC})$ is a chain of order-type $(\omega + 1)^\partial$	$\Lambda(\mathsf{bi\text{-}GD})$ is of size $2^{\aleph_0}$ and is not a chain

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