On interpretations between propositional logics

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Some sources of inspiration:

- **Matrix semantics** for logics (Łukasiewicz, Tarski, Łos, Suszko, Wójcicki ...)
- Blok and Pigozzi's seminal work on **algebraizable logics**
- **Leibniz hierarchy** of propositional logics (Czelakowski, Font, Herrmann, Jansana, Raftery ...)
- **Maltsev conditions** (Day, Maltsev, Jónsson, Pixley, Kiss, Kearnes, McKenzie, Szendrei ...)
- **Interpretations** between varieties (Taylor, Neumann, Garcia, Opršal, Tschantz ...)

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- What do we mean by an **interpretation** between logics?
- And what do we mean by **logic**?

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a \equiv c \iff\ & a \text{ and } c \text{ satisfies the same} \\
& \text{equality-free types with constants} \\
\iff\ & \text{for every non-equality atomic formula } \phi(x, y_1, \ldots, y_n) \\
& \text{and for every } b_1, \ldots, b_n \in M, \\
& \mathbf{M} \vDash \phi(a, b_1, \ldots, b_n) \text{ iff } \mathbf{M} \vDash \phi(c, b_1, \ldots, b_n).
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- Thus, the natural models of T are the ones whose indiscernibility relation is the identity relation.
- This setting subsumes model theory **with** equality.

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- Let $T_{\vdash}$ be the theory in the equality-free language obtained extending the algebraic language of $\vdash$ with $P(x)$, axiomatized by the infinitary universal Horn sentences

$$\forall \vec{x} \bigwedge_{\gamma \in \varGamma} P(\gamma(\vec{x})) \to P(\varphi(\vec{x}))$$

  for all valid inferences $\varGamma \vdash \varphi$ of $\vdash$.

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- Observe that $\langle \boldsymbol{A}, F \rangle$ is a model of $\vdash$ iff it is a model of $T_\vdash$ in the standard sense.

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- A **translation** of an algebraic language $\mathscr{L}$ into another $\mathscr{L}'$ is a map $\boldsymbol{\tau}$ that assigns an n-ary term $\boldsymbol{\tau}(f)(x_1, \ldots, x_n)$ of $\mathscr{L}'$ to every n-ary symbol $f(x_1, \ldots, x_n)$ of $\mathscr{L}$.

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Example. Let $\mathscr{L}_{\wedge\vee}$ be the language of lattices, and $\mathscr{L}_{\mathsf{BA}}$ that of Boolean algebras. If $\boldsymbol{\tau}$ is the inclusion map from $\mathscr{L}_{\wedge\vee}$ to $\mathscr{L}_{\mathsf{BA}}$, and $\boldsymbol{A}$ a Boolean algebra, then $\boldsymbol{A}^{\boldsymbol{\tau}}$ is its lattice reduct of $\boldsymbol{A}$.

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- Elements of Log are classes $[\![\vdash]\!]$ of equi-interpretable logics.

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- Do **infima** and **suprema** exist?

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Log is a **set-complete** meet-semilattice.

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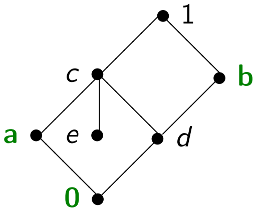

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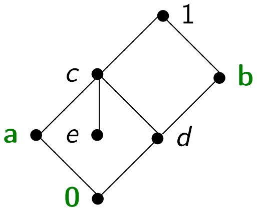

Then let $\mathbf{L}$ be the logic induced by the pair of matrices

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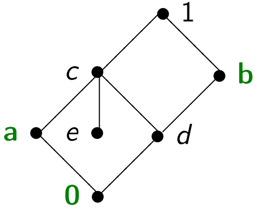
- Consider the algebra $\boldsymbol{A} = \langle A; \vee, \mathbf{a}, \mathbf{b}, \mathbf{0} \rangle$ depicted below.



Then let $\mathbf{L}$ be the logic induced by the pair of matrices

$$\{\langle \boldsymbol{A}, \{1\}\rangle, \langle \boldsymbol{A}, \{1, c\}\rangle\}.$$

- The **supremum** of $[\![\mathbf{CPC}_{\neg}]\!]$ and $[\![\mathbf{L}]\!]$ does **not** exist in Log.

Leibniz classes and hierarchy

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Basic question:

Leibniz classes and hierarchy

Basic question:

- What are **Leibniz classes** of logics?

What is the Leibniz hierarchy?

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- A classification of logics in terms **syntactic principles** that govern the behaviour of the **indiscernibility** relation.

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- A logic $\vdash$ is **equivalential** if there is a non-empty set of formulas $\Delta(x, y)$ s.t. for all models $\langle \boldsymbol{A}, F \rangle$ of $\vdash$ and $a, c \in A$,

$$a \equiv c \Longleftrightarrow \Delta^{\boldsymbol{A}}(a, c) \subseteq F.$$

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$$\emptyset \vdash \Delta(x, x) \qquad x, \Delta(x, y) \vdash y$$

$$\bigcup_{1 \leqslant i \leqslant n} \Delta(x_i, y_i) \vdash \Delta(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))$$

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- Equivalential logics form a **Leibniz class**.

- A **Leibniz condition** is a sequence $\Phi = \{\vdash_\alpha : \alpha \in \mathsf{OR}\}$ of logics, indexed by all ordinals, s.t.

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  $$\mathrm{Log}(\Phi) \coloneqq \{\vdash : \vdash \text{ is a logic and } \textbf{satisfies } \Phi\}.$$

- A **Leibniz class** is a class of logics of the form $\mathrm{Log}(\Phi)$, for some Leibniz condition Φ.

Theorem

Let K be a class of logics. TFAE:

1. K is a Leibniz class.
2. K is “essentially” a set-complete filter of Log.
3. K is closed under the formation of **term-equivalent** logics, **compatible expansions**, and **non-indexed products** indexed by arbitrarily large sets.

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- K is the class of logics satisfying Φ.

Indecomposable Leibniz classes

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Basic question:

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Basic question:

- Which of Leibniz classes are **primitive** or fundamental?

- When ordered under inclusion, Leibniz classes form a “lattice”.

Definition

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A Leibniz class K is said to be

- **meet-irreducible** if for every pair K_1 and K_2 of Leibniz classes (of logics with some tautology),

$$\text{if } K = K_1 \cap K_2, \text{ then either } K = K_1 \text{ or } K = K_2.$$

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- **meet-prime** if for every pair of Leibniz classes K_1 and K_2 (of logics with some tautology),

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- Intuitively, a Leibniz class is meet-prime (resp. irreducible) when it captures a **fundamental concept**.
- We shall apply this test to two conditions, i.e. the definability of truth-sets and of indiscernibility.

Definability of truth-sets.

- A logic $\vdash$ is **truth-equational** if there is a set of equations $E(x)$ s.t. for every $\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}^{\equiv}(\vdash)$

$$a \in F \iff \boldsymbol{A} \vDash E(a), \text{ for all } a \in A.$$

Definability of truth-sets.

- A logic $\vdash$ with tautologies is **truth-equational** if there are no $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{A}, G\rangle \in \mathrm{Mod}^{\equiv}(\vdash)$ such that $\emptyset \subsetneq F \subsetneq G$.

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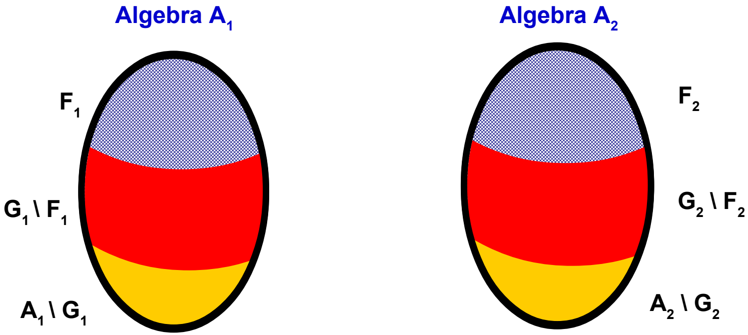
- As $\vdash_1$ and $\vdash_2$ are not truth-equational, there are matrices

$$\langle \boldsymbol{A}_1, F_1\rangle, \langle \boldsymbol{A}_1, G_1\rangle \in \mathrm{Mod}^{\equiv}(\vdash_1) \text{ s.t. } \emptyset \subsetneq F_1 \subsetneq G_1$$
$$\langle \boldsymbol{A}_2, F_2\rangle, \langle \boldsymbol{A}_2, G_2\rangle \in \mathrm{Mod}^{\equiv}(\vdash_2) \text{ s.t. } \emptyset \subsetneq F_2 \subsetneq G_2.$$



Algebra A1
F1
G1 \ F1
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Algebra A2
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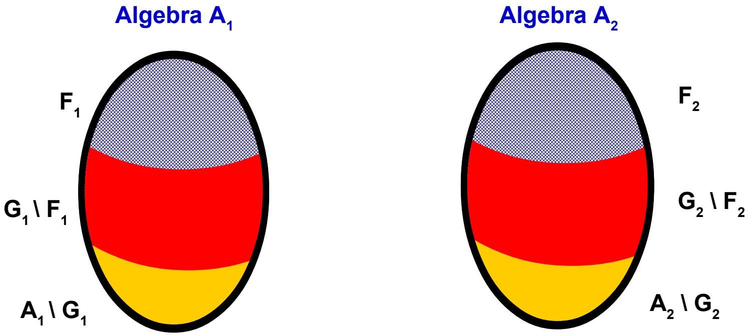

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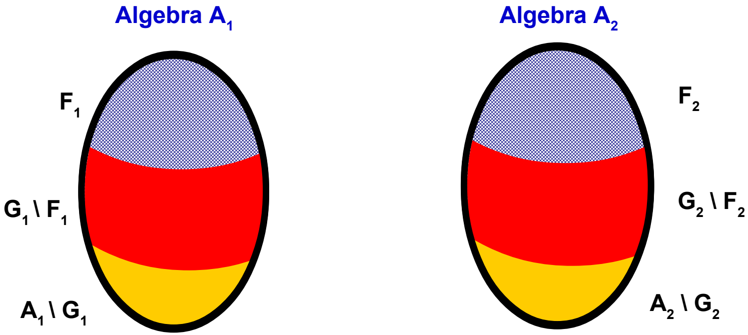

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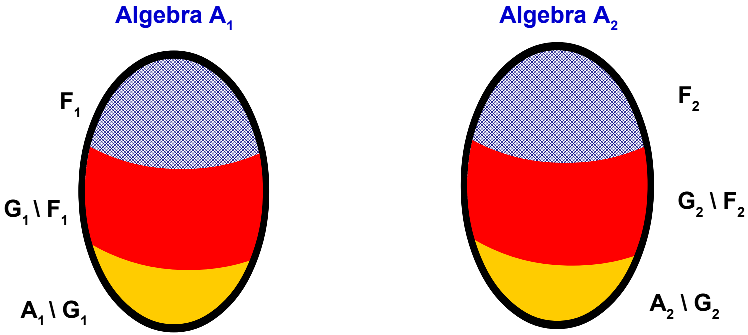
- The problem is that $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ have not the same universe.



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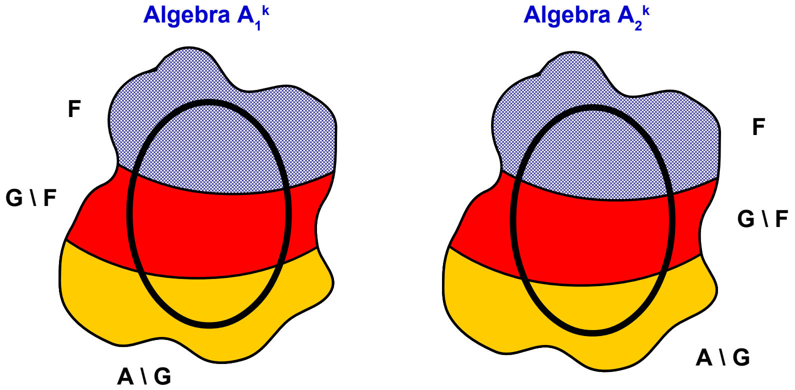

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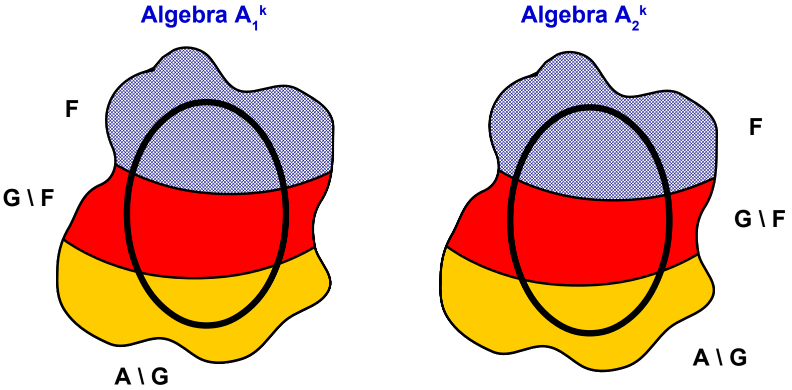

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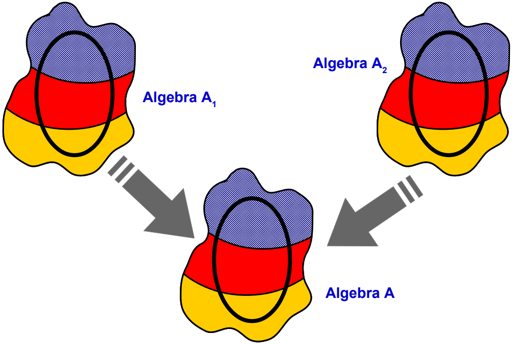

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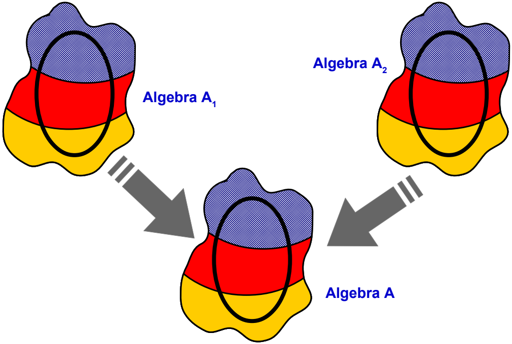

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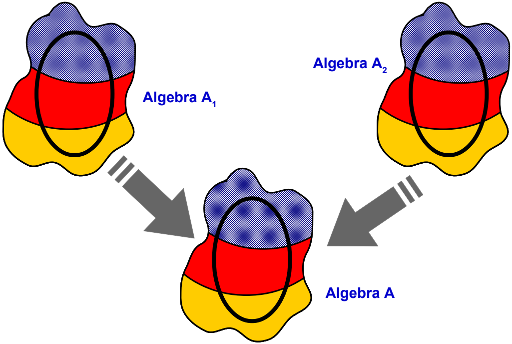

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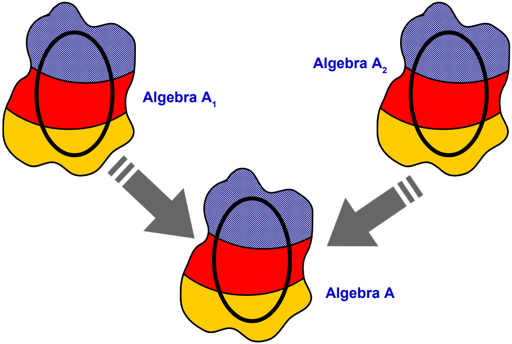

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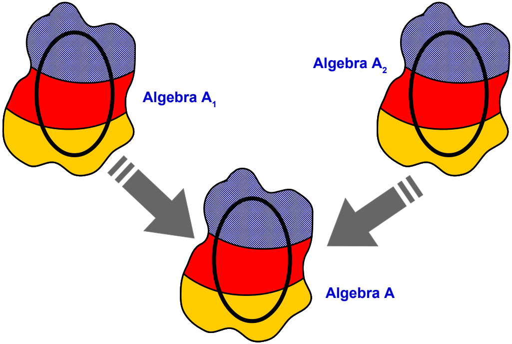

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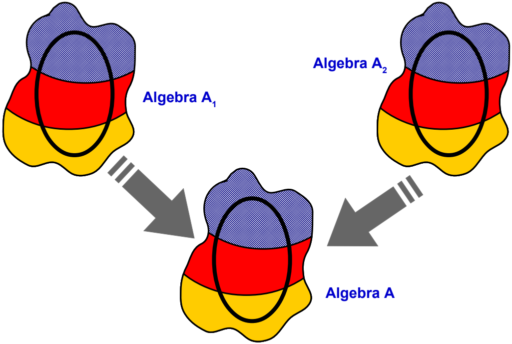

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  where $\vdash^{\mathsf{eq}}_{\alpha}$ is the logic in the language with binary symbols $\{\multimap_{\epsilon} : \epsilon < \max\{\omega, |\alpha|\}\}$ axiomatized by the rules

$$\emptyset \rhd \Delta_{\alpha}(x, x) \qquad x, \Delta_{\alpha}(x, y) \rhd y$$

$$\Delta_{\alpha}(x_1, y_1) \cup \Delta_{\alpha}(x_2, y_2) \rhd \Delta_{\alpha}(x_1 \multimap_{\epsilon} x_2, y_1 \multimap_{\epsilon} y_2)$$

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Theorem

The logic $\vdash^{\mathrm{eq}}_{\alpha}$ is **completely meet-prime** in Log. Thus equivalential logics are determined by a Leibniz condition consisting only of completely meet-prime logics.

Thank you for your attention!