Epimorphisms in varieties of Heyting algebras

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Definition

Let K be a variety of algebras and $\boldsymbol{A}, \boldsymbol{B} \in$ K. A homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a K-epimorphism if for every $\boldsymbol{C} \in$ K and every pair of homomorphisms $g, h \colon \boldsymbol{B} \rightrightarrows \boldsymbol{C}$,

$$\text{if } g \circ f = h \circ f, \text{ then } g = h.$$

- All surjective homomorphisms $f \colon \boldsymbol{A} \to \boldsymbol{B}$ are K-epimorphisms.
- The contrary however need not hold, and K is said to have the **ES property** when K-epimorphisms are surjective.
- This demand can be simplified: a subalgebra $\boldsymbol{A} \leqslant \boldsymbol{B} \in$ K is K-epic if the inclusion $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$ is a K-epimorphism.

Remark. K has the ES property iff its members have no **proper** K-epic subalgebra.

- When the variety K **algebraizes** a logic $\vdash$,

  K has the **ES property** iff $\vdash$ has the **Beth definability property**,

  where (informally) the latter means that implicit definitions in $\vdash$ can be turned explicit.

More precisely, for **arbitrarily large** disjoint sets of variables Z, X,

- Z is defined **implicitly** in terms of X by means of a set of formulas Γ over $X \cup Z$, if

  $$\Gamma \cup \sigma[\Gamma] \vdash z \leftrightarrow \sigma(z)$$

  for every $z \in Z$, and substitution σ such that $\sigma(x) = x$ for all $x \in X$;
- Z is defined **explicitly** in terms of X by means of Γ, when for every $z \in Z$, there exists a formula φ_z over X such that

  $$\Gamma \vdash z \leftrightarrow \varphi_z.$$

- **Aim of the talk.** Investigate varieties of Heyting algebras with the ES property (equiv. intermediate logics with the infinite Beth property).

- What is known about epimorphisms in Heyting varieties K?

Theorem (Maksimova)

There are only **finitely many** varieties K with the following **stronger** variant of the ES property:
if $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a hom. in K and $b \in B \smallsetminus f[A]$, then there are $\boldsymbol{C} \in$ K and $g, h \colon \boldsymbol{B} \rightrightarrows \boldsymbol{C}$ such that $g \circ f = h \circ f$ and $g(b) \neq h(b)$.

- Varieties satisfying this stronger property include Boolean algebras, Gödel algebras, and Heyting algebras.

Theorem (Kreisel)

Every variety K has the following **weaker** variant of the ES:
if $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a hom. in K such that $\boldsymbol{B}$ is generated by $f[A]$ plus finitely many elements of $B \smallsetminus f[A]$, then f is not a K-epimorphism.

- However, the standard ES property remains poorly understood.

- What can we infer about the standard ES property?

Theorem (Campercholi)

Let K be an arithmetical variety whose FSI members form a universal class. Then K has the ES property if and only if its **FSI members** lack proper K-epic subalgebras.

- We obtain the following:

Proposition

Fin. gen. varieties of Heyting algebras have the ES property.

Proof.

- Suppose, with a view to contradiction, that there is a finitely generated Heyting variety K without the ES property.
- By Campercholi's result there is a FSI $\boldsymbol{B} \in \mathsf{K}$ with a proper K-epic subalgebra $\boldsymbol{A}$.
- As K is fin. gen., Jónsson's lemma guarantees that $\boldsymbol{B}$ is finite.
- By Kreisel's result the inclusion $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$ isn't a K-epimorphism.
- This contradicts the fact that $\boldsymbol{A}$ is a K-epic subalgebra of $\boldsymbol{B}$.

- The challenge of understanding the ES property is concerned with non-finitely generated varieties only.
- One of the more general positive results is the following:

Theorem (G. Bezhanishvili, T.M., and J. Raftery)

Varieties of Heyting algebras of finite depth have the ES property.

- Hence there is a continuum of varieties with the ES property.
- What about varieties **without** the ES property? To spot them, we rely on Esakia duality for Heyting varieties K:

Observation

K lacks the ES property iff there are an Esakia space $\boldsymbol{X} \in \mathsf{K}_*$ with a correct partition R different from the identity relation s.t. for every $\boldsymbol{Y} \in \mathsf{K}_*$ and every pair of Esakia morphisms $g, h \colon \boldsymbol{Y} \rightrightarrows \boldsymbol{X}$, if $\langle g(y), h(y) \rangle \in R$ for every $y \in Y$, then $g = h$.

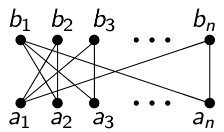
Definition

Let $0 < n \in \omega$. A poset $\langle X, \leqslant \rangle$ has **width $\leqslant n$** if for every $x \in X$, the upset $\uparrow x$ does not contain antichains of $n+1$ elements.
A Heyting algebra $\boldsymbol{A}$ has **width $\leqslant n$** if so does its Esakia dual, i.e. if the poset of prime filters of $\boldsymbol{A}$ has width $\leqslant n$.

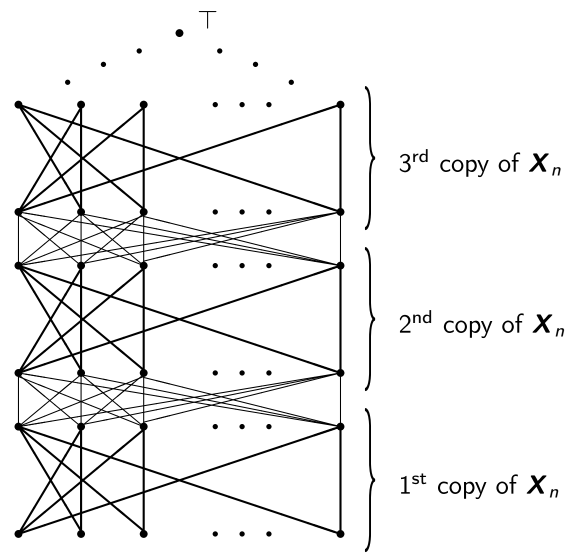
Theorem (Baker, Hosoi, Maksimova, and Ono)

Let $0 < n \in \omega$. The class W_n of Heyting algebras of width $\leqslant n$ is a variety. In particular, W_1 is the variety of Gödel algebras.

- **Aim.** To show that W_n lacks the ES property for every $n \geqslant 2$.
- Given $n \geqslant 2$, let $\boldsymbol{X}_n$ be the following poset:



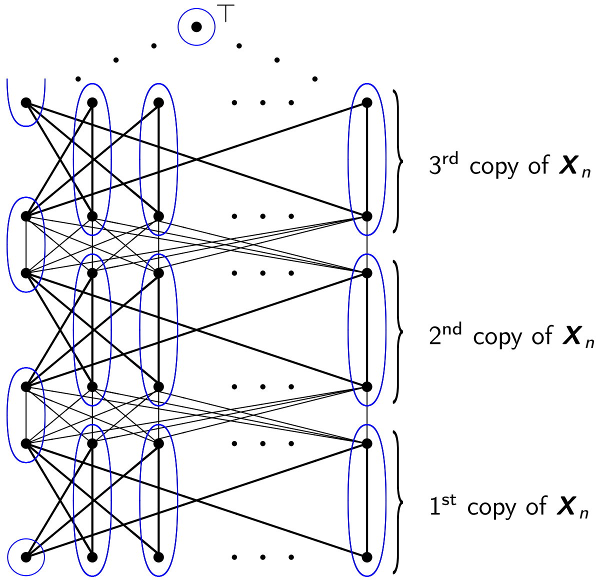
- Now, we construct a tower of ω copies of $\boldsymbol{X}_n$ as follows:



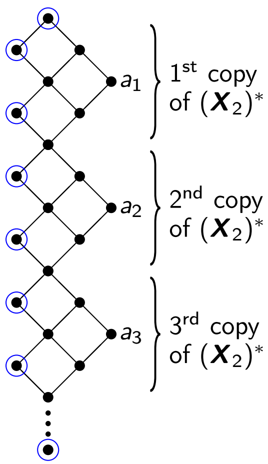
- Let $\boldsymbol{X}_n^\infty$ be the above poset endowed with the topology

$$\tau = \{U \colon \text{if } \top \in U, \text{ then } U \text{ extends an infinite upset}\}.$$

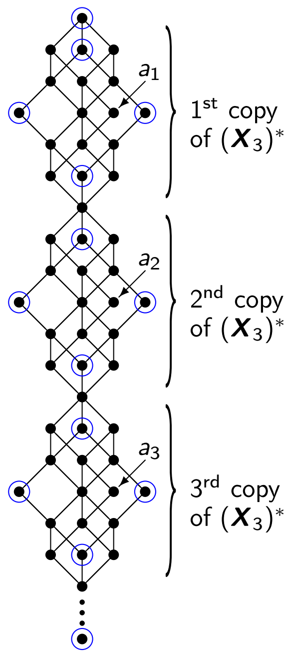
- **Remark.** $\boldsymbol{X}_n^\infty$ is an Esakia space, and the relation R depicted below is a correct partition on it.



$(\boldsymbol{X}_2^\infty)^*$:



$(\boldsymbol{X}_3^\infty)^*$:



Main observation

For every $\boldsymbol{Y} \in (\mathsf{W}_n)_*$ and every pair of Esakia morphisms $g, h \colon \boldsymbol{Y} \rightrightarrows \boldsymbol{X}_n^\infty$, if $\langle g(y), h(y)\rangle \in R$ for every $y \in Y$, then $g = h$.

- As a consequence we obtain the following:

Theorem

Let $2 \leqslant n \in \omega$ and $\mathsf{K} \subseteq \mathsf{W}_n$ be a variety. If $\boldsymbol{X}_n^\infty \in \mathsf{K}_*$, then K lacks the ES property. In particular,

1. W_n lacks the ES property;
2. W_2 has a continuum of subvarieties lacking the ES property.

- Can we understand this failure of the ES property in terms of a (more transparent) failure of the infinite Beth definability?

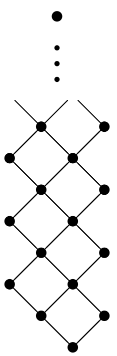
- The sum $\boldsymbol{A} + \boldsymbol{B}$ of two disjoint Heyting algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ is the unique Heyting algebra obtained pasting $\boldsymbol{A}$ on top of $\boldsymbol{B}$, and identifying the minimum of $\boldsymbol{A}$ with the maximum of $\boldsymbol{B}$.
- The sum $\sum \boldsymbol{A}_n$ of a family of disjoint Heyting algebras $\{\boldsymbol{A}_n \colon n \in \omega\}$ is the unique Heyting algebra with universe

$$\{\bot\} \cup \bigcup_{n\in\omega} (A_n \smallsetminus \{0^{\boldsymbol{A}_n}\})$$

and whose lattice order is defined for every $a, b \in \sum \boldsymbol{A}_n$ as:

$$a \leqslant b \iff \text{either } a = \bot \text{ or } (a, b \in A_n \text{ for some } n \in \omega \text{ and } a \leqslant^{\boldsymbol{A}_n} b)$$
$$\text{or } (a \in A_n \text{ and } b \in A_m \text{ for some } n, m \in \omega \text{ s.t. } m < n).$$

- The Rieger-Nishimura lattice $\boldsymbol{RN}$ is the one-generated free Heyting algebra depicted below:



Definition

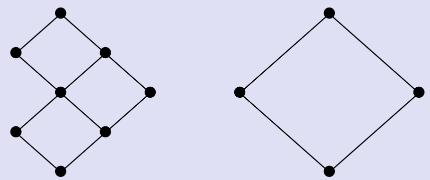
The Kuznetsov-Gerčiu variety is defined as

$$\mathsf{KG} := \mathbb{V}\{\boldsymbol{A}_1 + \cdots + \boldsymbol{A}_n \colon \boldsymbol{A}_1, \ldots, \boldsymbol{A}_n \in \mathbb{H}(\boldsymbol{RN}) \text{ and } 0 < n \in \omega\}.$$

Thank you for coming!

Theorem

A variety $\mathsf{K} \subseteq \mathsf{KG}$ has the ES property iff it excludes all sums of the form $\sum \boldsymbol{A}_n$ where $\{\boldsymbol{A}_n \colon n \in \omega\}$ is a family such that each $\boldsymbol{A}_n$ is one of the following:



As a consequence, we obtain that

- Gödel varieties have the ES property (already known).
- The ES property implies local finiteness in subvarieties of KG.
- The ES property is hereditary in subvarieties of KG.
- The variety $\mathbb{V}(\boldsymbol{RN})$ lacks the ES property, and has a continuum of locally finite such subvarieties.