Frames, ordered algebras, and quantifiers for deductive systems

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Some fascinating aspects of relational semantics, which are not immediately available in matrix semantics:

- **Logic**: a distinctive **freedom** of construction (e.g. the possibility of gluing models or adding points to them)
- **Algebra**: **representation** theory, and **dualities** for ordered algebras (e.g. canonical extensions, Priestley-style dualities)
- **Philosophy**: suggestive interpretations in terms of **possible worlds**, and relations with **constructive** mathematics

These and other observations prompted the general study of relational semantics from various perspectives:

- Jónsson and Tarski's seminal **representation of BAOs**
- **Canonical extensions** of ordered algebras (Gehrke, Harding, Jónsson and others)
- **Gaggle theory** (Bimbó, Dunn, Hardegree, and others)
- **Residuated frames** (Ciabattoni, Galatos, Jipsen, Terui and others)

Aim of the talk

To sketch an **abstract** approach to relational semantics, which encompasses the above ones and applies to arbitrary logics.

- To this end, we rely on the theory of

  Δ_1-**completions** of arbitrary posets

  due to Gehrke, Jansana, Palmigiano and Priestley.

Correspondence between frames and ordered algebras

Basic questions:

- What do we mean by **ordered algebras** and **frames**?
- How can we transform them one into the other?

- An **ordered language** is an algebraic language whose symbols $f(x_1, \ldots, x_n)$ are equipped with a map

$$\sigma_f \colon \{1, \ldots, n\} \to \{+, -\}.$$

Sometimes we write

$$f(x_1, \ldots, x_m; x_{m+1}, \ldots, x_n)$$

to denote that $\sigma_f(k) = +$ for $k \leqslant m$, and $\sigma_f(k) = -$ for $k > m$.

Definition

Let $\mathscr{L}$ be an ordered language. An **$\mathscr{L}$-ordered algebra** is a pair $\langle \boldsymbol{A}, \leqslant \rangle$ where $\boldsymbol{A}$ is an algebra, $\leqslant$ a partial order on A, and for every basic operation $f(x_1, \ldots, x_n)$ and $m \in \{1, \ldots, n\}$,

if $\sigma_f(m) = +$, then $f^{\boldsymbol{A}}(x_1, \ldots, x_n)$ is **increasing** in x_m w.r.t. $\leqslant$

if $\sigma_f(m) = -$, then $f^{\boldsymbol{A}}(x_1, \ldots, x_n)$ is **decreasing** in x_m w.r.t. $\leqslant$.

- **Residuated Lattices** and **Modal Algebras**, when ordered under the lattice order, are typical examples of ordered algebras for suitable ordered languages.

Definition

A **polarity** is a triple $\langle W, J, R\rangle$ such that W and J are non-empty sets and $R \subseteq W \times J$.

- Intuitively,

W is a set of **worlds**, i.e. states of **positive information**
J is a set of **co-worlds**, i.e. states of **negative information**.

More precisely, if $w \in W$ and $j \in J$, then

w is a set of information known to be **true**
j is a set of information known to be **false**.

Finally, R is an **incompatibility** relation, i.e.

$\langle w, j\rangle \in R \iff$ the positive information of w is incompatible with the negative information of j.

Definition

A **polarity** is a triple $\langle W, J, R\rangle$ such that W and J are non-empty sets and $R \subseteq W \times J$.

- We define a relation $\leqslant_W$ on W setting for every $w_1, w_2 \in W$,

$$w_1 \leqslant_W w_2 \iff \text{the positive info of } w_2 \textbf{ extends} \text{ that of } w_1$$
$$\iff \forall j \in J(\text{ if } \langle w_1, j\rangle \in R, \text{ then } \langle w_2, j\rangle \in R).$$

- Similarly, we define a relation $\leqslant_J$ on J whose meaning is

$$j_1 \leqslant_J j_2 \iff \text{ the negative info of } j_2 \textbf{ extends} \text{ that of } j_1.$$

In general, $\leqslant_W$ and $\leqslant_J$ are preorders.

- A **labelled language** is an ordered language $\mathscr{L}$ equipped with a map $\beta\colon \mathscr{L} \to \{\Box, \Diamond\}$.

Definition

Let $\mathscr{L}$ be a labelled language. An **$\mathscr{L}$-frame** is a structure

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

where $\langle W, J, R\rangle$ is a polarity s.t. $\leqslant_W$ and $\leqslant_J$ are **antisymmetric**, and for every operation $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ s.t. $\beta(f) = \Diamond$,

$$T_f \subseteq W^m \times J^n \times W$$

is a relation whose intuitive meaning is: $\langle \vec{w}, \vec{j}, u\rangle \in T_f$ iff

if $\varphi_1, \ldots, \varphi_m$ are **true** resp. w.r.t. the information at $w_1, \ldots, w_m$, and $\psi_1, \ldots, \psi_n$ are **false** resp. w.r.t. the information at $j_1, \ldots, j_n$, then $f(\vec{\varphi}, \vec{\psi})$ is true resp. w.r.t. the information at u.

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$$T_f \subseteq W^m \times J^n \times W$$

is a relation satisfying the following conditions:

- **Monotonicity**: For every $\vec{w}_1, \vec{w}_2 \in W^m$, $\vec{j}_1, \vec{j}_2 \in J^n$, and $u_1, u_2 \in W$ such that $\vec{w}_2 \leqslant_W \vec{w}_1$, $\vec{j}_2 \leqslant_J \vec{j}_1$ and $u_1 \leqslant_W u_2$,

$$\text{if } \langle \vec{w}_1, \vec{j}_1, u_1\rangle \in T_f, \text{ then } \langle \vec{w}_2, \vec{j}_2, u_2\rangle \in T_f.$$

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$$T_f \subseteq W^m \times J^n \times W$$

is a relation satisfying the following conditions:

- **Closedness**: For every $\vec{w} \in W^m$, $\vec{j} \in J^n$, and $u \in W$, if $\langle \vec{w}, \vec{j}, u\rangle \notin T_f$, there is $i \in J$ such that

  $$\langle u, i\rangle \notin R \text{ and } \langle v, i\rangle \in R$$

  for every $v \in W$ s.t. $\langle \vec{w}, \vec{j}, v\rangle \in T_f$.

From ordered algebras to frames

- Let $\langle \boldsymbol{A}, \leqslant \rangle$ be an $\mathscr{L}$-ordered algebra for a labelled language $\mathscr{L}$.
- Choose:

A. A set W of upsets of $\langle A, \leqslant \rangle$ containing the principal ones.

B. A set J of downsets of $\langle A, \leqslant \rangle$ containing the principal ones.

- Let $R \subseteq W \times J$ be defined as follows:

$$\langle w, j \rangle \in R \iff w \cap j \neq \emptyset.$$

Observation I

$\langle W, J, R \rangle$ is a **polarity** s.t. $\leqslant_W$ and $\leqslant_J$ are the inclusion relations.

- For every operation $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ s.t. $\beta(f) = \Diamond$, let $T_f \subseteq W^m \times J^n \times W$ be the relation defined as follows:

$$\langle \vec{w}, \vec{j}, u \rangle \in T_f \iff f^{\boldsymbol{A}}[w_1, \ldots, w_m, j_1, \ldots, j_n] \subseteq u.$$

Observation II

$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\} \rangle$ is an **$\mathscr{L}$-frame**.

From frames to ordered algebras

- Every polarity $\langle W, J, R\rangle$ induces a Galois connection

$$(\cdot)^{\triangleright} \colon \mathcal{P}(W) \longleftrightarrow \mathcal{P}(J) \colon (\cdot)^{\triangleleft}$$

  by setting for $A \subseteq W$ and $B \subseteq J$

$$\begin{aligned} A^{\triangleright} &:= \{j \in J : \langle w, j\rangle \in R \text{ for all } w \in A\} \\ B^{\triangleleft} &:= \{w \in W : \langle w, j\rangle \in R \text{ for all } j \in B\}. \end{aligned}$$

- Then $(\cdot)^{\triangleright\triangleleft} \colon \mathcal{P}(W) \to \mathcal{P}(W)$ is a closure operator on W. We denote the complete lattice of closed sets of $(\cdot)^{\triangleright\triangleleft}$ by

$$\mathcal{G}(W, J, R).$$

- Let $\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$ be an $\mathscr{L}$-frame.
- Given an operation $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ s.t. $\beta(f) = \Diamond$ set

$$f_{T_f} : \mathcal{G}(W, J, R)^{m+n} \to \mathcal{G}(W, J, R)$$

as follows:

$$f_{T_f}(\vec{a}, \vec{b}) = \bigvee^{\mathcal{G}(W,J,R)} \{u^{\rhd\lhd} : \text{ there are } w_1 \in a_1, \ldots, w_m \in a_m \text{ and } j_1 \in b_1^{\rhd}, \ldots, j_n \in b_n^{\rhd} \text{ s.t. } \langle \vec{w}, \vec{j}, u\rangle \in T_f\}.$$

Observation

$\boldsymbol{F}^+ = \langle \mathcal{G}(W, J, R), \{f_{T_f} : f \in \mathscr{L}\}, \subseteq\rangle$ is an **$\mathscr{L}$-ordered algebra**.

Definition

1. An **$\mathscr{L}$-general frame** is a pair $\langle \boldsymbol{F}, A\rangle$ where $\boldsymbol{F}$ is an $\mathscr{L}$-frame and A is the universe of a subalgebra of $\boldsymbol{F}^+$.
2. The **complex algebra** of a general frame $\langle \boldsymbol{F}, A\rangle$ is

$$\langle \boldsymbol{F}, A\rangle^+ := \langle \boldsymbol{A}, \subseteq\rangle \text{ where } \boldsymbol{A} \leqslant \boldsymbol{F}^+.$$

Completions of ordered algebras via frames

- Let $\langle \boldsymbol{A}, \leqslant \rangle$ be an ordered algebra.
- Suppose that we transform it into a frame $\boldsymbol{F}$ as before.
- Then define a map

$$\lambda \colon \langle \boldsymbol{A}, \leqslant \rangle \to \boldsymbol{F}^{+}$$

  setting

$$\lambda(a) \coloneqq \{w \in W : a \in w\}, \text{ for all } a \in A.$$

Observation

1. $\lambda \colon \langle \boldsymbol{A}, \leqslant \rangle \to \boldsymbol{F}^{+}$ is an order and algebraic **embedding**.
2. The completion $\boldsymbol{F}^{+}$ satisfies some **density** and **compactness** properties w.r.t. $\langle \boldsymbol{A}, \leqslant \rangle$.

Relational semantics for arbitrary propositional logics

Basic questions:

- What does it mean that a logic has a **relational semantics**?
- Can we make sense of **local** and **global** consequences?

Satisfaction and co-satisfaction in a frame

- A **valuation** in a general frame $\langle \boldsymbol{F}, A \rangle$ is a map $v \colon Var \to A$.
- We will define a relation of **satisfaction** of formulas φ under v at worlds $w \in W$, in symbols

$$w, v \Vdash \varphi.$$

  The intuitive reading is

$$w, v \Vdash \varphi \iff \varphi \text{ is } \textbf{true} \text{ according to the information of } w.$$

- Similarly, we will define a relation of **co-satisfaction** of formulas φ under v at co-worlds $j \in J$, whose meaning is

$$j, v \Vdash \varphi \iff \varphi \text{ is } \textbf{false} \text{ according to the information of } j.$$

- Let $v^+ \colon \boldsymbol{Fm} \to \boldsymbol{A}$ be the unique homomorphism extending v. We define formally satisfaction and co-satisfaction as follows:

$$w, v \Vdash \varphi \iff w^{\rhd\lhd} \subseteq v^+(\varphi)$$
$$j, v \Vdash \varphi \iff v^+(\varphi) \subseteq j^{\lhd}.$$

Let Fr be a class of $\mathscr{L}$-general frames.

1. The **local consequence** relation of Fr is:

$$\Gamma \vdash \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle \in \text{Fr and } w \in W \\ \text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.$$

2. The **co-local consequence** relation of Fr is:

$$\Gamma \vdash \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle \in \text{Fr and } j \in J \\ \text{if } j, v \Vdash \Gamma, \text{ then } j, v \Vdash \varphi.$$

Definition

A logic $\vdash$ is a **local consequence** if it is the local consequence of a class of $\mathscr{L}$-general frames.

Can we characterize arbitrary local consequences?

Syntactic characterization

A logic $\vdash$ is a local consequence iff there is an ordered language for $\vdash$ s.t. for every connective $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$, and every pair of formulas φ and ψ s.t. $\varphi \vdash \psi$,

$$f(\delta_1, \ldots, \delta_{i-1}, \varphi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon}) \vdash f(\delta_1, \ldots, \delta_{i-1}, \psi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon})$$

$$f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \psi, \epsilon_{j+1}, \ldots, \epsilon_n) \vdash f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \varphi, \epsilon_{j+1}, \ldots, \epsilon_n)$$

for every $\vec{\delta}$ and $\vec{\epsilon}$, and $i \leqslant m$, $j \leqslant n$.

- As a consequence, **fragments** of local consequences are still local consequences.
- Moreover, if $\vdash$ is a local consequence, then the **interderivability** relation $\dashv\vdash$ is a congruence of $\boldsymbol{Fm}$.
- **Non-Examples**: Łukasiewicz infinite-valued logic, Relevance logic $\mathbf{R_t}$, global consequence of the modal system $\mathbf{K}$ etc.

Examples

Definition

Let K be a class of $\mathscr{L}$-ordered algebras. The logic $\vdash^{\leqslant}_{\mathsf{K}}$ **preserving degrees of truth** of K is defined as follows:

$$\Gamma \vdash^{\leqslant}_{\mathsf{K}} \varphi \iff \text{for all } \langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{K},\ \text{hom } v\colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ and } a \in A$$
$$\text{if } a \leqslant v(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } a \leqslant v(\varphi).$$

- If K is the variety of Heyting algebras, then $\vdash^{\leqslant}_{\mathsf{K}}$ is **IPC**.
- If K is the variety of Modal Algebras, then $\vdash^{\leqslant}_{\mathsf{K}}$ is the local consequence of **K**.

Observation

If K is a class of $\mathscr{L}$-algebras, then the logic $\vdash^{\leqslant}_{\mathsf{K}}$ is a local consequence.

Global consequences

- Let Fr be a class of $\mathscr{L}$-general frames.
- The **global consequence** relation of Fr is:

$$\Gamma \vdash \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A \rangle \in \text{Fr}, \\ \text{if } w, v \Vdash \Gamma \text{ for every } w \in W, \\ \text{then } w, v \Vdash \varphi \text{ for every } w \in W.$$

Relative axiomatizations

Let Fr be a class of $\mathscr{L}$-general frames such that:

1. The local consequence of Fr is finitary and has either a conjunction or the deduction theorem.
2. If the loc. cons. of $\langle \boldsymbol{F}, A \rangle$ extends that of Fr, then $\langle \boldsymbol{F}, A \rangle \in$ Fr.

Then the global consequence of Fr is the extension of its local consequence with the so-called **Suszko rules**, i.e.

$$x, y, \varphi(x, \vec{z}) \rhd \varphi(y, \vec{z}), \text{ for every formula } \varphi(v, \vec{z}).$$

- Suszko rules generalize the **necessitation rule** in modal logic.

Examples of global consequences

- Let $\vdash$ be a substructural logic with **weakening**, i.e. one associated with a variety K of **integral** Residuated Lattices.
- Let $\vdash_{K}^{\leqslant}$ be the logic preserving degrees of truth of K equipped with the lattice order.

Recall that...

$\vdash_{K}^{\leqslant}$ is the **local** consequence of a class of $\mathscr{L}$-general frames.

- Let Fr be the class of $\mathscr{L}$-general frames, whose local consequence extends $\vdash_{K}^{\leqslant}$.

Observation

The substructural logic $\vdash$ is the **global** consequence of Fr. It is obtained extending $\vdash_{K}^{\leqslant}$ with the Suszko rules.

- All substructral logics with **weakening** (e.g. Łukasiewicz infinite-valued logic, **IPC** etc.) are **global** consequences.

Logic-based correspondences between ordered algebras and frames

Basic questions:

- Why do most logics have a semantics of **ordered algebras**?
- Are there **logic-based dualities** for ordered algebras?

Relational models of a logic

Definition

Let $\vdash$ be a logic, and $\mathscr{L}$ a labelled language for $\vdash$.

1. An $\mathscr{L}$-general frame $\langle \boldsymbol{F}, A\rangle$ is a **model** of $\vdash$ if its local consequence extends $\vdash$.
2. We set

$$\mathrm{Rel}_{\mathscr{L}}(\vdash) := \{\langle \boldsymbol{F}, A\rangle \colon \langle \boldsymbol{F}, A\rangle \text{ is an } \mathscr{L}\text{-general frame and a model of } \vdash\}.$$

Ordered algebras of a logic

Definition

Let $\vdash$ be a logic, and $\mathscr{L}$ an ordered language for $\vdash$.

1. An $\mathscr{L}$-ordered algebra $\langle \boldsymbol{A}, \leqslant \rangle$ is an $\mathscr{L}$-**ordered model** of $\vdash$ if for every $a \in A$ the upset $\uparrow a$ is a deductive filter of $\vdash$.
2. We set

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) := \{\langle \boldsymbol{A}, \leqslant \rangle : \langle \boldsymbol{A}, \leqslant \rangle \text{ is an } \mathscr{L}\text{-ordered model of } \vdash\}.$$

Observation

$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is closed under $\mathbb{S}$ and $\mathbb{P}$ (and $\mathbb{P}_{\mathrm{u}}$ if $\vdash$ is finitary).

- **Non-Mathematical Thesis**: $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ may be understood as the class of **distinguished** ordered models of $\vdash$ (from the point of view of the ordered language $\mathscr{L}$).

Theoretic justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$

Observation

Let $\vdash$ be a logic, and $\mathscr{L}$ a labelled language for $\vdash$.

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) = \{\langle \boldsymbol{F}, A\rangle^{+} : \langle \boldsymbol{F}, A\rangle \text{ is an } \mathscr{L}\text{-general frame and a model of } \vdash\}.$$

In other words, $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is the class of **complex algebras** of relational models of $\vdash$.

On general grounds, logics may have a semantics of **ordered algebras** because:

- either they **have** a **local** relational semantics (e.g. **IPC**)
- or they **extend** logics with a **local** relational semantics (e.g. Łukasiewicz infinite-valued logic).

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$

- Let K be a variety with a **semilattice** reduct s.t. when ordered under the meet-order is a class of $\mathscr{L}$-ordered algebras. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}}) = \{\langle \boldsymbol{A}, \leqslant\rangle : \boldsymbol{A} \in \mathsf{K} \text{ and } \leqslant \text{ is the meet-order of } \boldsymbol{A}\}.$$

- Let $\mathbf{IPC}_{\rightarrow}$ be the $\langle\rightarrow\rangle$-fragment of **IPC**. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\rightarrow}) = \text{ Hilbert algebras } + \text{ Hilbert-order.}$$

- Let $\mathbf{InFL}_{\mathsf{e}}^{\leqslant}$ be the $\langle\cdot,\rightarrow\rangle$-fragment of the logic **preserving** degrees of truth of commutative FL-algebras. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_{\mathsf{e}}^{\leqslant}) = \langle\cdot,\rightarrow,\leqslant\rangle\text{-subreducts of comm. FL-algebras.}$$

- Let $\mathbf{InR}^{\leqslant}$ be the $\langle\cdot,\rightarrow,\neg\rangle$-fragment of the logic **preserving** degrees of truth of De Morgan monoids. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InR}^{\leqslant}) = \langle\cdot,\rightarrow,\neg,\leqslant\rangle\text{-subreducts of De Morgan monoids.}$$

Logic-based correspondence between $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ and $\mathsf{Rel}_{\mathscr{L}}(\vdash)$

Definition

Let $\vdash$ be a logic and $\mathscr{L}$ an ordered language for $\vdash$. The $\mathscr{L}$-**cologic** of $\vdash$ is the logic $\vdash^{\partial}_{\mathscr{L}}$ preserving degrees of truth of

$$\{\langle \boldsymbol{A}, \leqslant^{\partial}\rangle : \langle \boldsymbol{A}, \leqslant\rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)\}.$$

- **Remark**: In known cases the co-logic is the expected dual of $\vdash$.

Let $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ and $\mathscr{L}$ be a labelled language.

- The $\mathscr{L}$-**canonical polarity** of $\langle \boldsymbol{A}, \leqslant \rangle$ is the polarity

$$\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle \coloneqq \langle W, J, R \rangle$$

  where $R \subseteq W \times J$ is the relation of non-empty intersection and

$$W = \{w \subseteq A : w \text{ is both an upset and a } \vdash\text{-filter}\}$$

$$J = \{j \subseteq A : w \text{ is both a downset and a } \vdash^{\partial}_{\mathscr{L}}\text{-filter}\}.$$

- Let $\langle \boldsymbol{A}, \leqslant \rangle^{g}_{+}$ be the $\mathscr{L}$-general frame based on $\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle$.

Correspondence between ordered algebras and frames

The following maps are well defined:

$$(\cdot)^{g}_{+} \colon \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) \longleftrightarrow \mathsf{Rel}_{\mathscr{L}}(\vdash) \colon (\cdot)^{+}$$

Moreover, $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is the class of complex algebras of $\mathsf{Rel}_{\mathscr{L}}(\vdash)$.

- Relational duals are constructed in a **logic-based** way.

Logics preserving degrees of truth of Lattice Expansions

- Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant\rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ we have:

A. $\boldsymbol{A} \in \mathsf{K}$ and $\leqslant$ is the lattice order of $\boldsymbol{A}$.

B. $\mathrm{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant\rangle = \langle W, J, R\rangle$ is s.t.

$$W = \text{ lattice filters} \quad \text{and} \quad J = \text{ lattice ideals}.$$

- Moreover, $(\langle \boldsymbol{A}, \leqslant\rangle_+)^+$ is the canonical extension of $\langle \boldsymbol{A}, \leqslant\rangle$.

Implicative fragment of IPC

- For all $\langle \boldsymbol{A}, \leqslant\rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\rightarrow})$ we have:

A. $\langle \boldsymbol{A}, \leqslant\rangle$ is a Hilbert algebra equipped with the Hilbert-order.

B. $\mathrm{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant\rangle = \langle W, J, R\rangle$ is s.t.

$$W = \text{ implicative filters} \quad \text{and} \quad J = \text{ downsets}.$$

- Moreover, $(\langle \boldsymbol{A}, \leqslant\rangle_+)^+$ is intrinsically a Heyting algebra.

Intensional fragment of $\mathbf{FL}_{\mathbf{e}}^{\leqslant}$

- For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_{\mathbf{e}}^{\leqslant})$ we have:

A. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \rightarrow, \leqslant \rangle$-subreduct of a commutative FL-algebra.

B. $\mathrm{Pol}_{\mathscr{L}} \langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ upsets } \quad \text{and} \quad J = \text{ downsets}.$$

- Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_{+})^{+}$ is intrinsically a commutative FL-algebra.

Intensional fragment of $\mathbf{R}^{\leqslant}$

- For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InR}^{\leqslant})$ we have:

A. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \rightarrow, \neg, \leqslant \rangle$-subreduct of a De Morgan monoid.

B. $\mathrm{Pol}_{\mathscr{L}} \langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ intensional filters } \quad \text{and} \quad J = \text{ intensional ideals}.$$

- Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_{+})^{+}$ is intrinsically a De Morgan monoid.

A sample of what comes next...

- One can give a relational semantics for every logic, inspired by the Routley-Meyer semantics for Relevance Logic.
- We can delete co-worlds from frames in nice cases, e.g. distributive substructural and modal logics.
- This approach suggests a semantic-based of expanding every local consequence to the first-order lever with quantifiers and identity, which is axiomatized very transparently by means of meta-rules.
- This yields a complete alternative relational semantics for all first-order modal and superintuitionistic logics.

...thank you for coming!