On Equational Completeness Theorems

Tommaso Moraschini

Department of Philosophy, University of Barcelona

virtUMA 2020

Equational completeness theorems

- The equational completeness theorem of classical propositional logic **CPC** w.r.t. the variety BA of Boolean algebras states that

- The equational completeness theorem of classical propositional logic **CPC** w.r.t. the variety BA of Boolean algebras states that

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \text{for every } \boldsymbol{A} \in \mathrm{BA} \text{ and every hom } f\colon \boldsymbol{Fm} \to \boldsymbol{A}, \\ \text{if } f(\gamma) = 1^{\boldsymbol{A}} \text{ for all } \gamma \in \Gamma, \text{ then } f(\varphi) = 1^{\boldsymbol{A}}.$$

- The equational completeness theorem of classical propositional logic **CPC** w.r.t. the variety BA of Boolean algebras states that

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{BA} \text{ and every hom } f\colon \boldsymbol{Fm} \to \boldsymbol{A},$$
$$\text{if } f(\gamma) = 1^{\boldsymbol{A}} \text{ for all } \gamma \in \Gamma, \text{ then } f(\varphi) = 1^{\boldsymbol{A}}.$$

- Given a class of similar algebras K and a set of equations $\Theta \cup \{\varphi \approx \psi\}$, we write $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$ when

- The equational completeness theorem of classical propositional logic **CPC** w.r.t. the variety BA of Boolean algebras states that

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{BA} \text{ and every hom } f\colon \boldsymbol{Fm} \to \boldsymbol{A},$$
$$\text{if } f(\gamma) = 1^{\boldsymbol{A}} \text{ for all } \gamma \in \Gamma, \text{ then } f(\varphi) = 1^{\boldsymbol{A}}.$$

- Given a class of similar algebras K and a set of equations $\Theta \cup \{\varphi \approx \psi\}$, we write $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$ when

$$\text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every hom } f\colon \boldsymbol{Fm} \to \boldsymbol{A},$$
$$\text{if } f(\epsilon) = f(\delta) \text{ for all } \epsilon \approx \delta \in \Theta, \text{ then } f(\varphi) = f(\psi).$$

- The equational completeness theorem of classical propositional logic **CPC** w.r.t. the variety BA of Boolean algebras states that

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \text{for every } \boldsymbol{A} \in \mathrm{BA} \text{ and every hom } f\colon \boldsymbol{Fm} \to \boldsymbol{A},$$
$$\text{if } f(\gamma) = 1^{\boldsymbol{A}} \text{ for all } \gamma \in \Gamma, \text{ then } f(\varphi) = 1^{\boldsymbol{A}}.$$

- Given a class of similar algebras K and a set of equations $\Theta \cup \{\varphi \approx \psi\}$, we write $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$ when

$$\text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every hom } f\colon \boldsymbol{Fm} \to \boldsymbol{A},$$
$$\text{if } f(\epsilon) = f(\delta) \text{ for all } \epsilon \approx \delta \in \Theta, \text{ then } f(\varphi) = f(\psi).$$

When viewed as a relation, $\vDash_{\mathsf{K}}$ is called the

equational consequence relative to K.

- The equational completeness theorem of classical propositional logic **CPC** w.r.t. the variety BA of Boolean algebras states that

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{BA} \text{ and every hom } f\colon \boldsymbol{Fm} \to \boldsymbol{A},$$
$$\text{if } f(\gamma) = 1^{\boldsymbol{A}} \text{ for all } \gamma \in \Gamma, \text{ then } f(\varphi) = 1^{\boldsymbol{A}}.$$

- Given a class of similar algebras K and a set of equations $\Theta \cup \{\varphi \approx \psi\}$, we write $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$ when

$$\text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every hom } f\colon \boldsymbol{Fm} \to \boldsymbol{A},$$
$$\text{if } f(\epsilon) = f(\delta) \text{ for all } \epsilon \approx \delta \in \Theta, \text{ then } f(\varphi) = f(\psi).$$

  When viewed as a relation, $\vDash_{\mathsf{K}}$ is called the

  equational consequence relative to K.

- In this terminology, the equational completeness theorem of **CPC** can be written, more concisely, as

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1.$$

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

$$\begin{aligned} \boldsymbol{\tau}(\varphi) &:= \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \boldsymbol{\tau}\} \\ \boldsymbol{\tau}[\Gamma] &:= \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma). \end{aligned}$$

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

$$\boldsymbol{\tau}(\varphi) := \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \boldsymbol{\tau}\}$$
$$\boldsymbol{\tau}[\Gamma] := \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma).$$

- Taking $\boldsymbol{\tau}(x) := \{x \approx 1\}$, we get

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

$$\boldsymbol{\tau}(\varphi) := \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \boldsymbol{\tau}\}$$
$$\boldsymbol{\tau}[\Gamma] := \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma).$$

- Taking $\boldsymbol{\tau}(x) := \{x \approx 1\}$, we get

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1$$
$$\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi).$$

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

$$\boldsymbol{\tau}(\varphi) := \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \boldsymbol{\tau}\}$$
$$\boldsymbol{\tau}[\Gamma] := \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma).$$

- Taking $\boldsymbol{\tau}(x) := \{x \approx 1\}$, we get

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1$$
$$\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi).$$

Observation.

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

$$\boldsymbol{\tau}(\varphi) := \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \boldsymbol{\tau}\}$$
$$\boldsymbol{\tau}[\Gamma] := \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma).$$

- Taking $\boldsymbol{\tau}(x) := \{x \approx 1\}$, we get

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1$$
$$\Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi).$$

Observation. The essence of the equational completeness theorem of **CPC** w.r.t. BA consists in that

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

$$\boldsymbol{\tau}(\varphi) := \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \boldsymbol{\tau}\}$$
$$\boldsymbol{\tau}[\Gamma] := \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma).$$

- Taking $\boldsymbol{\tau}(x) := \{x \approx 1\}$, we get

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathsf{BA}} \varphi \approx 1$$
$$\iff \boldsymbol{\tau}[\Gamma] \models_{\mathsf{BA}} \boldsymbol{\tau}(\varphi).$$

Observation. The essence of the equational completeness theorem of **CPC** w.r.t. BA consists in that

CPC can be **interpreted** into $\models_{\mathsf{BA}}$.

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

$$\boldsymbol{\tau}(\varphi) := \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \boldsymbol{\tau}\}$$
$$\boldsymbol{\tau}[\Gamma] := \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma).$$

- Taking $\boldsymbol{\tau}(x) := \{x \approx 1\}$, we get

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1$$
$$\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi).$$

Observation. The essence of the equational completeness theorem of **CPC** w.r.t. BA consists in that

CPC can be **interpreted** into $\vDash_{\mathsf{BA}}$.

This is made possible by translating formulas into equations by means of the set of equations $\boldsymbol{\tau}(x)$ as follows:

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

$$\boldsymbol{\tau}(\varphi) := \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \boldsymbol{\tau}\}$$
$$\boldsymbol{\tau}[\Gamma] := \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma).$$

- Taking $\boldsymbol{\tau}(x) := \{x \approx 1\}$, we get

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1$$
$$\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{BA}} \boldsymbol{\tau}(\varphi).$$

Observation. The essence of the equational completeness theorem of **CPC** w.r.t. BA consists in that

CPC can be **interpreted** into $\vDash_{\mathsf{BA}}$.

This is made possible by translating formulas into equations by means of the set of equations $\boldsymbol{\tau}(x)$ as follows:

$$\psi \longmapsto \boldsymbol{\tau}(\psi), \text{ i.e., } \{\psi \approx 1\}.$$

- A (propositional) **logic** $\vdash$ is a consequence relation on the set of formulas Fm of an arbitrary algebraic language

- A (propositional) **logic** $\vdash$ is a consequence relation on the set of formulas *Fm* of an arbitrary algebraic language that, moreover, is substitution invariant, i.e.,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi),$$

  for every set of formulas $\Gamma \cup \{\varphi\}$ and every substitution σ.

- A (propositional) **logic** $\vdash$ is a consequence relation on the set of formulas *Fm* of an arbitrary algebraic language that, moreover, is substitution invariant, i.e.,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi),$$

for every set of formulas $\Gamma \cup \{\varphi\}$ and every substitution σ.

Definition

A logic $\vdash$ admits an **equational completeness theorem** if there are a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$ such that

- A (propositional) **logic** $\vdash$ is a consequence relation on the set of formulas *Fm* of an arbitrary algebraic language that, moreover, is substitution invariant, i.e.,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi),$$

for every set of formulas $\Gamma \cup \{\varphi\}$ and every substitution σ.

Definition

A logic $\vdash$ admits an **equational completeness theorem** if there are a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$ such that

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi),$$

for every set of formulas $\Gamma \cup \{\varphi\}$.

- A (propositional) **logic** $\vdash$ is a consequence relation on the set of formulas *Fm* of an arbitrary algebraic language that, moreover, is substitution invariant, i.e.,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi),$$

for every set of formulas $\Gamma \cup \{\varphi\}$ and every substitution σ.

Definition

A logic $\vdash$ admits an **equational completeness theorem** if there are a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$ such that

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi),$$

for every set of formulas $\Gamma \cup \{\varphi\}$.

Examples. **CPC** admits an equational completeness theorem w.r.t. Boolean algebras.

- A (propositional) **logic** $\vdash$ is a consequence relation on the set of formulas *Fm* of an arbitrary algebraic language that, moreover, is substitution invariant, i.e.,

$$\text{if } \Gamma \vdash \varphi \text{, then } \sigma[\Gamma] \vdash \sigma(\varphi),$$

for every set of formulas $\Gamma \cup \{\varphi\}$ and every substitution σ.

Definition

A logic $\vdash$ admits an **equational completeness theorem** if there are a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$ such that

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi),$$

for every set of formulas $\Gamma \cup \{\varphi\}$.

Examples. **CPC** admits an equational completeness theorem w.r.t. Boolean algebras. Indeed, any extension of **FL** admits an equational completeness theorems w.r.t. some ISP-class of FL-algebras.

A more unsettling example.

A more unsettling example.

- Glivenko's Theorem connects **CPC** and **IPC** as follows:

A more unsettling example.

- Glivenko's Theorem connects **CPC** and **IPC** as follows:

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

A more unsettling example.

- Glivenko's Theorem connects **CPC** and **IPC** as follows:

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

Thus, taking $\boldsymbol{\tau}(x) := \{\neg\neg x \approx 1\}$, we get

A more unsettling example.

- Glivenko's Theorem connects **CPC** and **IPC** as follows:

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

Thus, taking $\boldsymbol{\tau}(x) := \{\neg\neg x \approx 1\}$, we get

$$\begin{aligned} \Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1 \\ &\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi), \end{aligned}$$

where HA is the variety of Heyting algebras.

A more unsettling example.

- Glivenko's Theorem connects **CPC** and **IPC** as follows:

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

Thus, taking $\boldsymbol{\tau}(x) := \{\neg\neg x \approx 1\}$, we get

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1 \\ &\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi),\end{aligned}$$

where HA is the variety of Heyting algebras.

Observation

CPC admits an equational completeness theorem w.r.t. the variety of **Heyting** algebras

A more unsettling example.

- Glivenko's Theorem connects **CPC** and **IPC** as follows:

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

Thus, taking $\boldsymbol{\tau}(x) := \{\neg\neg x \approx 1\}$, we get

$$\begin{aligned} \Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1 \\ &\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi), \end{aligned}$$

where HA is the variety of Heyting algebras.

Observation

CPC admits an equational completeness theorem w.r.t. the variety of **Heyting** algebras (although certainly not the intended one).

A more unsettling example.

- Glivenko's Theorem connects **CPC** and **IPC** as follows:

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

Thus, taking $\boldsymbol{\tau}(x) := \{\neg\neg x \approx 1\}$, we get

$$\begin{aligned} \Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1 \\ &\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi), \end{aligned}$$

where HA is the variety of Heyting algebras.

Observation

CPC admits an equational completeness theorem w.r.t. the variety of **Heyting** algebras (although certainly not the intended one).

- Notably, the situation does not improve if we restrict to the case where $\boldsymbol{\tau}(x) = \{x \approx 1\}$.

A more unsettling example.

- Glivenko's Theorem connects **CPC** and **IPC** as follows:

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

Thus, taking $\boldsymbol{\tau}(x) := \{\neg\neg x \approx 1\}$, we get

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1 \\ &\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi),\end{aligned}$$

where HA is the variety of Heyting algebras.

Observation

CPC admits an equational completeness theorem w.r.t. the variety of **Heyting** algebras (although certainly not the intended one).

- Notably, the situation does not improve if we restrict to the case where $\boldsymbol{\tau}(x) = \{x \approx 1\}$. Actually, there is no escape from nonstandard equational completeness theorems.

Sometimes nonstandard equational completeness theorems are the sole possible ones.

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle \wedge, \vee \rangle$-fragment of $\mathbf{CPC}$.

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle\wedge,\vee\rangle$-fragment of $\mathbf{CPC}$.

Observation

$\mathbf{CPC}_{\wedge\vee}$ does **not** admit any equational completeness theorem w.r.t. the variety of distributive lattices.

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle\wedge,\vee\rangle$-fragment of $\mathbf{CPC}$.

Observation

$\mathbf{CPC}_{\wedge\vee}$ does **not** admit any equational completeness theorem w.r.t. the variety of distributive lattices.

Proof.

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle \wedge, \vee \rangle$-fragment of $\mathbf{CPC}$.

Observation

$\mathbf{CPC}_{\wedge\vee}$ does **not** admit any equational completeness theorem w.r.t. the variety of distributive lattices.

Proof.

- Suppose the contrary.

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle \wedge, \vee \rangle$-fragment of $\mathbf{CPC}$.

Observation

$\mathbf{CPC}_{\wedge\vee}$ does **not** admit any equational completeness theorem w.r.t. the variety of distributive lattices.

Proof.

- Suppose the contrary. Then there exists a set of equations $\boldsymbol{\tau}(x)$ witnessing an equational completeness theorem of $\mathbf{CPC}_{\wedge\vee}$ w.r.t. the variety DL of distributive lattices.

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle \wedge, \vee \rangle$-fragment of $\mathbf{CPC}$.

Observation

$\mathbf{CPC}_{\wedge\vee}$ does **not** admit any equational completeness theorem w.r.t. the variety of distributive lattices.

Proof.

- Suppose the contrary. Then there exists a set of equations $\boldsymbol{\tau}(x)$ witnessing an equational completeness theorem of $\mathbf{CPC}_{\wedge\vee}$ w.r.t. the variety DL of distributive lattices.
- As all equations in a single variable are valid in DL, we get

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle \wedge, \vee \rangle$-fragment of $\mathbf{CPC}$.

Observation

$\mathbf{CPC}_{\wedge\vee}$ does **not** admit any equational completeness theorem w.r.t. the variety of distributive lattices.

Proof.

- Suppose the contrary. Then there exists a set of equations $\boldsymbol{\tau}(x)$ witnessing an equational completeness theorem of $\mathbf{CPC}_{\wedge\vee}$ w.r.t. the variety DL of distributive lattices.
- As all equations in a single variable are valid in DL, we get

$$\mathsf{DL} \vDash \boldsymbol{\tau}(x), \text{ that is, } \emptyset \vDash_{\mathsf{DL}} \boldsymbol{\tau}(x).$$

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle \wedge, \vee \rangle$-fragment of **CPC**.

Observation

$\mathbf{CPC}_{\wedge\vee}$ does **not** admit any equational completeness theorem w.r.t. the variety of distributive lattices.

Proof.

- Suppose the contrary. Then there exists a set of equations $\boldsymbol{\tau}(x)$ witnessing an equational completeness theorem of $\mathbf{CPC}_{\wedge\vee}$ w.r.t. the variety DL of distributive lattices.
- As all equations in a single variable are valid in DL, we get

$$\mathsf{DL} \vDash \boldsymbol{\tau}(x), \text{ that is, } \emptyset \vDash_{\mathsf{DL}} \boldsymbol{\tau}(x).$$

- By the equational completeness theorem, $\emptyset \vDash_{\mathsf{DL}} \boldsymbol{\tau}(x)$ implies $\emptyset \vdash_{\mathbf{CPC}_{\wedge\vee}} x$, which is of course false.

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle\wedge, \vee\rangle$-fragment of **CPC**.

Observation

$\mathbf{CPC}_{\wedge\vee}$ does **not** admit any equational completeness theorem w.r.t. the variety of distributive lattices.

Proof.

- Suppose the contrary. Then there exists a set of equations $\boldsymbol{\tau}(x)$ witnessing an equational completeness theorem of $\mathbf{CPC}_{\wedge\vee}$ w.r.t. the variety DL of distributive lattices.
- As all equations in a single variable are valid in DL, we get

$$\mathsf{DL} \vDash \boldsymbol{\tau}(x), \text{ that is, } \emptyset \vDash_{\mathsf{DL}} \boldsymbol{\tau}(x).$$

- By the equational completeness theorem, $\emptyset \vDash_{\mathsf{DL}} \boldsymbol{\tau}(x)$ implies $\emptyset \vdash_{\mathbf{CPC}_{\wedge\vee}} x$, which is of course false. QED

Observation

$\mathbf{CPC}_{\wedge\vee}$ admits a (**nonstandard**) equational completeness theorem.

Observation

$\mathbf{CPC}_{\wedge\vee}$ admits a (**nonstandard**) equational completeness theorem.

Proof sketch.

Observation

$\mathbf{CPC}_{\wedge\vee}$ admits a (**nonstandard**) equational completeness theorem.

Proof sketch.

- Consider the three-element algebra

$$\boldsymbol{A} = \langle \{1, 0^+, 0^-\}; \wedge, \vee \rangle$$

Observation

$\mathbf{CPC}_{\wedge\vee}$ admits a (**nonstandard**) equational completeness theorem.

Proof sketch.

- Consider the three-element algebra

$$\boldsymbol{A} = \langle \{1, 0^+, 0^-\}; \wedge, \vee \rangle$$

with commutative operations defined by the tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

Observation

$\mathbf{CPC}_{\wedge\vee}$ admits a (**nonstandard**) equational completeness theorem.

Proof sketch.

- Consider the three-element algebra

$$\boldsymbol{A} = \langle \{1, 0^+, 0^-\}; \wedge, \vee \rangle$$

with commutative operations defined by the tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

- Then $\mathbf{CPC}_{\wedge\vee}$ admits an equational completeness theorem w.r.t. $\mathsf{K} := \{\boldsymbol{A}\}$ witnessed by the set of equations

$$\boldsymbol{\tau}(x) = \{x \approx x \wedge x\}.$$

Observation

$\mathbf{CPC}_{\wedge\vee}$ admits a (**nonstandard**) equational completeness theorem.

Proof sketch.

- Consider the three-element algebra

$$\boldsymbol{A} = \langle \{1, 0^+, 0^-\}; \wedge, \vee \rangle$$

  with commutative operations defined by the tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

- Then $\mathbf{CPC}_{\wedge\vee}$ admits an equational completeness theorem w.r.t. $\mathsf{K} := \{\boldsymbol{A}\}$ witnessed by the set of equations

$$\boldsymbol{\tau}(x) = \{x \approx x \wedge x\}.$$

- Why?

Observation

$\mathbf{CPC}_{\wedge\vee}$ admits a (**nonstandard**) equational completeness theorem.

Proof sketch.

- Consider the three-element algebra

$$\boldsymbol{A} = \langle \{1, 0^+, 0^-\}; \wedge, \vee \rangle$$

with commutative operations defined by the tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

- Then $\mathbf{CPC}_{\wedge\vee}$ admits an equational completeness theorem w.r.t. $\mathsf{K} := \{\boldsymbol{A}\}$ witnessed by the set of equations

$$\boldsymbol{\tau}(x) = \{x \approx x \wedge x\}.$$

- Why? Because there is a surjective hom $f \colon \boldsymbol{A} \to \boldsymbol{D}_2$ such that $f^{-1}(1) = \{1\}$, and 1 is the sole solution of $\boldsymbol{\tau}(x)$ in $\boldsymbol{A}$.

Observation

$\mathbf{CPC}_{\wedge\vee}$ admits a (**nonstandard**) equational completeness theorem.

Proof sketch.

- Consider the three-element algebra

$$\boldsymbol{A} = \langle \{1, 0^+, 0^-\}; \wedge, \vee \rangle$$

with commutative operations defined by the tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

- Then $\mathbf{CPC}_{\wedge\vee}$ admits an equational completeness theorem w.r.t. $\mathsf{K} := \{\boldsymbol{A}\}$ witnessed by the set of equations

$$\boldsymbol{\tau}(x) = \{x \approx x \wedge x\}.$$

- Why? Because there is a surjective hom $f \colon \boldsymbol{A} \to \boldsymbol{D}_2$ such that $f^{-1}(1) = \{1\}$, and 1 is the sole solution of $\boldsymbol{\tau}(x)$ in $\boldsymbol{A}$. QED

- On the other hand, some logics **lack** any equational completeness theorem, e.g., the **relevance** logic P–W axiomatized by the rules

- On the other hand, some logics **lack** any equational completeness theorem, e.g., the **relevance** logic **P–W** axiomatized by the rules

$$
\begin{aligned}
\emptyset \rhd (x \to y) &\to ((z \to x) \to (z \to y)) \\
\emptyset \rhd (x \to y) &\to ((y \to z) \to (x \to z)) \\
\emptyset \rhd x &\to x \\
x, x &\to y \rhd y.
\end{aligned}
$$

- On the other hand, some logics **lack** any equational completeness theorem, e.g., the **relevance** logic **P–W** axiomatized by the rules

$$\emptyset \rhd (x \to y) \to ((z \to x) \to (z \to y))$$
$$\emptyset \rhd (x \to y) \to ((y \to z) \to (x \to z))$$
$$\emptyset \rhd x \to x$$
$$x, x \to y \rhd y.$$

- Despite the simplicity of the concept, equational completeness theorems are poorly understood in general.

- On the other hand, some logics **lack** any equational completeness theorem, e.g., the **relevance** logic **P–W** axiomatized by the rules

$$\emptyset \rhd (x \to y) \to ((z \to x) \to (z \to y))$$
$$\emptyset \rhd (x \to y) \to ((y \to z) \to (x \to z))$$
$$\emptyset \rhd x \to x$$
$$x, x \to y \rhd y.$$

- Despite the simplicity of the concept, equational completeness theorems are poorly understood in general.

Aim of the talk

Characterize logics admitting an equational completeness theorem

- On the other hand, some logics **lack** any equational completeness theorem, e.g., the **relevance** logic **P–W** axiomatized by the rules

$$\emptyset \rhd (x \to y) \to ((z \to x) \to (z \to y))$$
$$\emptyset \rhd (x \to y) \to ((y \to z) \to (x \to z))$$
$$\emptyset \rhd x \to x$$
$$x, x \to y \rhd y.$$

- Despite the simplicity of the concept, equational completeness theorems are poorly understood in general.

Aim of the talk

Characterize logics admitting an equational completeness theorem or, equivalently, understand which logics can be interpreted into relative equational consequences.

Nonstandard equational completeness theorems: a general construction

Definition

Let $\vdash$ be a logic.

Definition

Let $\vdash$ be a logic. Two formulas φ and ψ are **logically equivalent** if

$$\delta(\varphi, \vec{y}) \dashv\vdash \delta(\psi, \vec{y}),$$

for every formula $\delta(x, \vec{y})$.

Definition

Let $\vdash$ be a logic. Two formulas φ and ψ are **logically equivalent** if

$$\delta(\varphi, \vec{y}) \dashv\vdash \delta(\psi, \vec{y}),$$

for every formula $\delta(x, \vec{y})$. In this case, we write $\varphi \equiv_{\vdash} \psi$.

Definition

Let $\vdash$ be a logic. Two formulas φ and ψ are **logically equivalent** if

$$\delta(\varphi, \vec{y}) \dashv\vdash \delta(\psi, \vec{y}),$$

for every formula $\delta(x, \vec{y})$. In this case, we write $\varphi \equiv_{\vdash} \psi$.

- In extensions $\vdash$ of **FL**, this specializes to the following:

$$\varphi \equiv_{\vdash} \psi \Longleftrightarrow \emptyset \vdash (\varphi \backslash \psi) \wedge (\psi \backslash \varphi).$$

Definition

Let $\vdash$ be a logic. Two formulas φ and ψ are **logically equivalent** if

$$\delta(\varphi, \vec{y}) \dashv\vdash \delta(\psi, \vec{y}),$$

for every formula $\delta(x, \vec{y})$. In this case, we write $\varphi \equiv_{\vdash} \psi$.

- In extensions $\vdash$ of **FL**, this specializes to the following:

$$\varphi \equiv_{\vdash} \psi \iff \emptyset \vdash (\varphi \backslash \psi) \wedge (\psi \backslash \varphi).$$

- Even more concretely, in **CPC** we have

$$\varphi \equiv_{\mathbf{CPC}} \psi \iff \emptyset \vdash_{\mathbf{CPC}} (\varphi \to \psi) \wedge (\psi \to \varphi) \iff \varphi \dashv\vdash_{\mathbf{CPC}} \psi.$$

Definition

Let $\vdash$ be a logic. Two formulas φ and ψ are **logically equivalent** if

$$\delta(\varphi, \vec{y}) \dashv\vdash \delta(\psi, \vec{y}),$$

for every formula $\delta(x, \vec{y})$. In this case, we write $\varphi \equiv_{\vdash} \psi$.

- In extensions $\vdash$ of **FL**, this specializes to the following:

$$\varphi \equiv_{\vdash} \psi \iff \emptyset \vdash (\varphi \backslash \psi) \wedge (\psi \backslash \varphi).$$

- Even more concretely, in **CPC** we have

$$\varphi \equiv_{\mathbf{CPC}} \psi \iff \emptyset \vdash_{\mathbf{CPC}} (\varphi \to \psi) \wedge (\psi \to \varphi) \iff \varphi \dashv\vdash_{\mathbf{CPC}} \psi.$$

Definition

A logic $\vdash$ is said to be **graph-based** if the arity of its connective is bounded above by one and, moreover, $\vdash$ has at most one unary connective.

Definition

Let $\vdash$ be a logic. Two formulas φ and ψ are **logically equivalent** if

$$\delta(\varphi, \vec{y}) \dashv\vdash \delta(\psi, \vec{y}),$$

for every formula $\delta(x, \vec{y})$. In this case, we write $\varphi \equiv_{\vdash} \psi$.

- In extensions $\vdash$ of **FL**, this specializes to the following:

$$\varphi \equiv_{\vdash} \psi \iff \emptyset \vdash (\varphi \backslash \psi) \wedge (\psi \backslash \varphi).$$

- Even more concretely, in **CPC** we have

$$\varphi \equiv_{\mathbf{CPC}} \psi \iff \emptyset \vdash_{\mathbf{CPC}} (\varphi \to \psi) \wedge (\psi \to \varphi) \iff \varphi \dashv\vdash_{\mathbf{CPC}} \psi.$$

Definition

A logic $\vdash$ is said to be **graph-based** if the arity of its connective is bounded above by one and, moreover, $\vdash$ has at most one unary connective.

Example. The $\langle \Diamond, 0, 1 \rangle$-fragment of any modal logic is graph-based,

Definition

Let $\vdash$ be a logic. Two formulas φ and ψ are **logically equivalent** if

$$\delta(\varphi, \vec{y}) \dashv\vdash \delta(\psi, \vec{y}),$$

for every formula $\delta(x, \vec{y})$. In this case, we write $\varphi \equiv_{\vdash} \psi$.

- In extensions $\vdash$ of **FL**, this specializes to the following:

$$\varphi \equiv_{\vdash} \psi \iff \emptyset \vdash (\varphi \backslash \psi) \wedge (\psi \backslash \varphi).$$

- Even more concretely, in **CPC** we have

$$\varphi \equiv_{\mathbf{CPC}} \psi \iff \emptyset \vdash_{\mathbf{CPC}} (\varphi \to \psi) \wedge (\psi \to \varphi) \iff \varphi \dashv\vdash_{\mathbf{CPC}} \psi.$$

Definition

A logic $\vdash$ is said to be **graph-based** if the arity of its connective is bounded above by one and, moreover, $\vdash$ has at most one unary connective.

Example. The $\langle \Diamond, 0, 1 \rangle$-fragment of any modal logic is graph-based, while the $\langle \Diamond, \Box, 0, 1 \rangle$-one is not.

Theorem

Let $\vdash$ a logic that is not graph-based.

Theorem

Let $\vdash$ a logic that is not graph-based. If there are two distinct formulas φ and ψ such that

$$\varphi \equiv_{\vdash} \psi \text{ and } \mathsf{Var}(\varphi) \cup \mathsf{Var}(\psi) = \{x\},$$

then $\vdash$ admits an equational completeness theorem.

Theorem

Let $\vdash$ a logic that is not graph-based. If there are two distinct formulas φ and ψ such that

$$\varphi \equiv_{\vdash} \psi \text{ and } \mathsf{Var}(\varphi) \cup \mathsf{Var}(\psi) = \{x\},$$

then $\vdash$ admits an equational completeness theorem.

Proof (rough) idea.

Theorem

Let $\vdash$ a logic that is not graph-based. If there are two distinct formulas φ and ψ such that

$$\varphi \equiv_{\vdash} \psi \text{ and } \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi) = \{x\},$$

then $\vdash$ admits an equational completeness theorem.

Proof (rough) idea.

- As $\vdash$ is not graph-based, it has either two distinct unary connectives $\Box$ and $\Diamond$, or an n-ary connective f with $n \geqslant 2$.

Theorem

Let $\vdash$ a logic that is not graph-based. If there are two distinct formulas φ and ψ such that

$$\varphi \equiv_{\vdash} \psi \text{ and } \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi) = \{x\},$$

then $\vdash$ admits an equational completeness theorem.

Proof (rough) idea.

- As $\vdash$ is not graph-based, it has either two distinct unary connectives $\Box$ and $\Diamond$, or an n-ary connective f with $n \geqslant 2$.
- Suppose the first case holds and let $k \in \omega$ be greater than the length of all branches in the subformula trees of φ and ψ.

Theorem

Let $\vdash$ a logic that is not graph-based. If there are two distinct formulas φ and ψ such that

$$\varphi \equiv_{\vdash} \psi \text{ and } \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi) = \{x\},$$

then $\vdash$ admits an equational completeness theorem.

Proof (rough) idea.

- As $\vdash$ is not graph-based, it has either two distinct unary connectives $\Box$ and $\Diamond$, or an n-ary connective f with $n \geqslant 2$.
- Suppose the first case holds and let $k \in \omega$ be greater than the length of all branches in the subformula trees of φ and ψ.
- Take

$$\varphi' := \Box^{2k} \Diamond \varphi(\Box^{k} \Diamond x) \text{ and } \psi' := \Box^{2k} \Diamond \psi(\Box^{k} \Diamond x).$$

Theorem

Let $\vdash$ a logic that is not graph-based. If there are two distinct formulas φ and ψ such that

$$\varphi \equiv_{\vdash} \psi \text{ and } \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi) = \{x\},$$

then $\vdash$ admits an equational completeness theorem.

Proof (rough) idea.

- As $\vdash$ is not graph-based, it has either two distinct unary connectives $\Box$ and $\Diamond$, or an n-ary connective f with $n \geqslant 2$.
- Suppose the first case holds and let $k \in \omega$ be greater than the length of all branches in the subformula trees of φ and ψ.
- Take

$$\varphi' := \Box^{2k}\Diamond\varphi(\Box^{k}\Diamond x) \text{ and } \psi' := \Box^{2k}\Diamond\psi(\Box^{k}\Diamond x).$$

and

$$\boldsymbol{\tau}(x) := \{\varphi' \approx \psi'\}.$$

Theorem

Let $\vdash$ a logic that is not graph-based. If there are two distinct formulas φ and ψ such that

$$\varphi \equiv_{\vdash} \psi \text{ and } \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi) = \{x\},$$

then $\vdash$ admits an equational completeness theorem.

Proof (rough) idea.

- As $\vdash$ is not graph-based, it has either two distinct unary connectives $\Box$ and $\Diamond$, or an n-ary connective f with $n \geqslant 2$.
- Suppose the first case holds and let $k \in \omega$ be greater than the length of all branches in the subformula trees of φ and ψ.
- Take

$$\varphi' := \Box^{2k}\Diamond\varphi(\Box^{k}\Diamond x) \text{ and } \psi' := \Box^{2k}\Diamond\psi(\Box^{k}\Diamond x).$$

and

$$\boldsymbol{\tau}(x) := \{\varphi' \approx \psi'\}.$$

Notice that from $\varphi \equiv_{\vdash} \psi$ it follows $\varphi' \equiv_{\vdash} \psi'$.

- To prove that $\vdash$ admits an equational completeness theorem, it suffices to show that for all set of formulas $\Gamma \cup \{\epsilon\}$,

- To prove that $\vdash$ admits an equational completeness theorem, it suffices to show that for all set of formulas $\Gamma \cup \{\epsilon\}$,

$$\text{if } \boldsymbol{\tau}(\epsilon) \subseteq \mathrm{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]), \text{ then } \Gamma \vdash \epsilon, \qquad (*)$$

- To prove that $\vdash$ admits an equational completeness theorem, it suffices to show that for all set of formulas $\Gamma \cup \{\epsilon\}$,

$$\text{if } \boldsymbol{\tau}(\epsilon) \subseteq \mathrm{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]), \text{ then } \Gamma \vdash \epsilon, \qquad (*)$$

since, in this case, $\vdash$ has an equational completenss thm w.r.t.

$$\mathsf{K} := \{\boldsymbol{Fm}/\mathrm{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]) : \Gamma \text{ is a theory of } \vdash\}$$

witnessed by the set of equations $\boldsymbol{\tau}$.

- To prove that $\vdash$ admits an equational completeness theorem, it suffices to show that for all set of formulas $\Gamma \cup \{\epsilon\}$,

$$\text{if } \boldsymbol{\tau}(\epsilon) \subseteq \mathrm{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]), \text{ then } \Gamma \vdash \epsilon, \qquad (*)$$

since, in this case, $\vdash$ has an equational completenss thm w.r.t.

$$\mathsf{K} := \{\boldsymbol{Fm}/\mathrm{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]) : \Gamma \text{ is a theory of } \vdash\}$$

witnessed by the set of equations $\boldsymbol{\tau}$.
- To prove $(*)$, we use the following description of congruence generation:

- To prove that $\vdash$ admits an equational completeness theorem, it suffices to show that for all set of formulas $\Gamma \cup \{\epsilon\}$,

$$\text{if } \boldsymbol{\tau}(\epsilon) \subseteq \mathsf{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]), \text{ then } \Gamma \vdash \epsilon, \qquad (*)$$

since, in this case, $\vdash$ has an equational completenss thm w.r.t.

$$\mathsf{K} := \{\boldsymbol{Fm}/\mathsf{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]) : \Gamma \text{ is a theory of } \vdash\}$$

witnessed by the set of equations $\boldsymbol{\tau}$.
- To prove $(*)$, we use the following description of congruence generation:

Maltsev's Lemma

Let $\boldsymbol{A}$ be an algebra, $X \subseteq A \times A$, and $a, c \in A$. Then $\langle a, c\rangle \in \mathsf{Cg}^{\boldsymbol{A}}(X)$ if and only if there are $e_0, \ldots, e_n \in A$, $\langle b_0, d_0\rangle, \ldots, \langle b_{n-1}, d_{n-1}\rangle \in X$, and unary polynomial functions $p_0, \ldots, p_{n-1}$ of $\boldsymbol{A}$ such that

$$a = e_0, c = e_n, \text{ and } \{e_i, e_{i+1}\} = \{p_i(b_i), p_i(d_i)\}, \text{ for every } i < n.$$

- To prove that $\vdash$ admits an equational completeness theorem, it suffices to show that for all set of formulas $\Gamma \cup \{\epsilon\}$,

$$\text{if } \boldsymbol{\tau}(\epsilon) \subseteq \mathrm{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]), \text{ then } \Gamma \vdash \epsilon, \qquad (*)$$

since, in this case, $\vdash$ has an equational completenss thm w.r.t.

$$\mathsf{K} := \{\boldsymbol{Fm}/\mathrm{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]) : \Gamma \text{ is a theory of } \vdash\}$$

witnessed by the set of equations $\boldsymbol{\tau}$.
- To prove $(*)$, we use the following description of congruence generation:

Maltsev's Lemma

Let $\boldsymbol{A}$ be an algebra, $X \subseteq A \times A$, and $a, c \in A$. Then $\langle a, c\rangle \in \mathrm{Cg}^{\boldsymbol{A}}(X)$ if and only if there are $e_0, \dots, e_n \in A$, $\langle b_0, d_0\rangle, \dots, \langle b_{n-1}, d_{n-1}\rangle \in X$, and unary polynomial functions $p_0, \dots, p_{n-1}$ of $\boldsymbol{A}$ such that

$$a = e_0, c = e_n, \text{ and } \{e_i, e_{i+1}\} = \{p_i(b_i), p_i(d_i)\}, \text{ for every } i < n.$$

plus a combinatorial inductive argument on subformula trees.

Corollary

Every logic is term-equivalent to one admitting an equational completeness theorem.

Corollary

Every logic is term-equivalent to one admitting an equational completeness theorem.

Proof.

Corollary

Every logic is term-equivalent to one admitting an equational completeness theorem.

Proof.

- Take a logic $\vdash$ and add to its language two unary operations $\Box$ and $\Diamond$ that behave like the identity map.

Corollary

Every logic is term-equivalent to one admitting an equational completeness theorem.

Proof.

- Take a logic $\vdash$ and add to its language two unary operations $\Box$ and $\Diamond$ that behave like the identity map.
- Clearly, $\vdash$ and the new logic $\vdash^+$ are term-equivalent.

Corollary

Every logic is term-equivalent to one admitting an equational completeness theorem.

Proof.

- Take a logic $\vdash$ and add to its language two unary operations $\Box$ and $\Diamond$ that behave like the identity map.
- Clearly, $\vdash$ and the new logic $\vdash^{+}$ are term-equivalent.
- The new logic $\vdash^{+}$ is not graph-based (because its language comprises $\Box$ and $\Diamond$) and

$$\Box x \equiv_{\vdash} x \equiv_{\vdash} \Diamond x.$$

Corollary

Every logic is term-equivalent to one admitting an equational completeness theorem.

Proof.

- Take a logic $\vdash$ and add to its language two unary operations $\Box$ and $\Diamond$ that behave like the identity map.
- Clearly, $\vdash$ and the new logic $\vdash^+$ are term-equivalent.
- The new logic $\vdash^+$ is not graph-based (because its language comprises $\Box$ and $\Diamond$) and

$$\Box x \equiv_\vdash x \equiv_\vdash \Diamond x.$$

- By the previous Theorem, $\vdash^+$ admits an equational completeness theorem.

Corollary

Every logic is term-equivalent to one admitting an equational completeness theorem.

Proof.

- Take a logic $\vdash$ and add to its language two unary operations $\Box$ and $\Diamond$ that behave like the identity map.
- Clearly, $\vdash$ and the new logic $\vdash^+$ are term-equivalent.
- The new logic $\vdash^+$ is not graph-based (because its language comprises $\Box$ and $\Diamond$) and

$$\Box x \equiv_{\vdash} x \equiv_{\vdash} \Diamond x.$$

- By the previous Theorem, $\vdash^+$ admits an equational completeness theorem. QED

Corollary

Every logic is term-equivalent to one admitting an equational completeness theorem.

Proof.

- Take a logic $\vdash$ and add to its language two unary operations $\Box$ and $\Diamond$ that behave like the identity map.
- Clearly, $\vdash$ and the new logic $\vdash^{+}$ are term-equivalent.
- The new logic $\vdash^{+}$ is not graph-based (because its language comprises $\Box$ and $\Diamond$) and

$$\Box x \equiv_{\vdash} x \equiv_{\vdash} \Diamond x.$$

- By the previous Theorem, $\vdash^{+}$ admits an equational completeness theorem. QED

Observation. Admitting an equational completeness theorem is not a property of clones, free algebras etc.

Examples: modal and substructural logics.

Examples: modal and substructural logics.

Corollary (essentially Blok & Rebgliato 2003)

The following holds:

Examples: modal and substructural logics.

Corollary (essentially Blok & Rebgliato 2003)

The following holds:

- Extensions of fragments of the local consequence of the modal system **K** in which a connective among $\{\wedge, \vee, \rightarrow\}$ is term-definable admit an equational completeness theorem.

Examples: **modal and substructural logics.**

Corollary (essentially Blok & Rebgliato 2003)

The following holds:

- Extensions of fragments of the local consequence of the modal system **K** in which a connective among $\{\wedge, \vee, \to\}$ is term-definable admit an equational completeness theorem.
- Extensions of fragments of $\mathbf{FL_e}$ in which a connective among $\{\wedge, \vee, \cdot, \to\}$ is term-definable admit an equational completeness theorem.

Examples: modal and substructural logics.

Corollary (essentially Blok & Rebgliato 2003)

The following holds:

- Extensions of fragments of the local consequence of the modal system **K** in which a connective among $\{\wedge, \vee, \to\}$ is term-definable admit an equational completeness theorem.
- Extensions of fragments of $\mathbf{FL_e}$ in which a connective among $\{\wedge, \vee, \cdot, \to\}$ is term-definable admit an equational completeness theorem.

- Notice that many of these equational completeness theorems are necessarily nonstandard.

Examples: modal and substructural logics.

Corollary (essentially Blok & Rebgliato 2003)

The following holds:

- Extensions of fragments of the local consequence of the modal system **K** in which a connective among $\{\wedge, \vee, \rightarrow\}$ is term-definable admit an equational completeness theorem.
- Extensions of fragments of $\mathbf{FL_e}$ in which a connective among $\{\wedge, \vee, \cdot, \rightarrow\}$ is term-definable admit an equational completeness theorem.

- Notice that many of these equational completeness theorems are necessarily nonstandard. Why?

Examples: modal and substructural logics.

Corollary (essentially Blok & Rebgliato 2003)

The following holds:

- Extensions of fragments of the local consequence of the modal system **K** in which a connective among $\{\wedge, \vee, \to\}$ is term-definable admit an equational completeness theorem.
- Extensions of fragments of $\mathbf{FL_e}$ in which a connective among $\{\wedge, \vee, \cdot, \to\}$ is term-definable admit an equational completeness theorem.

- Notice that many of these equational completeness theorems are necessarily nonstandard. Why? Because these fragments need not be algebraizable.

The case of locally tabular logics

Definition

A logic $\vdash$ is **locally tabular** if for every $1 \leqslant n \in \omega$ there are only finitely many non-logically equivalent formulas in variables $x_1, \ldots, x_n$.

Definition

A logic $\vdash$ is **locally tabular** if for every $1 \leqslant n \in \omega$ there are only finitely many non-logically equivalent formulas in variables $x_1, \ldots, x_n$.

Example. Any logic induced by a class of matrices whose algebraic reducts belong to a locally finite variety.

Definition

A logic $\vdash$ is **locally tabular** if for every $1 \leqslant n \in \omega$ there are only finitely many non-logically equivalent formulas in variables $x_1, \ldots, x_n$.

Example. Any logic induced by a class of matrices whose algebraic reducts belong to a locally finite variety.

Corollary

Every locally tabular logic $\vdash$ that is not graph-based admits an equational completeness theorem.

Definition

A logic $\vdash$ is **locally tabular** if for every $1 \leqslant n \in \omega$ there are only finitely many non-logically equivalent formulas in variables $x_1, \dots, x_n$.

Example. Any logic induced by a class of matrices whose algebraic reducts belong to a locally finite variety.

Corollary

Every locally tabular logic $\vdash$ that is not graph-based admits an equational completeness theorem.

Proof.

Definition

A logic $\vdash$ is **locally tabular** if for every $1 \leqslant n \in \omega$ there are only finitely many non-logically equivalent formulas in variables $x_1, \dots, x_n$.

Example. Any logic induced by a class of matrices whose algebraic reducts belong to a locally finite variety.

Corollary

Every locally tabular logic $\vdash$ that is not graph-based admits an equational completeness theorem.

Proof.

- Take a basic connective $f(x_1, \dots, x_k)$ and define

$$\Box x := f(x, \dots, x).$$

Definition

A logic $\vdash$ is **locally tabular** if for every $1 \leqslant n \in \omega$ there are only finitely many non-logically equivalent formulas in variables $x_1, \ldots, x_n$.

Example. Any logic induced by a class of matrices whose algebraic reducts belong to a locally finite variety.

Corollary

Every locally tabular logic $\vdash$ that is not graph-based admits an equational completeness theorem.

Proof.

- Take a basic connective $f(x_1, \ldots, x_k)$ and define

$$\Box x := f(x, \ldots, x).$$

- By local tabularity, there are nonnegative integers $m \leqslant n$ s.t.

$$\Box^m x \equiv_\vdash \Box^{n+1} x.$$

Definition

A logic $\vdash$ is **locally tabular** if for every $1 \leqslant n \in \omega$ there are only finitely many non-logically equivalent formulas in variables $x_1, \dots, x_n$.

Example. Any logic induced by a class of matrices whose algebraic reducts belong to a locally finite variety.

Corollary

Every locally tabular logic $\vdash$ that is not graph-based admits an equational completeness theorem.

Proof.

- Take a basic connective $f(x_1, \dots, x_k)$ and define
$$\Box x := f(x, \dots, x).$$
- By local tabularity, there are nonnegative integers $m \leqslant n$ s.t.
$$\Box^m x \equiv_{\vdash} \Box^{n+1} x.$$
- By the previous Theorem, we are done.

Definition

A logic $\vdash$ is **locally tabular** if for every $1 \leqslant n \in \omega$ there are only finitely many non-logically equivalent formulas in variables $x_1, \ldots, x_n$.

Example. Any logic induced by a class of matrices whose algebraic reducts belong to a locally finite variety.

Corollary

Every locally tabular logic $\vdash$ that is not graph-based admits an equational completeness theorem.

Proof.

- Take a basic connective $f(x_1, \ldots, x_k)$ and define
$$\Box x := f(x, \ldots, x).$$
- By local tabularity, there are nonnegative integers $m \leqslant n$ s.t.
$$\Box^m x \equiv_{\vdash} \Box^{n+1} x.$$
- By the previous Theorem, we are done. QED

What about locally tabular graph-based logics?

What about locally tabular graph-based logics? Take one such $\vdash$.

What about locally tabular graph-based logics? Take one such $\vdash$.

- The set of connectives of $\vdash$ comprises a single unary connective $\Box$ and some constants $\{\mathbf{c}_i : i < \alpha\}$.

What about locally tabular graph-based logics? Take one such $\vdash$.

- The set of connectives of $\vdash$ comprises a single unary connective $\Box$ and some constants $\{\mathbf{c}_i : i < \alpha\}$.
- By local tabularity, there are $m \leqslant n$ such that $\Box^m x \equiv_\vdash \Box^{n+1} x$.

What about locally tabular graph-based logics? Take one such $\vdash$.

- The set of connectives of $\vdash$ comprises a single unary connective $\Box$ and some constants $\{\mathbf{c}_i : i < \alpha\}$.
- By local tabularity, there are $m \leqslant n$ such that $\Box^m x \equiv_{\vdash} \Box^{n+1} x$.

Theorem

$\vdash$ admits an equational completeness theorem iff there are $i < \alpha$ and $k \leqslant n$ s.t.

What about locally tabular graph-based logics? Take one such $\vdash$.

- The set of connectives of $\vdash$ comprises a single unary connective $\Box$ and some constants $\{\mathbf{c}_i : i < \alpha\}$.
- By local tabularity, there are $m \leqslant n$ such that $\Box^m x \equiv_{\vdash} \Box^{n+1} x$.

Theorem

$\vdash$ admits an equational completeness theorem iff there are $i < \alpha$ and $k \leqslant n$ s.t. either $x \vdash \Box x$

What about locally tabular graph-based logics? Take one such $\vdash$.

- The set of connectives of $\vdash$ comprises a single unary connective $\Box$ and some constants $\{\mathbf{c}_i : i < \alpha\}$.
- By local tabularity, there are $m \leqslant n$ such that $\Box^m x \equiv_{\vdash} \Box^{n+1} x$.

Theorem

$\vdash$ admits an equational completeness theorem iff there are $i < \alpha$ and $k \leqslant n$ s.t. either $x \vdash \Box x$ or $y \vdash \Box^k x$

What about locally tabular graph-based logics? Take one such $\vdash$.

- The set of connectives of $\vdash$ comprises a single unary connective $\Box$ and some constants $\{\mathbf{c}_i : i < \alpha\}$.
- By local tabularity, there are $m \leqslant n$ such that $\Box^m x \equiv_{\vdash} \Box^{n+1} x$.

Theorem

$\vdash$ admits an equational completeness theorem iff there are $i < \alpha$ and $k \leqslant n$ s.t. either $x \vdash \Box x$ or $y \vdash \Box^k x$ or $y \vdash \Box^k \mathbf{c}_i$

What about locally tabular graph-based logics? Take one such $\vdash$.

- The set of connectives of $\vdash$ comprises a single unary connective $\Box$ and some constants $\{\mathbf{c}_i : i < \alpha\}$.
- By local tabularity, there are $m \leqslant n$ such that $\Box^m x \equiv_\vdash \Box^{n+1} x$.

Theorem

$\vdash$ admits an equational completeness theorem iff there are $i < \alpha$ and $k \leqslant n$ s.t. either $x \vdash \Box x$ or $y \vdash \Box^k x$ or $y \vdash \Box^k \mathbf{c}_i$ or

1. $x, \Box^{t+k} x \dashv\vdash \Box^t \mathbf{c}_i, x$ for all $t \leqslant n$ and
2. for all $s, g, h, t \leqslant (2n - m + 1)^2$ and $\{u_j \colon s > j \in \omega\} \cup \{v_j \colon s > j \in \omega\} \subseteq \omega$,

$$\{\Box^t x\} \cup \{\Box^{u_j} x \colon s > j \in \omega\} \cup \{\Box^{v_j} x \colon s > j \in \omega\} \vdash \Box^{t+g} x$$

$$\{\Box^{u_j} x \colon s > j \in \omega\} \cup \{\Box^{v_j} x \colon s > j \in \omega\} \vdash \Box^h \mathbf{c}_i,$$

provided that $u_j < v_j \leqslant n + n - m + 1$ for all $s > j \in \omega$, and that $\gcd(\{v_j - u_j \colon s > j \in \omega\})$ divides g and $h + k$.

- Thus, we have a complete description of locally tabular logics admitting an equational completeness theorem.

- Thus, we have a complete description of locally tabular logics admitting an equational completeness theorem.
- Using the bounds for the graph-based case, one gets

- Thus, we have a complete description of locally tabular logics admitting an equational completeness theorem.
- Using the bounds for the graph-based case, one gets

Corollary

The problem of determining whether a locally tabular logic admits an equational completeness theorem is decidable.

- Thus, we have a complete description of locally tabular logics admitting an equational completeness theorem.
- Using the bounds for the graph-based case, one gets

Corollary

The problem of determining whether a locally tabular logic admits an equational completeness theorem is decidable.

Observation. In the above result logics can be presented either by a finite set of finite matrices or by a finite Hilbert calculus.

- Thus, we have a complete description of locally tabular logics admitting an equational completeness theorem.
- Using the bounds for the graph-based case, one gets

Corollary

The problem of determining whether a locally tabular logic admits an equational completeness theorem is decidable.

Observation. In the above result logics can be presented either by a finite set of finite matrices or by a finite Hilbert calculus.

- The naive decision procedure runs in exponential time.

- Thus, we have a complete description of locally tabular logics admitting an equational completeness theorem.
- Using the bounds for the graph-based case, one gets

Corollary

The problem of determining whether a locally tabular logic admits an equational completeness theorem is decidable.

Observation. In the above result logics can be presented either by a finite set of finite matrices or by a finite Hilbert calculus.

- The naive decision procedure runs in exponential time.

Open problem. Is this problem is complete for EXPTIME?

The case of logics with theorems

- A formula φ is a theorem of a logic $\vdash$ if $\emptyset \vdash \varphi$.

- A formula φ is a theorem of a logic $\vdash$ if $\emptyset \vdash \varphi$.

Definition

A logic $\vdash$ is **assertional** if there is a class of algebras K with a term-definable element $\top$ s.t. for all set of formulas $\Gamma \cup \{\varphi\}$,

- A formula φ is a theorem of a logic $\vdash$ if $\emptyset \vdash \varphi$.

Definition

A logic $\vdash$ is **assertional** if there is a class of algebras K with a term-definable element $\top$ s.t. for all set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \models_{\mathsf{K}} \boldsymbol{\tau}(\varphi)$$

where $\boldsymbol{\tau}(x) := \{x \approx \top\}$.

- A formula φ is a theorem of a logic $\vdash$ if $\emptyset \vdash \varphi$.

Definition

A logic $\vdash$ is **assertional** if there is a class of algebras K with a term-definable element $\top$ s.t. for all set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi)$$

where $\boldsymbol{\tau}(x) := \{x \approx \top\}$.

- Assertional logics admit an equational completeness theorem by definition.

- A formula φ is a theorem of a logic $\vdash$ if $\emptyset \vdash \varphi$.

Definition

A logic $\vdash$ is **assertional** if there is a class of algebras K with a term-definable element $\top$ s.t. for all set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi)$$

where $\boldsymbol{\tau}(x) := \{x \approx \top\}$.

- Assertional logics admit an equational completeness theorem by definition. They have theorems, since $\emptyset \vdash \top$.

- A formula φ is a theorem of a logic $\vdash$ if $\emptyset \vdash \varphi$.

Definition

A logic $\vdash$ is **assertional** if there is a class of algebras K with a term-definable element $\top$ s.t. for all set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi)$$

where $\boldsymbol{\tau}(x) := \{x \approx \top\}$.

- Assertional logics admit an equational completeness theorem by definition. They have theorems, since $\emptyset \vdash \top$.

Theorem (essentially Suszko)

A logic $\vdash$ is assertional iff it has theorems and

$$x, y, \delta(x, \vec{z}) \vdash \delta(y, \vec{z}),$$

for every formula $\delta(v, \vec{z})$.

- Logics with theorems admitting an equational completeness theorem can be described as follows:

- Logics with theorems admitting an equational completeness theorem can be described as follows:

Theorem

Let $\vdash$ a logic with a theorem ϵ such that $\mathrm{Var}(\epsilon) \neq \emptyset$.

- Logics with theorems admitting an equational completeness theorem can be described as follows:

Theorem

Let $\vdash$ a logic with a theorem ϵ such that $\mathrm{Var}(\epsilon) \neq \emptyset$. Then $\vdash$ admits an equational completeness theorem iff one of the following conditions holds:

- Logics with theorems admitting an equational completeness theorem can be described as follows:

Theorem

Let $\vdash$ a logic with a theorem ϵ such that $\mathsf{Var}(\epsilon) \neq \emptyset$. Then $\vdash$ admits an equational completeness theorem iff one of the following conditions holds:

1. $\vdash$ is graph-based and assertional; or

- Logics with theorems admitting an equational completeness theorem can be described as follows:

Theorem

Let $\vdash$ a logic with a theorem ϵ such that $\mathsf{Var}(\epsilon) \neq \emptyset$. Then $\vdash$ admits an equational completeness theorem iff one of the following conditions holds:

1. $\vdash$ is graph-based and assertional; or
2. $\vdash$ is not graph-based and there are distinct φ and ψ s.t.

$$\mathsf{Var}(\varphi) \cup \mathsf{Var}(\psi) = \{x\} \text{ and } \varphi \equiv_{\vdash} \psi.$$

- Logics with theorems admitting an equational completeness theorem can be described as follows:

Theorem

Let $\vdash$ a logic with a theorem ϵ such that $\mathsf{Var}(\epsilon) \neq \emptyset$. Then $\vdash$ admits an equational completeness theorem iff one of the following conditions holds:

1. $\vdash$ is graph-based and assertional; or
2. $\vdash$ is not graph-based and there are distinct φ and ψ s.t.

$$\mathsf{Var}(\varphi) \cup \mathsf{Var}(\psi) = \{x\} \text{ and } \varphi \equiv_{\vdash} \psi.$$

Open problem. Extend this characterization beyond logics with theorems (ideally, to all logics).

Definition

A logic $\vdash$ is **protoalgebraic** if there is a set of formulas $\Delta(x, y)$ s.t.

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

Definition

A logic $\vdash$ is **protoalgebraic** if there is a set of formulas $\Delta(x, y)$ s.t.

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

Example. Almost every logic with a respectable implication connective $x \to y$. To see this, take

$$\Delta := \{x \to y\}.$$

Definition

A logic $\vdash$ is **protoalgebraic** if there is a set of formulas $\Delta(x, y)$ s.t.

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

Example. Almost every logic with a respectable implication connective $x \to y$. To see this, take

$$\Delta := \{x \to y\}.$$

Theorem

A protoalgebraic logic $\vdash$ admits an equational completeness theorem iff it has two distinct logically equivalent formulas.

Definition

A logic $\vdash$ is **protoalgebraic** if there is a set of formulas $\Delta(x, y)$ s.t.

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

Example. Almost every logic with a respectable implication connective $x \to y$. To see this, take

$$\Delta := \{x \to y\}.$$

Theorem

A protoalgebraic logic $\vdash$ admits an equational completeness theorem iff it has two distinct logically equivalent formulas.

- Essentially all reasonable protoalgebraic logics admit equational completeness theorems.

Definition

A logic $\vdash$ is **protoalgebraic** if there is a set of formulas $\Delta(x, y)$ s.t.

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

Example. Almost every logic with a respectable implication connective $x \to y$. To see this, take

$$\Delta := \{x \to y\}.$$

Theorem

A protoalgebraic logic $\vdash$ admits an equational completeness theorem iff it has two distinct logically equivalent formulas.

- Essentially all reasonable protoalgebraic logics admit equational completeness theorems.
- However, **P–W** is a protoalgebraic logic lacking any equational completeness theorem.

- While most protoalgebraic logics admit an equational completeness theorem, this might be necessarily nonstandard.

- While most protoalgebraic logics admit an equational completeness theorem, this might be necessarily nonstandard.
- This happens already in normal modal logics:

- While most protoalgebraic logics admit an equational completeness theorem, this might be necessarily nonstandard.
- This happens already in normal modal logics:

Observation

The local consequence of the modal system **K** (resp. **K4**, **S4**) does **not** admit an equational completeness theorem w.r.t. the variety of modal algebras (resp. of K4-algebras, resp. of interior algebras).

- While most protoalgebraic logics admit an equational completeness theorem, this might be necessarily nonstandard.
- This happens already in normal modal logics:

Observation

The local consequence of the modal system **K** (resp. **K4**, **S4**) does **not** admit an equational completeness theorem w.r.t. the variety of modal algebras (resp. of K4-algebras, resp. of interior algebras).

Proof sketch.

- While most protoalgebraic logics admit an equational completeness theorem, this might be necessarily nonstandard.
- This happens already in normal modal logics:

Observation

The local consequence of the modal system **K** (resp. **K4**, **S4**) does **not** admit an equational completeness theorem w.r.t. the variety of modal algebras (resp. of K4-algebras, resp. of interior algebras).

Proof sketch.

- The local consequence of the modal system **K** is defined as

- While most protoalgebraic logics admit an equational completeness theorem, this might be necessarily nonstandard.
- This happens already in normal modal logics:

Observation

The local consequence of the modal system **K** (resp. **K4**, **S4**) does **not** admit an equational completeness theorem w.r.t. the variety of modal algebras (resp. of K4-algebras, resp. of interior algebras).

Proof sketch.

- The local consequence of the modal system **K** is defined as

$$\Gamma \vdash^{\mathscr{L}}_{\mathbf{K}} \varphi \iff \text{for every Kripke frame } \langle W, R\rangle, \text{ valuation } v\colon \mathrm{Var} \to W, \text{ and world } w \in W, \text{ if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.$$

- Suppose, by contradiction, that $\vdash_{\mathbf{K}}^{\mathscr{L}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.

- Suppose, by contradiction, that $\vdash_{\mathbf{K}}^{\mathscr{L}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash_{\mathbf{K}}^{\mathscr{L}} \delta$.

- Suppose, by contradiction, that $\vdash^{\mathscr{L}}_{\mathbf{K}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash^{\mathscr{L}}_{\mathbf{K}} \delta$.
- Since $\epsilon \approx \delta \in \boldsymbol{\tau}$, we get

$$x, \Box(\delta \to \delta) \vdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta).$$

- Suppose, by contradiction, that $\vdash^{\mathscr{L}}_{\mathbf{K}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash^{\mathscr{L}}_{\mathbf{K}} \delta$.
- Since $\epsilon \approx \delta \in \boldsymbol{\tau}$, we get

$$x, \Box(\delta \to \delta) \vdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta).$$

- Then there are a Kripke frame $\langle W, R \rangle$, a valuation v, and a world $w \in W$ such that

$$w, v \Vdash \epsilon \text{ and } w, v \nVdash \delta.$$

- Suppose, by contradiction, that $\vdash^{\mathscr{L}}_{\mathbf{K}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash^{\mathscr{L}}_{\mathbf{K}} \delta$.
- Since $\epsilon \approx \delta \in \boldsymbol{\tau}$, we get

$$x, \Box(\delta \to \delta) \vdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta).$$

- Then there are a Kripke frame $\langle W, R \rangle$, a valuation v, and a world $w \in W$ such that

$$w, v \Vdash \epsilon \text{ and } w, v \nVdash \delta.$$

- Attach to $\langle W, R \rangle$ a new point w^+ that access everything.

- Suppose, by contradiction, that $\vdash^{\mathscr{L}}_{\mathbf{K}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash^{\mathscr{L}}_{\mathbf{K}} \delta$.
- Since $\epsilon \approx \delta \in \boldsymbol{\tau}$, we get

$$x, \Box(\delta \to \delta) \vdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta).$$

- Then there are a Kripke frame $\langle W, R \rangle$, a valuation v, and a world $w \in W$ such that

$$w, v \Vdash \epsilon \text{ and } w, v \nVdash \delta.$$

- Attach to $\langle W, R \rangle$ a new point w^+ that access everything. Call the new frame $\langle W^+, R^+ \rangle$.

- Suppose, by contradiction, that $\vdash^{\mathscr{L}}_{\mathbf{K}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash^{\mathscr{L}}_{\mathbf{K}} \delta$.
- Since $\epsilon \approx \delta \in \boldsymbol{\tau}$, we get

$$x, \Box(\delta \to \delta) \vdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta).$$

- Then there are a Kripke frame $\langle W, R \rangle$, a valuation v, and a world $w \in W$ such that

$$w, v \Vdash \epsilon \text{ and } w, v \nVdash \delta.$$

- Attach to $\langle W, R \rangle$ a new point w^+ that access everything. Call the new frame $\langle W^+, R^+ \rangle$.
- Extend the valuation v to v^+ into $\langle W^+, R^+ \rangle$, stipulating that x holds at w^+.

- Suppose, by contradiction, that $\vdash^{\mathscr{L}}_{\mathbf{K}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash^{\mathscr{L}}_{\mathbf{K}} \delta$.
- Since $\epsilon \approx \delta \in \boldsymbol{\tau}$, we get

$$x, \Box(\delta \to \delta) \vdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta).$$

- Then there are a Kripke frame $\langle W, R \rangle$, a valuation v, and a world $w \in W$ such that

$$w, v \Vdash \epsilon \text{ and } w, v \nVdash \delta.$$

- Attach to $\langle W, R \rangle$ a new point w^+ that access everything. Call the new frame $\langle W^+, R^+ \rangle$.
- Extend the valuation v to v^+ into $\langle W^+, R^+ \rangle$, stipulating that x holds at w^+. Then

$$w^+, v^+ \Vdash x \text{ and } w^+, v^+ \nVdash \Box(\epsilon \to \delta).$$

- Suppose, by contradiction, that $\vdash^{\mathscr{L}}_{\mathbf{K}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash^{\mathscr{L}}_{\mathbf{K}} \delta$.
- Since $\epsilon \approx \delta \in \boldsymbol{\tau}$, we get

$$x, \Box(\delta \to \delta) \vdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta).$$

- Then there are a Kripke frame $\langle W, R \rangle$, a valuation v, and a world $w \in W$ such that

$$w, v \Vdash \epsilon \text{ and } w, v \nVdash \delta.$$

- Attach to $\langle W, R \rangle$ a new point w^+ that access everything. Call the new frame $\langle W^+, R^+ \rangle$.
- Extend the valuation v to v^+ into $\langle W^+, R^+ \rangle$, stipulating that x holds at w^+. Then

$$w^+, v^+ \Vdash x \text{ and } w^+, v^+ \nVdash \Box(\epsilon \to \delta).$$

- Thus, $x, \Box(\delta \to \delta) \nvdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta)$, a contradiction.

- Suppose, by contradiction, that $\vdash^{\mathscr{L}}_{\mathbf{K}}$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash^{\mathscr{L}}_{\mathbf{K}} \delta$.
- Since $\epsilon \approx \delta \in \boldsymbol{\tau}$, we get

$$x, \Box(\delta \to \delta) \vdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta).$$

- Then there are a Kripke frame $\langle W, R \rangle$, a valuation v, and a world $w \in W$ such that

$$w, v \Vdash \epsilon \text{ and } w, v \nVdash \delta.$$

- Attach to $\langle W, R \rangle$ a new point w^+ that access everything. Call the new frame $\langle W^+, R^+ \rangle$.
- Extend the valuation v to v^+ into $\langle W^+, R^+ \rangle$, stipulating that x holds at w^+. Then

$$w^+, v^+ \Vdash x \text{ and } w^+, v^+ \nVdash \Box(\epsilon \to \delta).$$

- Thus, $x, \Box(\delta \to \delta) \nvdash^{\mathscr{L}}_{\mathbf{K}} \Box(\epsilon \to \delta)$, a contradiction. QED

Open problems.

Open problems.

- Characterize normal modal logics whose local consequences admit a standard equational completeness theorem, i.e., one w.r.t. their corresponding variety of modal algebras.

Open problems.

- Characterize normal modal logics whose local consequences admit a standard equational completeness theorem, i.e., one w.r.t. their corresponding variety of modal algebras.
- Investigate the same problem for the incarnation of local consequences in the substructural setting:

Open problems.

- Characterize normal modal logics whose local consequences admit a standard equational completeness theorem, i.e., one w.r.t. their corresponding variety of modal algebras.
- Investigate the same problem for the incarnation of local consequences in the substructural setting:

  logics **preserving** **degrees of truth**.

Open problems.

- Characterize normal modal logics whose local consequences admit a standard equational completeness theorem, i.e., one w.r.t. their corresponding variety of modal algebras.
- Investigate the same problem for the incarnation of local consequences in the substructural setting:

  logics **preserving degrees of truth**.
- Given a variety K of FL-algebras, define

Open problems.

- Characterize normal modal logics whose local consequences admit a standard equational completeness theorem, i.e., one w.r.t. their corresponding variety of modal algebras.
- Investigate the same problem for the incarnation of local consequences in the substructural setting:

  logics **preserving degrees of truth**.

- Given a variety K of FL-algebras, define

$$\Gamma \vdash^{\leqslant}_{\mathsf{K}} \varphi \Longleftrightarrow \text{for every } \boldsymbol{A} \in \mathsf{K}, \text{ hom } f \colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ and } a \in A, \text{ if } a \leqslant f(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } a \leqslant f(\varphi).$$

Open problems.

- Characterize normal modal logics whose local consequences admit a standard equational completeness theorem, i.e., one w.r.t. their corresponding variety of modal algebras.
- Investigate the same problem for the incarnation of local consequences in the substructural setting:

  logics **preserving degrees of truth**.

- Given a variety K of FL-algebras, define

$$\Gamma \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{K}, \text{ hom } f \colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ and } a \in A, \text{ if } a \leqslant f(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } a \leqslant f(\varphi).$$

For instance, does Łukasiewicz logic preserving degrees of truth admit an equational completeness theorem w.r.t. MV-algebras?

Open problems.

- Characterize normal modal logics whose local consequences admit a standard equational completeness theorem, i.e., one w.r.t. their corresponding variety of modal algebras.
- Investigate the same problem for the incarnation of local consequences in the substructural setting:

  logics **preserving degrees of truth**.
- Given a variety K of FL-algebras, define

$$\Gamma \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{K}, \text{ hom } f \colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ and } a \in A, \text{ if } a \leqslant f(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } a \leqslant f(\varphi).$$

For instance, does Łukasiewicz logic preserving degrees of truth admit an equational completeness theorem w.r.t. MV-algebras?

- More in general, characterize logics admitting a standard equational completeness theorem.

Undecidability: coding the halting problem

We know that:

We know that:

- Determining whether a logic admits an equational completeness theorem is decidable for locally tabular logics.

We know that:

- Determining whether a logic admits an equational completeness theorem is decidable for locally tabular logics.

We shall see that:

We know that:

- Determining whether a logic admits an equational completeness theorem is decidable for locally tabular logics.

We shall see that:

- The same problem becomes undecidable for logics presented by an arbitrary Hilbert calculus.

We know that:

- Determining whether a logic admits an equational completeness theorem is decidable for locally tabular logics.

We shall see that:

- The same problem becomes undecidable for logics presented by an arbitrary Hilbert calculus.
- Undecidability persists if we restrict to protoalgebraic logics.

We know that:

- Determining whether a logic admits an equational completeness theorem is decidable for locally tabular logics.

We shall see that:

- The same problem becomes undecidable for logics presented by an arbitrary Hilbert calculus.
- Undecidability persists if we restrict to protoalgebraic logics.

Strategy:

We know that:

- Determining whether a logic admits an equational completeness theorem is decidable for locally tabular logics.

We shall see that:

- The same problem becomes undecidable for logics presented by an arbitrary Hilbert calculus.
- Undecidability persists if we restrict to protoalgebraic logics.

Strategy:

- We code the halting problem inside that of determining whether a protoalgebraic logic (presented by a finite Hilbert calculus) admits an equational completeness theorem.

We know that:

- Determining whether a logic admits an equational completeness theorem is decidable for locally tabular logics.

We shall see that:

- The same problem becomes undecidable for logics presented by an arbitrary Hilbert calculus.
- Undecidability persists if we restrict to protoalgebraic logics.

Strategy:

- We code the halting problem inside that of determining whether a protoalgebraic logic (presented by a finite Hilbert calculus) admits an equational completeness theorem.

Difficulty:

We know that:

- Determining whether a logic admits an equational completeness theorem is decidable for locally tabular logics.

We shall see that:

- The same problem becomes undecidable for logics presented by an arbitrary Hilbert calculus.
- Undecidability persists if we restrict to protoalgebraic logics.

Strategy:

- We code the halting problem inside that of determining whether a protoalgebraic logic (presented by a finite Hilbert calculus) admits an equational completeness theorem.

Difficulty:

- A protoalgebraic logic admits an equational completeness theorem iff its has two distinct logically equivalent formulas. We need to code the halting problem without letting the logic know that word composition is associative.

Turing machines.

Turing machines.

- A Turing machine **M** is a tuple $\langle P, Q, q_0, \delta \rangle$ where P and Q are sets of states, $q_0 \in Q$ is the initial state, Q the set of non-final states, P the set of final states, and

$$\delta\colon Q \times \{0, 1, \varnothing\} \to (Q \cup P) \times \{0, 1\} \times \{L, R\}.$$

Turing machines.

- A Turing machine **M** is a tuple $\langle P, Q, q_0, \delta\rangle$ where P and Q are sets of states, $q_0 \in Q$ is the initial state, Q the set of non-final states, P the set of final states, and

$$\delta\colon Q \times \{0, 1, \varnothing\} \to (Q \cup P) \times \{0, 1\} \times \{L, R\}.$$

- Instruction of the form $\delta(q, a) = \langle q', b, L\rangle$ mean: if the machine **M** reads a at state q, then it replaces a with b, moves left, and switches to state q'.

Turing machines.

- A Turing machine **M** is a tuple $\langle P, Q, q_0, \delta \rangle$ where P and Q are sets of states, $q_0 \in Q$ is the initial state, Q the set of non-final states, P the set of final states, and

$$\delta \colon Q \times \{0, 1, \emptyset\} \to (Q \cup P) \times \{0, 1\} \times \{L, R\}.$$

- Instruction of the form $\delta(q, a) = \langle q', b, L \rangle$ mean: if the machine **M** reads a at state q, then it replaces a with b, moves left, and switches to state q'.
- Our Turing machines can write only zeros and ones, but can read 0, 1, and the empty symbol $\emptyset$.

Turing machines.

- A Turing machine **M** is a tuple $\langle P, Q, q_0, \delta \rangle$ where P and Q are sets of states, $q_0 \in Q$ is the initial state, Q the set of non-final states, P the set of final states, and

$$\delta\colon Q \times \{0, 1, \varnothing\} \to (Q \cup P) \times \{0, 1\} \times \{L, R\}.$$

- Instruction of the form $\delta(q, a) = \langle q', b, L \rangle$ mean: if the machine **M** reads a at state q, then it replaces a with b, moves left, and switches to state q'.
- Our Turing machines can write only zeros and ones, but can read 0, 1, and the empty symbol $\varnothing$.

Configurations.

Turing machines.

- A Turing machine **M** is a tuple $\langle P, Q, q_0, \delta \rangle$ where P and Q are sets of states, $q_0 \in Q$ is the initial state, Q the set of non-final states, P the set of final states, and

$$\delta\colon Q \times \{0, 1, \varnothing\} \to (Q \cup P) \times \{0, 1\} \times \{L, R\}.$$

- Instruction of the form $\delta(q, a) = \langle q', b, L \rangle$ mean: if the machine **M** reads a at state q, then it replaces a with b, moves left, and switches to state q'.
- Our Turing machines can write only zeros and ones, but can read 0, 1, and the empty symbol $\varnothing$.

Configurations.

- A configuration for **M** is a tuple $\langle q, \vec{w}, v, \vec{u} \rangle$ where $q \in Q \cup P$, $\vec{w}$ and $\vec{u}$ are either finite non-empty sequences of zeros and ones or the one-element sequence $\langle \varnothing \rangle$, and

$$v \in \{\langle 0 \rangle, \langle 1 \rangle, \langle \varnothing \rangle\}.$$

Turing machines.

- A Turing machine $\mathbf{M}$ is a tuple $\langle P, Q, q_0, \delta \rangle$ where P and Q are sets of states, $q_0 \in Q$ is the initial state, Q the set of non-final states, P the set of final states, and

$$\delta \colon Q \times \{0, 1, \varnothing\} \to (Q \cup P) \times \{0, 1\} \times \{L, R\}.$$

- Instruction of the form $\delta(q, a) = \langle q', b, L \rangle$ mean: if the machine $\mathbf{M}$ reads a at state q, then it replaces a with b, moves left, and switches to state q'.
- Our Turing machines can write only zeros and ones, but can read 0, 1, and the empty symbol $\varnothing$.

Configurations.

- A configuration for $\mathbf{M}$ is a tuple $\langle q, \vec{w}, v, \vec{u} \rangle$ where $q \in Q \cup P$, $\vec{w}$ and $\vec{u}$ are either finite non-empty sequences of zeros and ones or the one-element sequence $\langle \varnothing \rangle$, and

$$v \in \{\langle 0 \rangle, \langle 1 \rangle, \langle \varnothing \rangle\}.$$

- Given two configurations $\mathtt{c}$ and $\mathtt{d}$ for $\mathbf{M}$, we say that $\mathtt{c}$ yields $\mathtt{d}$ if $\mathbf{M}$ allows to move from $\mathtt{c}$ to $\mathtt{d}$ in a single step.

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in
$$P \cup Q \cup \{0, 1, \emptyset\},$$
a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in

$$P \cup Q \cup \{0, 1, \emptyset\},$$

  a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.
- The **logic of M** $\vdash_{\mathbf{M}}$ is axiomatized by the rules

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in

$$P \cup Q \cup \{0, 1, \varnothing\},$$

  a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.
- The **logic of M** $\vdash_{\mathbf{M}}$ is axiomatized by the rules

$$q \cdot \lambda(x \cdot y, a, z) \rhd q' \cdot \lambda(x, y, b \cdot z)$$

$$\hat{q} \cdot \lambda(x, \hat{a}, y \cdot z) \rhd \hat{q}' \cdot \lambda(x \cdot \hat{b}, y, z)$$

$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(\varnothing \cdot x, y, z)$$

$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(x, y, z \cdot \varnothing)$$

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in
$$P \cup Q \cup \{0, 1, \varnothing\},$$
a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.
- The **logic of M** $\vdash_{\mathbf{M}}$ is axiomatized by the rules
$$q \cdot \lambda(x \cdot y, a, z) \rhd q' \cdot \lambda(x, y, b \cdot z)$$
$$\hat{q} \cdot \lambda(x, \hat{a}, y \cdot z) \rhd \hat{q}' \cdot \lambda(x \cdot \hat{b}, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd\rhd p \cdot \lambda(\varnothing \cdot x, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd\rhd p \cdot \lambda(x, y, z \cdot \varnothing)$$
for all $p, q, q', \hat{q}, \hat{q}' \in P \cup Q$ and $a, \hat{a}, b, \hat{b} \in \{0, 1, \varnothing\}$ s.t.
$$\delta(q, a) = \langle q', b, L\rangle \text{ and } \delta(\hat{q}, \hat{a}) = \langle \hat{q}', \hat{b}, R\rangle.$$

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in

$$P \cup Q \cup \{0, 1, \emptyset\},$$

a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.

- The **logic of M** $\vdash_{\mathbf{M}}$ is axiomatized by the rules

$$q \cdot \lambda(x \cdot y, a, z) \rhd q' \cdot \lambda(x, y, b \cdot z)$$

$$\hat{q} \cdot \lambda(x, \hat{a}, y \cdot z) \rhd \hat{q}' \cdot \lambda(x \cdot \hat{b}, y, z)$$

$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(\emptyset \cdot x, y, z)$$

$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(x, y, z \cdot \emptyset)$$

for all $p, q, q', \hat{q}, \hat{q}' \in P \cup Q$ and $a, \hat{a}, b, \hat{b} \in \{0, 1, \emptyset\}$ s.t.

$$\delta(q, a) = \langle q', b, L\rangle \text{ and } \delta(\hat{q}, \hat{a}) = \langle \hat{q}', \hat{b}, R\rangle.$$

- We can code configurations $c = \langle q, \vec{w}, v, \vec{u}\rangle$ for $\mathbf{M}$, where

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in

$$P \cup Q \cup \{0, 1, \varnothing\},$$

a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.

- The **logic of M** $\vdash_{\mathbf{M}}$ is axiomatized by the rules

$$q \cdot \lambda(x \cdot y, a, z) \rhd q' \cdot \lambda(x, y, b \cdot z)$$
$$\hat{q} \cdot \lambda(x, \hat{a}, y \cdot z) \rhd \hat{q}' \cdot \lambda(x \cdot \hat{b}, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(\varnothing \cdot x, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(x, y, z \cdot \varnothing)$$

for all $p, q, q', \hat{q}, \hat{q}' \in P \cup Q$ and $a, \hat{a}, b, \hat{b} \in \{0, 1, \varnothing\}$ s.t.

$$\delta(q, a) = \langle q', b, L\rangle \text{ and } \delta(\hat{q}, \hat{a}) = \langle \hat{q}', \hat{b}, R\rangle.$$

- We can code configurations $c = \langle q, \vec{w}, v, \vec{u}\rangle$ for **M**, where

$$\vec{w} = \langle w_1, \ldots, w_n\rangle, v = \langle a\rangle, \text{ and } \vec{u} = \langle u_1, \ldots, u_m\rangle,$$

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in

$$P \cup Q \cup \{0, 1, \emptyset\},$$

a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.

- The **logic of M** $\vdash_{\mathbf{M}}$ is axiomatized by the rules

$$q \cdot \lambda(x \cdot y, a, z) \rhd q' \cdot \lambda(x, y, b \cdot z)$$

$$\hat{q} \cdot \lambda(x, \hat{a}, y \cdot z) \rhd \hat{q}' \cdot \lambda(x \cdot \hat{b}, y, z)$$

$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(\emptyset \cdot x, y, z)$$

$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(x, y, z \cdot \emptyset)$$

for all $p, q, q', \hat{q}, \hat{q}' \in P \cup Q$ and $a, \hat{a}, b, \hat{b} \in \{0, 1, \emptyset\}$ s.t.

$$\delta(q, a) = \langle q', b, L\rangle \text{ and } \delta(\hat{q}, \hat{a}) = \langle \hat{q}', \hat{b}, R\rangle.$$

- We can code configurations $c = \langle q, \vec{w}, v, \vec{u}\rangle$ for $\mathbf{M}$, where

$$\vec{w} = \langle w_1, \ldots, w_n\rangle, v = \langle a\rangle, \text{ and } \vec{u} = \langle u_1, \ldots, u_m\rangle,$$

with formulas of $\mathscr{L}(\mathbf{M})$ as follows:

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in

$$P \cup Q \cup \{0, 1, \emptyset\},$$

a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.

- The **logic of M** $\vdash_{\mathbf{M}}$ is axiomatized by the rules

$$q \cdot \lambda(x \cdot y, a, z) \rhd q' \cdot \lambda(x, y, b \cdot z)$$
$$\hat{q} \cdot \lambda(x, \hat{a}, y \cdot z) \rhd \hat{q}' \cdot \lambda(x \cdot \hat{b}, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(\emptyset \cdot x, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(x, y, z \cdot \emptyset)$$

for all $p, q, q', \hat{q}, \hat{q}' \in P \cup Q$ and $a, \hat{a}, b, \hat{b} \in \{0, 1, \emptyset\}$ s.t.

$$\delta(q, a) = \langle q', b, L\rangle \text{ and } \delta(\hat{q}, \hat{a}) = \langle \hat{q}', \hat{b}, R\rangle.$$

- We can code configurations $\mathsf{c} = \langle q, \vec{w}, v, \vec{u}\rangle$ for $\mathbf{M}$, where

$$\vec{w} = \langle w_1, \dots, w_n\rangle, v = \langle a\rangle, \text{ and } \vec{u} = \langle u_1, \dots, u_m\rangle,$$

with formulas of $\mathscr{L}(\mathbf{M})$ as follows:

$$\varphi_{\mathsf{c}} := q \cdot \lambda((\cdots(w_1 \cdot w_2)\cdots) \cdot w_n, a, u_1 \cdot (\dots(u_{m-1} \cdot u_m)\dots)).$$

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in
$$P \cup Q \cup \{0, 1, \varnothing\},$$
a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.
- The **logic of M** $\vdash_{\mathbf{M}}$ is axiomatized by the rules
$$q \cdot \lambda(x \cdot y, a, z) \rhd q' \cdot \lambda(x, y, b \cdot z)$$
$$\hat{q} \cdot \lambda(x, \hat{a}, y \cdot z) \rhd \hat{q}' \cdot \lambda(x \cdot \hat{b}, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(\varnothing \cdot x, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(x, y, z \cdot \varnothing)$$
for all $p, q, q', \hat{q}, \hat{q}' \in P \cup Q$ and $a, \hat{a}, b, \hat{b} \in \{0, 1, \varnothing\}$ s.t.
$$\delta(q, a) = \langle q', b, L\rangle \text{ and } \delta(\hat{q}, \hat{a}) = \langle \hat{q}', \hat{b}, R\rangle.$$
- We can code configurations $\mathtt{c} = \langle q, \vec{w}, v, \vec{u}\rangle$ for $\mathbf{M}$, where
$$\vec{w} = \langle w_1, \ldots, w_n\rangle, v = \langle a\rangle, \text{ and } \vec{u} = \langle u_1, \ldots, u_m\rangle,$$
with formulas of $\mathscr{L}(\mathbf{M})$ as follows:
$$\varphi_{\mathtt{c}} := q \cdot \lambda((\cdots(w_1 \cdot w_2)\cdots) \cdot w_n, a, u_1 \cdot (\ldots(u_{m-1} \cdot u_m)\ldots)).$$
- If $\mathtt{c}$ yields $\mathtt{d}$, then $\varphi_{\mathtt{c}} \vdash_{\mathbf{M}} \varphi_{\mathtt{d}}$.

Let **M** be a Turing machine and $\vec{t}$ and input.

Let M be a Turing machine and $\vec{t}$ and input. Let also $\mathsf{In}(\mathsf{M}, \vec{t}\,)$ be the initial configuration of M on input $\vec{t}$.

Let **M** be a Turing machine and $\vec{t}$ and input. Let also $\mathsf{In}(\mathbf{M}, \vec{t}\,)$ be the initial configuration of **M** on input $\vec{t}$.

- Let $\vdash^{\vec{t}}_{\mathbf{M}}$ be the expansion of the logic of **M** with a new connective $\leftrightarrow$ and rules

Let $\mathbf{M}$ be a Turing machine and $\vec{t}$ and input. Let also $\mathsf{In}(\mathbf{M}, \vec{t}\,)$ be the initial configuration of $\mathbf{M}$ on input $\vec{t}$.

- Let $\vdash^{\vec{t}}_{\mathbf{M}}$ be the expansion of the logic of $\mathbf{M}$ with a new connective $\leftrightarrow$ and rules

$$\emptyset \rhd \varphi_{\mathsf{In}(\mathbf{M},\vec{t}\,)}$$

$$p \cdot y \rhd x \leftrightarrow (x \cdot x)$$

$$\emptyset \rhd x \leftrightarrow x$$

$$x, x \leftrightarrow y \rhd y$$

$$x_1 \leftrightarrow y_1, \dots, x_n \leftrightarrow y_n \rhd *(x_1, \dots, x_n) \leftrightarrow *(y_1, \dots, y_n)$$

Let $\mathbf{M}$ be a Turing machine and $\vec{t}$ and input. Let also $\mathsf{In}(\mathbf{M}, \vec{t})$ be the initial configuration of $\mathbf{M}$ on input $\vec{t}$.

- Let $\vdash^{\vec{t}}_{\mathbf{M}}$ be the expansion of the logic of $\mathbf{M}$ with a new connective $\leftrightarrow$ and rules

$$\emptyset \rhd \varphi_{\mathsf{In}(\mathbf{M},\vec{t})}$$

$$p \cdot y \rhd x \leftrightarrow (x \cdot x)$$

$$\emptyset \rhd x \leftrightarrow x$$

$$x, x \leftrightarrow y \rhd y$$

$$x_1 \leftrightarrow y_1, \ldots, x_n \leftrightarrow y_n \rhd *(x_1, \ldots, x_n) \leftrightarrow *(y_1, \ldots, y_n)$$

  for every final state p, every positive integer n, and every n-ary connective $*$.

Notice that

Let **M** be a Turing machine and $\vec{t}$ and input. Let also $\mathsf{In}(\mathbf{M}, \vec{t})$ be the initial configuration of **M** on input $\vec{t}$.

- Let $\vdash_{\mathbf{M}}^{\vec{t}}$ be the expansion of the logic of **M** with a new connective $\leftrightarrow$ and rules

$$\emptyset \rhd \varphi_{\mathsf{In}(\mathbf{M},\vec{t})}$$
$$p \cdot y \rhd x \leftrightarrow (x \cdot x)$$
$$\emptyset \rhd x \leftrightarrow x$$
$$x, x \leftrightarrow y \rhd y$$
$$x_1 \leftrightarrow y_1, \ldots, x_n \leftrightarrow y_n \rhd *(x_1, \ldots, x_n) \leftrightarrow *(y_1, \ldots, y_n)$$

  for every final state p, every positive integer n, and every n-ary connective $*$.

Notice that

- The last three rules guarantee that $\leftrightarrow$ captures logical equivalence, in the sense that

Let $\mathbf{M}$ be a Turing machine and $\vec{t}$ and input. Let also $\mathsf{In}(\mathbf{M}, \vec{t})$ be the initial configuration of $\mathbf{M}$ on input $\vec{t}$.

- Let $\vdash_{\mathbf{M}}^{\vec{t}}$ be the expansion of the logic of $\mathbf{M}$ with a new connective $\leftrightarrow$ and rules

$$\emptyset \rhd \varphi_{\mathsf{In}(\mathbf{M},\vec{t})}$$
$$p \cdot y \rhd x \leftrightarrow (x \cdot x)$$
$$\emptyset \rhd x \leftrightarrow x$$
$$x, x \leftrightarrow y \rhd y$$
$$x_1 \leftrightarrow y_1, \dots, x_n \leftrightarrow y_n \rhd *(x_1, \dots, x_n) \leftrightarrow *(y_1, \dots, y_n)$$

  for every final state p, every positive integer n, and every n-ary connective $*$.

Notice that

- The last three rules guarantee that $\leftrightarrow$ captures logical equivalence, in the sense that

$$\epsilon \equiv_{\vdash_{\mathbf{M}}^{\vec{t}}} \delta \Longleftrightarrow \emptyset \vdash_{\mathbf{M}}^{\vec{t}} \epsilon \leftrightarrow \delta.$$

Let $\mathbf{M}$ be a Turing machine and $\vec{t}$ and input. Let also $\mathsf{In}(\mathbf{M}, \vec{t}\,)$ be the initial configuration of $\mathbf{M}$ on input $\vec{t}$.

- Let $\vdash_{\mathbf{M}}^{\vec{t}}$ be the expansion of the logic of $\mathbf{M}$ with a new connective $\leftrightarrow$ and rules

$$\emptyset \rhd \varphi_{\mathsf{In}(\mathbf{M},\vec{t}\,)}$$
$$p \cdot y \rhd x \leftrightarrow (x \cdot x)$$
$$\emptyset \rhd x \leftrightarrow x$$
$$x, x \leftrightarrow y \rhd y$$
$$x_1 \leftrightarrow y_1, \dots, x_n \leftrightarrow y_n \rhd *(x_1, \dots, x_n) \leftrightarrow *(y_1, \dots, y_n)$$

  for every final state p, every positive integer n, and every n-ary connective $*$.

Notice that

- The last three rules guarantee that $\leftrightarrow$ captures logical equivalence, in the sense that

$$\epsilon \equiv_{\vdash_{\mathbf{M}}^{\vec{t}}} \delta \Longleftrightarrow \emptyset \vdash_{\mathbf{M}}^{\vec{t}} \epsilon \leftrightarrow \delta.$$

- The logic $\vdash_{\mathbf{M}}^{\vec{t}}$ is protoalgebraic with $\Delta(x, y) := \{x \leftrightarrow y\}$.

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input.

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash_{\mathbf{M}}^{\vec{t}}$ admits an equational completeness theorem.

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash^{\vec{t}}_{\mathbf{M}}$ admits an equational completeness theorem.

Proof sketch (easy part only).

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash_{\mathbf{M}}^{\vec{t}}$ admits an equational completeness theorem.

Proof sketch (easy part only).

- If $\mathbf{M}$ halts on $\vec{t}$, there is a sequence of configurations $\mathsf{c}_1, \ldots, \mathsf{c}_k$ s.t. $\mathsf{c}_1 = \mathsf{In}(\mathbf{M}, \vec{t})$, each c_i yields c_{i+1}, and the state p of c_k is final.

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash^{\vec{t}}_{\mathbf{M}}$ admits an equational completeness theorem.

Proof sketch (easy part only).

- If $\mathbf{M}$ halts on $\vec{t}$, there is a sequence of configurations $\mathsf{c}_1, \ldots, \mathsf{c}_k$ s.t. $\mathsf{c}_1 = \mathsf{In}(\mathbf{M}, \vec{t})$, each c_i yields c_{i+1}, and the state p of c_k is final.
- Since each c_i yields c_{i+1},

$$\varphi_{\mathsf{In}(\mathbf{M},\vec{t})} \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathsf{c}_2} \vdash^{\vec{t}}_{\mathbf{M}} \cdots \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathsf{c}_k}.$$

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash^{\vec{t}}_{\mathbf{M}}$ admits an equational completeness theorem.

Proof sketch (easy part only).

- If $\mathbf{M}$ halts on $\vec{t}$, there is a sequence of configurations $\mathsf{c}_1, \ldots, \mathsf{c}_k$ s.t. $\mathsf{c}_1 = \mathsf{In}(\mathbf{M}, \vec{t})$, each c_i yields c_{i+1}, and the state p of c_k is final.
- Since each c_i yields c_{i+1},

$$\varphi_{\mathsf{In}(\mathbf{M},\vec{t})} \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathsf{c}_2} \vdash^{\vec{t}}_{\mathbf{M}} \cdots \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathsf{c}_k}.$$

- By definition, $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathsf{In}(\mathbf{M},\vec{t})}$, whence $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathsf{c}_k}$.

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash^{\vec{t}}_{\mathbf{M}}$ admits an equational completeness theorem.

Proof sketch (easy part only).

- If $\mathbf{M}$ halts on $\vec{t}$, there is a sequence of configurations $\mathsf{c}_1, \ldots, \mathsf{c}_k$ s.t. $\mathsf{c}_1 = \mathrm{In}(\mathbf{M}, \vec{t})$, each c_i yields c_{i+1}, and the state p of c_k is final.
- Since each c_i yields c_{i+1},

$$\varphi_{\mathrm{In}(\mathbf{M},\vec{t})} \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathsf{c}_2} \vdash^{\vec{t}}_{\mathbf{M}} \cdots \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathsf{c}_k}.$$

- By definition, $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathrm{In}(\mathbf{M},\vec{t})}$, whence $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathsf{c}_k}$.
- Since $\varphi_{\mathsf{c}_k} = p \cdot \psi$ for some ψ, we can apply the rule $p \cdot y \rhd x \leftrightarrow (x \cdot x)$, obtaining

$$\emptyset \vdash^{\vec{t}}_{\mathbf{M}} x \leftrightarrow (x \cdot x).$$

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash^{\vec{t}}_{\mathbf{M}}$ admits an equational completeness theorem.

Proof sketch (easy part only).

- If $\mathbf{M}$ halts on $\vec{t}$, there is a sequence of configurations $c_1, \ldots, c_k$ s.t. $c_1 = \mathrm{In}(\mathbf{M}, \vec{t})$, each c_i yields c_{i+1}, and the state p of c_k is final.
- Since each c_i yields c_{i+1},

$$\varphi_{\mathrm{In}(\mathbf{M},\vec{t})} \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_2} \vdash^{\vec{t}}_{\mathbf{M}} \cdots \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_k}.$$

- By definition, $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathrm{In}(\mathbf{M},\vec{t})}$, whence $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_k}$.
- Since $\varphi_{c_k} = p \cdot \psi$ for some ψ, we can apply the rule $p \cdot y \rhd x \leftrightarrow (x \cdot x)$, obtaining

$$\emptyset \vdash^{\vec{t}}_{\mathbf{M}} x \leftrightarrow (x \cdot x).$$

- Thus, x and $x \cdot x$ are logically equivalent distinct formulas.

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash^{\vec{t}}_{\mathbf{M}}$ admits an equational completeness theorem.

Proof sketch (easy part only).

- If $\mathbf{M}$ halts on $\vec{t}$, there is a sequence of configurations $c_1, \ldots, c_k$ s.t. $c_1 = \mathrm{In}(\mathbf{M}, \vec{t})$, each c_i yields c_{i+1}, and the state p of c_k is final.
- Since each c_i yields c_{i+1},

$$\varphi_{\mathrm{In}(\mathbf{M},\vec{t})} \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_2} \vdash^{\vec{t}}_{\mathbf{M}} \cdots \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_k}.$$

- By definition, $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathrm{In}(\mathbf{M},\vec{t})}$, whence $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_k}$.
- Since $\varphi_{c_k} = p \cdot \psi$ for some ψ, we can apply the rule $p \cdot y \rhd x \leftrightarrow (x \cdot x)$, obtaining

$$\emptyset \vdash^{\vec{t}}_{\mathbf{M}} x \leftrightarrow (x \cdot x).$$

- Thus, x and $x \cdot x$ are logically equivalent distinct formulas. As $\vdash^{\vec{t}}_{\mathbf{M}}$ is protoalgebraic, it admits an equational completeness theorem.

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash^{\vec{t}}_{\mathbf{M}}$ admits an equational completeness theorem.

Proof sketch (easy part only).

- If $\mathbf{M}$ halts on $\vec{t}$, there is a sequence of configurations $c_1, \ldots, c_k$ s.t. $c_1 = \mathrm{In}(\mathbf{M}, \vec{t})$, each c_i yields c_{i+1}, and the state p of c_k is final.
- Since each c_i yields c_{i+1},

$$\varphi_{\mathrm{In}(\mathbf{M},\vec{t})} \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_2} \vdash^{\vec{t}}_{\mathbf{M}} \cdots \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_k}.$$

- By definition, $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathrm{In}(\mathbf{M},\vec{t})}$, whence $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_k}$.
- Since $\varphi_{c_k} = p \cdot \psi$ for some ψ, we can apply the rule $p \cdot y \rhd x \leftrightarrow (x \cdot x)$, obtaining

$$\emptyset \vdash^{\vec{t}}_{\mathbf{M}} x \leftrightarrow (x \cdot x).$$

- Thus, x and $x \cdot x$ are logically equivalent distinct formulas. As $\vdash^{\vec{t}}_{\mathbf{M}}$ is protoalgebraic, it admits an equational completeness theorem. QED

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics,

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems,

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems, protoalgebraic logics.

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems, protoalgebraic logics.

The problem of determining whether a logic admits an equational completeness theorem is:

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems, protoalgebraic logics.

The problem of determining whether a logic admits an equational completeness theorem is:

- Decidable for logics presented by a finite set of finite matrices and locally tabular logics presented by a finite Hilbert calculus;

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems, protoalgebraic logics.

The problem of determining whether a logic admits an equational completeness theorem is:

- Decidable for logics presented by a finite set of finite matrices and locally tabular logics presented by a finite Hilbert calculus;
- Undecidable for arbitrary (protoalgebraic) logics presented by a finite Hilbert calculus.

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems, protoalgebraic logics.

The problem of determining whether a logic admits an equational completeness theorem is:

- Decidable for logics presented by a finite set of finite matrices and locally tabular logics presented by a finite Hilbert calculus;
- Undecidable for arbitrary (protoalgebraic) logics presented by a finite Hilbert calculus.

Open problems.

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems, protoalgebraic logics.

The problem of determining whether a logic admits an equational completeness theorem is:

- Decidable for logics presented by a finite set of finite matrices and locally tabular logics presented by a finite Hilbert calculus;
- Undecidable for arbitrary (protoalgebraic) logics presented by a finite Hilbert calculus.

Open problems. Standard equational completeness theorem,

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems, protoalgebraic logics.

The problem of determining whether a logic admits an equational completeness theorem is:

- Decidable for logics presented by a finite set of finite matrices and locally tabular logics presented by a finite Hilbert calculus;
- Undecidable for arbitrary (protoalgebraic) logics presented by a finite Hilbert calculus.

Open problems. Standard equational completeness theorem, complexity issues,

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.

We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems, protoalgebraic logics.

The problem of determining whether a logic admits an equational completeness theorem is:

- Decidable for logics presented by a finite set of finite matrices and locally tabular logics presented by a finite Hilbert calculus;
- Undecidable for arbitrary (protoalgebraic) logics presented by a finite Hilbert calculus.

Open problems. Standard equational completeness theorem, complexity issues, logics lacking theorems etc.

Thank you very much for your attention!