Sahlqvist theory for deductive systems

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Intuitionistic logic IPC is complete with respect to

- Intuitionistic Kripke frames, i.e., posets $\mathbb{X} = \langle X, \leqslant \rangle$; and
- Heyting algebras, i.e., structures $\boldsymbol{A} = \langle A; \wedge, \vee, \rightarrow, 0, 1 \rangle$ that comprise a bounded lattice $\langle A; \wedge, \vee, 0, 1 \rangle$ and satisfy

$$a \wedge b \leqslant c \iff a \leqslant b \rightarrow c, \text{ for every } a, b, c \in A.$$

1965–1974: Kripke and Esakia connected the two as follows:

- With every Heyting algebra $\boldsymbol{A}$ we associate the poset $\boldsymbol{A}_*$ of meet irreducible (a.k.a. prime) filters of $\boldsymbol{A}$.
- With every poset $\mathbb{X}$ we associate a Heyting algebra

$$\mathsf{Up}(\mathbb{X}) := \langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \rightarrow, \emptyset, X \rangle$$

whose universe is the set of upsets of $\mathbb{X}$ and $\rightarrow$ is defied as

$$U \rightarrow V := \{x \in X : \text{for every } y \geqslant x, \text{ if } y \in U, \text{ then } y \in V\}.$$

In addition, every Heyting algebra $\boldsymbol{A}$ embeds into $\mathsf{Up}(\boldsymbol{A}_*)$ via the map defined by the rule

$$a \longmapsto \{F \in \boldsymbol{A}_* : a \in F\}.$$

Given a formula φ, two natural questions arise:

- Canonicity: Is it true that for every Heyting algebra A,

$$\text{if } A \vDash \varphi, \text{ then } \mathsf{Up}(A_*) \vDash \varphi?$$

- Correspondence: Is there a sentence $\mathsf{tr}(\varphi)$ in the language of posets such that for every poset $\mathbb{X}$,

$$\mathsf{Up}(\mathbb{X}) \vDash \varphi \iff \mathbb{X} \vDash \mathsf{tr}(\varphi)?$$

This holds true for the following formulas:

Definition

A modal formula is

- a Sahlqvist antecedent (SA) if it is constructed from atoms, negative formulas, and 0 and 1 using only $\wedge$ and $\vee$;
- a Sahlqvist implication (SI) if it is positive, or of the form $\neg\varphi$ for a SA φ, or of the form $\varphi \to \psi$ for a SA φ and a positive ψ;
- Sahlqvist if it is constructed from SI using only $\wedge$ and $\vee$.

- Example. $x \vee \neg x$ and $(x \to y) \vee (y \to x)$ are Sahlqvist, whence CPC and LC are axiomatized by Sahlqvist formulas.

Definition

A Sahlqvist quasiequation is an expression of the form

$$\Phi = \varphi_1 \wedge y \leqslant z \,\&\ldots\&\, \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z,$$

where $\varphi_1, \ldots, \varphi_n$ are Sahlqvist formulas.

Sahlqvist Theorem for IPC

The following holds for every Sahlqvist quasiequation Φ:

- Canonicity: For every Heyting algebra $\boldsymbol{A}$,

$$\text{if } \boldsymbol{A} \vDash \Phi, \text{ then } \mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi;$$

- Correspondence: There exists an effectively computable sentence $\mathsf{tr}(\Phi)$ such that for every poset $\mathbb{X}$:

$$\mathsf{Up}(\mathbb{X}) \vDash \Phi \Longleftrightarrow \mathbb{X} \vDash \mathsf{tr}(\Phi).$$

Example. The meaning of $\mathsf{tr}(x \vee \neg x)$ is "$\mathbb{X}$ is discrete" and that of $\mathsf{tr}((x \to y) \vee (y \to x))$ is "$\mathbb{X}$ is a root system".

Aim: To extend Sahlqvist canonicity to fragments of IPC with $\wedge$.

- The case of pseudocomplemented semilattices (PSL):

Let $\boldsymbol{A} \in$ PSL and

$$\Phi = \varphi_1 \wedge y \leqslant z \,\&\, \ldots \,\&\, \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z$$

a Sahlqvist quasiequation in $\wedge, \neg, 0$ only valid in $\boldsymbol{A}$.

- There's an embedding $f \colon \boldsymbol{A} \to \boldsymbol{B}$ for a HA $\boldsymbol{B}$ s.t. $\mathsf{Up}(\boldsymbol{B}_*) \vDash \Phi$.
- The partial map $f_* \colon \boldsymbol{B}_* \rightsquigarrow \boldsymbol{A}_*$ with

  $$\mathsf{dom}(f_*) \coloneqq \{F \in \boldsymbol{B}_* : f^{-1}[F] \in \boldsymbol{A}_*\}$$

  defined by the rule $F \longmapsto f^{-1}[F]$ is a surjective

  partial negative p-morphism,

  i.e., order preserving and s.t. for all $F \in \mathsf{dom}(f_*)$ and $G \in \boldsymbol{A}_*$,

  $$f_*(F) \subseteq G \Longrightarrow \exists H \in \mathsf{dom}(f_*) \text{ s.t. } F \subseteq H \text{ and } G \subseteq f_*(H).$$

- By duality, the map $(f_*)^* \colon \mathsf{Up}(\boldsymbol{A}_*) \to \mathsf{Up}(\boldsymbol{B}_*)$, defined by

  $$U \longmapsto \boldsymbol{B}_* \smallsetminus {\downarrow} f_*[\boldsymbol{A}_* \smallsetminus U],$$

  is an embedding of pseudocomplemented semilattices.
- Since $\mathsf{Up}(\boldsymbol{B}_*)$ validates Φ, we conclude that so does $\mathsf{Up}(\boldsymbol{A}_*)$.

Sahlqvist Canonicity for fragments of IPC with $\wedge$

Let Φ be a Sahlqvist quasiequation in the language of a fragment L of IPC comprising $\wedge$. For every L-subreduct $\boldsymbol{A}$ of a Heyting algebra,

if $\boldsymbol{A} \vDash \Phi$, then $\mathrm{Up}(\boldsymbol{A}_*) \vDash \Phi$.

- The excluded middle $x \vee \neg x$ can be rendered as

$$\Psi = x \wedge y \leqslant z \,\&\, \neg x \wedge y \leqslant z \Longrightarrow y \leqslant z$$

which is canonical in PSL. Furthermore, for every poset $\mathbb{X}$,

$$\mathrm{Up}(\mathbb{X}) \vDash \Psi \iff \text{the order of } \mathbb{X} \text{ is the identity}.$$

- The bounded top width n formula btw_n can be rendered as

$$\Phi_n = \underset{1 \leqslant i \leqslant n+1}{\&} \Big(\neg(\neg x_i \wedge \bigwedge_{0 < j < i} x_j) \wedge y \leqslant z \Big) \Longrightarrow y \leqslant z,$$

which is canonical in PSLs and for every poset $\mathbb{X}$,

$\mathrm{Up}(\mathbb{X}) \vDash \Phi_n \iff$ in principal upsets in $\mathbb{X}$, every $(n+1)$-element antichain is below an n-element one.

Remark. The formula btw_n cannot be rendered as an equation!

The intuitionistic Sahlqvist theory can be extended to any logic.

- A logic $\vdash$ is a substitution invariant finitary consequence relation on the set of formulas of some algebraic language.
- Given an algebra $\boldsymbol{A}$, let $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ be the lattice of deductive filters of $\vdash$ on $\boldsymbol{A}$, i.e., subsets of A closed under the rules of $\vdash$.
- The lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ is algebraic with semilattice of compact elements $\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})$.
- A logic $\vdash$ is protoalgebraic if there exists a set of formulas $\Delta(x, y)$ such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$.

Most logics with a very weak implication are protoalgebraic as witnessed by the set $\Delta = \{x \to y\}$.

A logic $\vdash$ is said to have:

- the inconsistency lemma (IL) when for every $n \in \mathbb{Z}^+$ there exists a finite set $\sim_n(x_1, \ldots, x_n)$ of formulas s.t.

$$\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \text{ is inconsistent} \quad \text{iff} \quad \Gamma \vdash \sim_n(\varphi_1, \ldots, \varphi_n);$$

- the deduction theorem (DT) when there exists a finite set $x \Rightarrow y$ of formulas s.t.

$$\Gamma, \varphi \vdash \psi \quad \text{iff} \quad \Gamma \vdash \varphi \Rightarrow \psi;$$

- the proof by cases (PC) when there exists a finite set $x \curlyvee y$ of formulas s.t.

$$\Gamma, \varphi \vdash \gamma \text{ and } \Gamma, \psi \vdash \gamma \quad \text{iff} \quad \Gamma, \varphi \bigcurlyvee \psi \vdash \gamma.$$

The structure of the semilattices $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ determines the validity of these properties in a logic $\vdash$.

Theorem (Blok & Pigozzi, Czelakowski & Dziobiak, Raftery)

A protoalgebraic logic $\vdash$ has the IL (resp. DT, PC) iff the semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is pseudocomplemented (resp. implicative semilattice, distributive lattice) for every algebra $\boldsymbol{A}$.

A formula $\varphi(x_1, \ldots, x_n)$ of IPC is compatible with a logic $\vdash$ when

- If $\neg$ occurs in φ, then $\vdash$ has the IL;
- If $\rightarrow$ occurs in φ, then $\vdash$ has the DT;
- If $\vee$ occurs in φ, then $\vdash$ has the PC.

In this case, for every $k \in \mathbb{Z}^+$ we define a finite set

$$\boldsymbol{\varphi}^k(x_1^1, \ldots, x_1^k, \ldots, x_n^1, \ldots, x_n^k)$$

of formulas of $\vdash$ as follows:

- If $\varphi = x_i$, then $\boldsymbol{\varphi}^k := \{x_i^1, \ldots, x_i^k\}$;
- If $\varphi = \psi_1 \wedge \psi_2$, then $\boldsymbol{\varphi}^k := \boldsymbol{\psi}_1^k \cup \boldsymbol{\psi}_2^k$;
- If $\varphi = \neg\psi$ and $\boldsymbol{\psi}^k = \{\chi_1, \ldots, \chi_m\}$, then

  $$\boldsymbol{\varphi}^k := \sim_m(\chi_1, \ldots, \chi_m),$$

  where $\sim_m(z_1, \ldots, z_m)$ is the set witnessing the IL for $\vdash$;
- Similarly, for $\vee$ and $\rightarrow$.

The spectrum of an algebra $\boldsymbol{A}$ relative to a logic $\vdash$ is the poset $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ of the meet irreducible deductive filters of $\vdash$ on $\boldsymbol{A}$.

General Sahlqvist Theorem

TFEA for a Sahlqvist quasiequation

$$\Phi = \varphi_1 \wedge y \leqslant z \,\&\ldots\&\, \varphi_m \wedge y \leqslant z \Longrightarrow y \leqslant z$$

compatible with a protoalgebraic logic $\vdash$:

- The logic $\vdash$ validates the metarules

$$\frac{\Gamma, \boldsymbol{\varphi}_{\mathbf{1}}^{k}(\vec{\gamma}_1, \ldots, \vec{\gamma}_n) \rhd \psi \quad \ldots \quad \Gamma, \boldsymbol{\varphi}_{\boldsymbol{m}}^{k}(\vec{\gamma}_1, \ldots, \vec{\gamma}_n) \rhd \psi}{\Gamma \rhd \psi}$$

  for all $k \in \mathbb{Z}^{+}$ and finite sets of formulas $\Gamma \cup \{\psi, \vec{\gamma}_1, \ldots, \vec{\gamma}_n\}$;
- For every algebra $\boldsymbol{A}$, we have $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$.

Proof by example. Suppose that the Sahlqvist quasiequation

$$\Phi = x \wedge y \leqslant z \,\&\, \neg x \wedge y \leqslant z \Longrightarrow y \leqslant z.$$

corresponding to the excluded middle $x \vee \neg x$ is compatible with $\vdash$.
Remark. The semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is pseudocomplemented, for all $\boldsymbol{A}$.

- Suppose that $\vdash$ validates the metarules

$$\frac{\Gamma, \gamma_1, \dots, \gamma_n \rhd \psi \qquad \Gamma, \sim_n(\gamma_1, \dots, \gamma_n) \rhd \psi}{\Gamma \rhd \psi.}$$

- Then $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{Fm})$ validates Φ.
- By protoalgebraicity, $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ validates Φ, for every $\boldsymbol{A}$.
- By Canonicity, $\mathsf{Up}(\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})_*) \vDash \Phi$.
- As $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ is an algebraic lattice,

$$\mathsf{Fi}_{\vdash}(\boldsymbol{A}) \cong \text{the lattice of filters of the semilattice } \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A}).$$

- Consequently, $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \cong \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})_*$.
- Thus, $\mathsf{Up}(\mathsf{Spec}_{\vdash}(\boldsymbol{A})) \vDash \Phi$.
- By Correspondence, $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathrm{tr}(\Phi)$.

Examples. Let $\vdash$ be a protoalgebraic logic with the IL.

Corollary (Lávička & Přenosil)

The logic $\vdash$ validates the following metarules for $n \in \mathbb{Z}^+$:

$$\frac{\Gamma, \gamma_1, \ldots, \gamma_n \rhd \psi \qquad \Gamma, \sim_n(\gamma_1, \ldots, \gamma_n) \rhd \psi}{\Gamma \rhd \psi}$$

iff it is semisimple: the poset $\mathrm{Spec}_{\vdash}(\boldsymbol{A})$ is discrete, for every $\boldsymbol{A}$.

Corollary (for $n = 1$, Lávička, M., Raftery)

The logic $\vdash$ validates the following metarules for $n \in \mathbb{Z}^+$:

$$\frac{\Gamma, \sim(\vec{\gamma}_1 \cup \cdots \cup \vec{\gamma}_{i-1} \cup \sim \vec{\gamma}_i) \rhd \psi \text{ for every } 1 \leqslant i \leqslant n+1}{\Gamma \rhd \psi}$$

iff it has bounded top width n: the principal upsets in $\mathrm{Spec}_{\vdash}(\boldsymbol{A})$ have at most n maximal elements, for every $\boldsymbol{A}$.

A concrete example.

- For every $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and φ, we write

$$\Gamma \to \varphi := (\gamma_1 \to (\gamma_2 \to (\ldots(\gamma_{n-1} \to \gamma_n)\ldots))) \to \varphi$$

- Then for every Sahlqvist quasiequation

$$\Phi = \varphi_1 \wedge y \leqslant z \,\&\ldots\&\, \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z$$

compatible with a logic $\vdash$, we define a set of formulas

$$\Phi^* := \bigcup_{k \in \mathbb{Z}^+} ((\boldsymbol{\varphi}_1^k \to x) \cup \cdots \cup (\boldsymbol{\varphi}_n^k \to x)) \to x.$$

Salhqvist Canonicity for fragments of IPC with $\to$

Let L be a fragment of IPC comprising $\to$. For every L-subreduct $\boldsymbol{A}$ of a Heyting algebra,

$$\text{if } \boldsymbol{A} \vDash \Phi^*, \text{ then } \mathrm{Up}(\mathrm{Spec}_{\mathsf{L}}(\boldsymbol{A})) \vDash \Phi^*,$$

where $\mathrm{Spec}_{\mathsf{L}}(\boldsymbol{A})$ is the poset of meet irr. implicative filters of $\boldsymbol{A}$.

Thank you very much for your attention!