Profiniteness and spectra of Heyting algebras

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Joint work with
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- The prime spectrum $\boldsymbol{A}_+$ of a Heyting algebra $\boldsymbol{A}$ becomes an Esakia space, when endowed with the topology generated by

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- If X is an Esakia space, then the collection $\mathsf{ClUp}(X)$ of clopen upsets of X can be viewed as a Heyting algebra

$$\langle \mathsf{ClUp}(X); \cap, \cup, \to, \emptyset, X\rangle$$

in which $U \to V$ is defined as $X \smallsetminus {\downarrow}(U \smallsetminus V)$.

The representation problem

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- Esakia representable posets have enough gaps. Consequently, no nontrivial dense linear order is Esakia representable.

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A poset X is said to be **order compact** when, for every family $\{U_i : i \in I\}$ of order closed sets,

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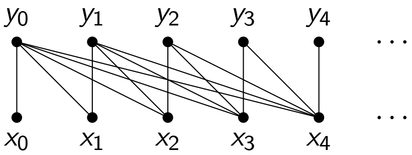

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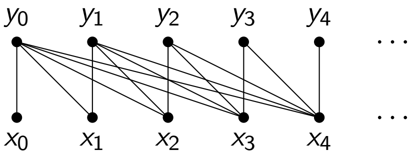

Remark. If a poset X is order compact, then **infima** and **suprema** of nonempty chains exist in X.

Trees and well-orders

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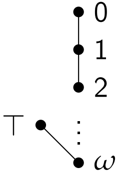

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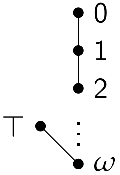

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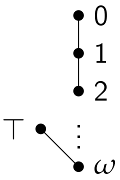

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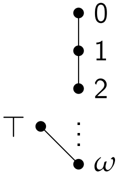

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A forest is said to be **well-ordered** when its principal downsets are well-ordered, i.e., do not contain infinite descending chains.

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$$X_\alpha := \{x \in X : o(x) = \alpha\} \text{ and } X_{\leqslant \alpha} := \bigcup_{\gamma \leqslant \alpha} X_\gamma.$$

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- For every ordinal $\alpha \leqslant \kappa$, we define a topology τ_α on $X_{\leqslant \alpha}$ such that $\langle X_{\leqslant \alpha}, \leqslant, \tau_\alpha \rangle$ is an Esakia space.

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The problem for arbitrary forests is still open.

Profinite algebras and completions

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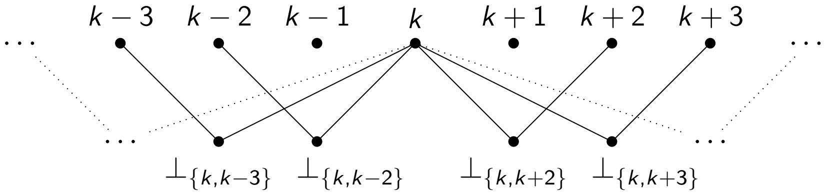

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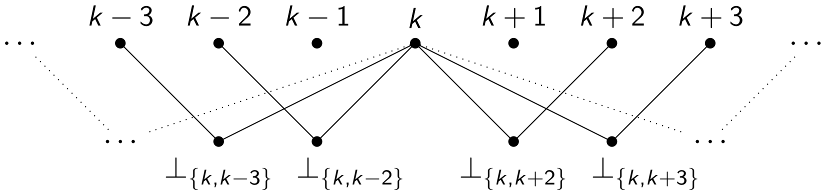

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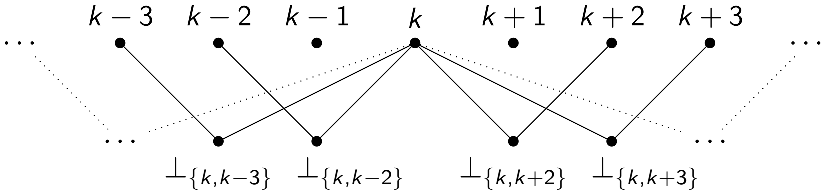

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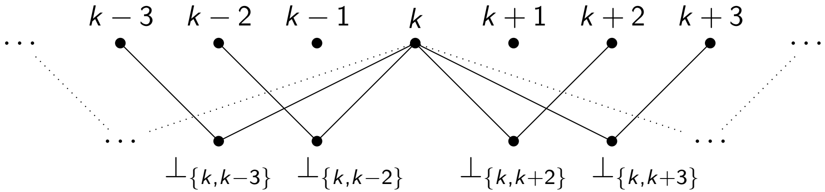

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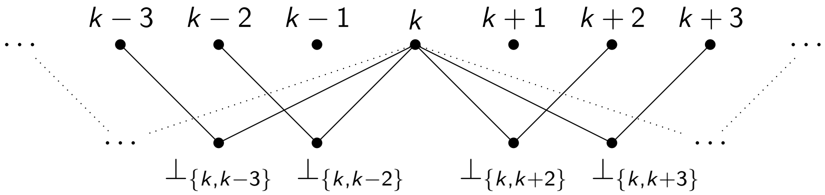
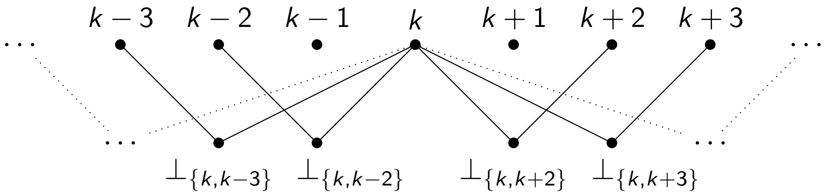
- Thus, X is Esakia representable, whence order compact.



- Observe that

$$\bigcap_{k \in Z} \uparrow\downarrow k = \emptyset.$$



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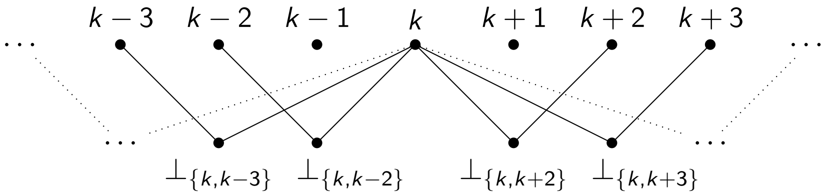
which is impossible, because $\max\{k_1, \ldots, k_n\} + 2$ belongs to the above intersection. QED



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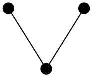
Corollary

There are profinite Heyting algebras that are not profinite completions, e.g., $\mathrm{Up}(X)$.

Remark. The algebra Up(X) is a embeds into a direct power of the Heyting algebra of upsets of the **two-fork** F_2.



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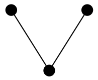

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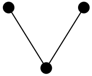

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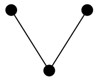

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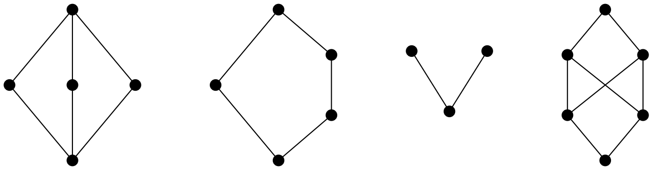

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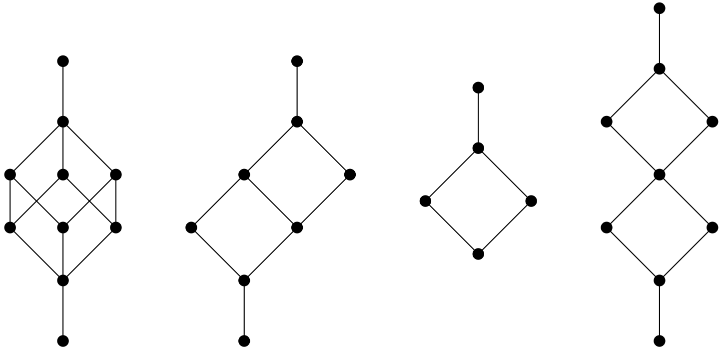

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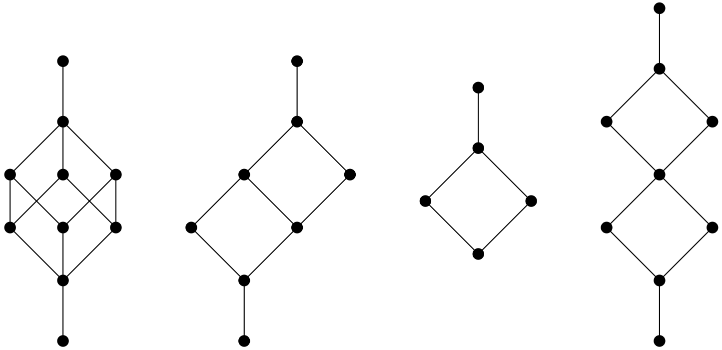

Theorem (Jankov 1963)

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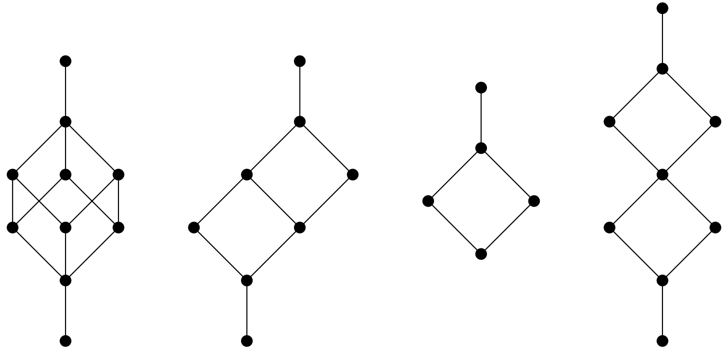

Theorem (Jankov 1963)

Every finite subdirectly irreducible Heyting algebra $\boldsymbol{A}$ is **splitting**: there exists the largest variety K of Heyting algebras omitting $\boldsymbol{A}$. K is axiomatized by the equation $J(\boldsymbol{A}) \approx 1$, where $J(\boldsymbol{A})$ is the **Jankov formula** of $\boldsymbol{A}$.

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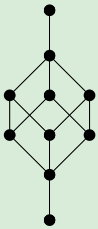
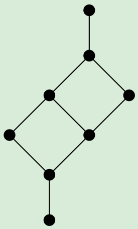
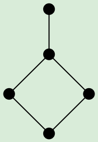
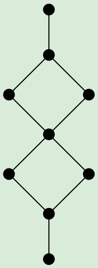

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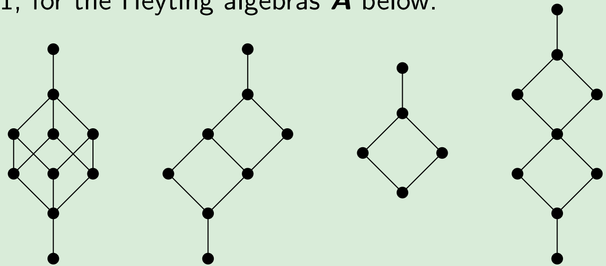

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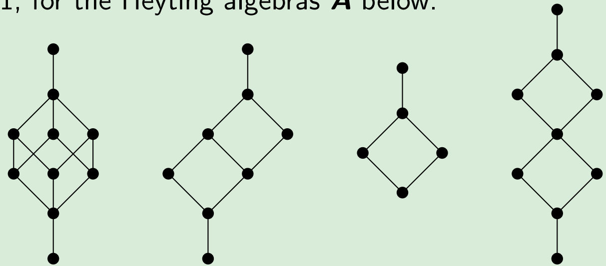

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Aim. Prove the converse of the Proposition.

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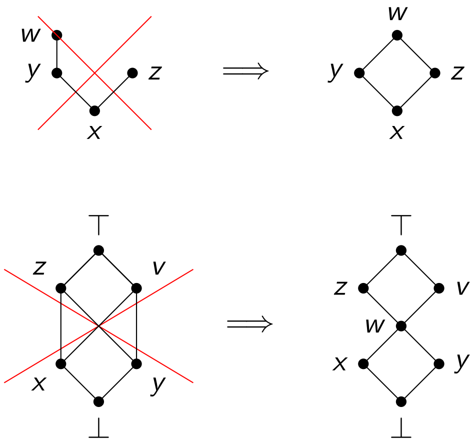
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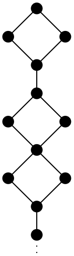

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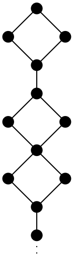

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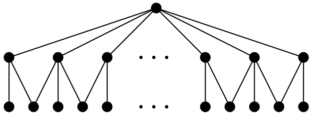

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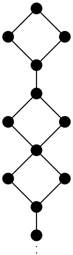

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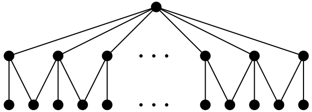

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A diamond system is Esakia representable iff it has enough gaps and infinima and suprema of its nonempty chains exist.

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- As $X = Y_{\text{fin}}$, $\mathsf{Up}(X)$ is a profinite completion. QED

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Corollary

Intermediate logics algebraized by varieties of Heyting algebras whose profinite members are profinite completions are locally tabular, finitely axiomatizable, have the infinite Beth definability property, and are hereditarily structurally complete.

Thank you very much for your attention!