Varieties of De Morgan monoids and axiomatic extension of Relevance Logic

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Definition

An involutive residuated lattice is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \neg, t\rangle$ such that

- $\langle A, \wedge, \vee\rangle$ is a lattice.
- $\langle A, \cdot, t\rangle$ is a commutative monoid.
- $\neg\neg a = a$ for every $a \in A$.
- For every $a, b, c \in A$ we have

$$a \cdot b \leq c \Longleftrightarrow b \cdot \neg c \leq \neg a.$$

- Setting $x \to y := \neg(x \cdot \neg y)$, we have:

$$\begin{aligned} x \cdot y \leq z &\Longleftrightarrow y \cdot \neg z \leq \neg x \\ &\Longleftrightarrow \neg(\neg x) \leq \neg(y \cdot \neg z) \\ &\Longleftrightarrow x \leq y \to z. \end{aligned}$$

- Therefore $\langle A, \wedge, \vee, \cdot, \to, t\rangle$ is a residuated lattice.

Definition

A De Morgan monoid is an involutive and distributive residuated lattice that satisfies $x \leq x \cdot x$.

- De Morgan monoids form a variety, i.e. an equational class, which we denote by DMM.
- In a De Morgan monoid $\boldsymbol{A}$ we write $f := \neg t$. Then

$$\neg a = a \to f, \text{ for every } a \in A.$$

Example (Sugihara monoids)

- Consider the non-zero integers

$$Z^* := \{a \in Z : a \neq 0\}.$$

- Equip it with the lattice operations corresponding to the standard ordering $\langle Z, \leq \rangle$ and let $\neg$ be the additive inversion.
- Set $t := 1$ and define the monoidal operation

$$b \cdot a = a \cdot b = \begin{cases} a \wedge b & \text{if } |a| = |b| \\ a & \text{if } |a| > |b|. \end{cases}$$

- The algebra $\boldsymbol{Z}^* = \langle Z^*, \wedge, \vee, \cdot, \neg, t \rangle$ is a De Morgan monoid.
- $\mathbb{V}(\boldsymbol{Z}^*)$ is the variety SM of Sugihara monoids, i.e. idempotent De Morgan monoids.
- Boolean algebras are Sugihara monoids.

Remark

Every variety of De Morgan monoids K determines a logic $\vdash_{\mathsf{K}}$.

- More precisely, for every ser $\Gamma \cup \{\varphi\}$ of formulas, we set

$$\begin{aligned} \Gamma \vdash_{\mathsf{K}} \varphi \Longleftrightarrow\ & \text{there is } n \in \omega \text{ and } \gamma_1, \dots, \gamma_n \in \Gamma \text{ s.t.} \\ & \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and hom } v\colon \boldsymbol{Fm} \to \boldsymbol{A}, \\ & \text{if } t \leq v(\gamma_1), \dots, v(\gamma_n), \text{ then } t \leq v(\varphi). \end{aligned}$$

- For $\mathsf{K} = \mathsf{DMM}$, we write $\mathbf{R^t}$ instead of $\vdash_{\mathsf{K}}$.
- The logic $\vdash_{\mathsf{K}}$ is algebraizable, with K as its unique equivalent quasi-variety. This fact allows us to study most metalogical properties of $\vdash_{\mathsf{K}}$ is terms of purely algebraic properties of K.

- There is a dual lattice isomorphism between the lattice $\mathrm{Ext}(\mathbf{R^t})$ of axiomatic extensions of $\mathbf{R^t}$ and the lattice $\mathrm{Var}(\mathsf{DMM})$ of varieties of De Morgan monoids:

$$\mathrm{Alg}(\cdot)\colon \mathrm{Ext}(\mathbf{R^t}) \longleftrightarrow \mathrm{Var}(\mathsf{DMM})\colon \vdash_{(\cdot)}$$

- Thus $\mathrm{Ext}(\mathbf{R^t})$ can be studied through the lenses of $\mathrm{Var}(\mathsf{DMM})$.

Details

- Given $\vdash$ in $\mathrm{Ext}(\mathbf{R^t})$, the class $\mathrm{Alg}(\vdash)$ is the variety of De Morgan monoids axiomatized by $\{t \leq \varphi \colon \emptyset \vdash \varphi\}$.
- Given K in $\mathrm{Var}(\mathsf{DMM})$, the logic $\vdash_{\mathsf{K}}$ is obtained extending $\mathbf{R^t}$ with the axioms $\{\varphi \colon \mathsf{K} \vDash t \leq \varphi\}$.
- For instance, $\vdash_{\mathsf{SM}}$ is the axiomatic extension obtained adding the Mingle Axiom $x \to (x \to x)$ to $\mathbf{R^t}$.

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- Let $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \neg, t \rangle$ be a De Morgan monoid. The negative cone of $\boldsymbol{A}$ is the algebra $\boldsymbol{A}^{-} = \langle A^{-}, \wedge, \vee, \to^{*}, t \rangle$ s.t.

$$A^{-} \coloneqq \{a \in A \colon a \leq t\}$$

and for every $a, b \in A^{-}$,

$$a \to^{*} b \coloneqq (a \to^{\boldsymbol{A}} b) \wedge t$$

- Remark: The negative cone $\boldsymbol{A}^{-}$ is a Brouwerian algebra, i.e. the $\langle \wedge, \vee, \to, t \rangle$-subreduct of a Heyting algebra.
- A seducing idea is to try to represent some De Morgan monoids, in terms of the more transparent structure of their negative cones.
- Accordingly, in this talk will describe some constructions which produce De Morgan monoids out of Brouwerian algebras.
- Let us start with some category equivalence, which work in the idempotent case, i.e. for Sugihara monoids.

- Let X be a class of similar algebras and $\kappa > 0$ be a cardinal.
- Consider the language $\mathscr{L}_{\mathsf{X}}^{\kappa}$ whose n-ary operations are the κ-sequences

$$\langle t_i : i < \kappa \rangle \text{ where each } t_i \text{ is a term of } \mathsf{X} \text{ in variables } \vec{x}_1, \ldots, \vec{x}_n.$$

Definition

Given $\boldsymbol{A} \in \mathsf{X}$, let $\boldsymbol{A}^{[\kappa]}$ be the $\mathscr{L}_{\mathsf{X}}^{\kappa}$-algebra with universe A^{κ} s.t.

$$\langle t_i : i < \kappa \rangle^{\boldsymbol{A}^{[\kappa]}}(\vec{a}_1, \ldots, \vec{a}_n) = \langle t_i^{\boldsymbol{A}}(\vec{a}_1/\vec{x}_1, \ldots, \vec{a}_n/\vec{x}_n) : i < \kappa \rangle.$$

The κ-th matrix power of X is the class

$$\mathsf{X}^{[\kappa]} \coloneqq \mathbb{I}\{\boldsymbol{A}^{[\kappa]} : \boldsymbol{A} \in \mathsf{X}\}.$$

- This construction extends to a functor $[\kappa] \colon \mathsf{X} \to \mathsf{X}^{[\kappa]}$.

Definition

Let X be a class of algebras of language $\mathscr{L}_{\mathsf{X}}$ and $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. A set of equations θ in one variable is compatible with $\mathscr{L}$ in X if for every n-ary operation $\varphi \in \mathscr{L}$ we have that:

$$\theta(x_1) \cup \cdots \cup \theta(x_n) \vDash_{\mathsf{X}} \theta(\varphi(x_1, \ldots, x_n)).$$

- For every $\boldsymbol{A} \in \mathsf{X}$, we let $\boldsymbol{A}(\theta, \mathscr{L})$ be the algebra

$$A(\theta, \mathscr{L}) = \{a \in A : \boldsymbol{A} \vDash \theta(a)\}$$

equipped with the restriction of the operations in $\mathscr{L}$.
- We obtain a functor $\theta_{\mathscr{L}} \colon \mathsf{X} \to \{\boldsymbol{A}(\theta, \mathscr{L}) : \boldsymbol{A} \in \mathsf{X}\}$.

Theorem (McKenzie 96, M. 16)

If $\mathcal{F} \colon \mathsf{K} \to \mathsf{M}$ is a right adjoint functor between quasi-varieties, then $\mathcal{F}$ is naturally isomorphic to $\theta_{\mathscr{L}} \circ [\kappa]$ for some κ, θ and $\mathscr{L}$.

- Rephrasing: right adjoints = twist-product constructions.

- Let's come back to Sugihara monoids:

Definition

1. Relative Stone algebras are $\langle \wedge, \vee, \to, t \rangle$-subreduct of subdirect products of Heyting chains.
2. A relative Stone algebra with a Boolean constant is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \to, t, f \rangle$ where $\langle A, \wedge, \vee, \to, t \rangle$ is a relative Stone algebra and f is an element such that

$$a \vee (a \to f) = t, \text{ for every } a \in A.$$

We denote by bRSA the variety of relative Stone algebras with a Boolean constant.

- If $\boldsymbol{A} \in \mathsf{SM}$, then the algebra $\downarrow(\boldsymbol{A}) \coloneqq \langle \boldsymbol{A}^{-}, f \rangle$ is a relative Stone algebra with a Boolean constant. This negative-cone construction is indeed a category equivalence $\downarrow \colon \mathsf{SM} \to \mathsf{bRSA}$.
- What about the corresponding equivalence functor (twist-product construction) which transforms bRSA into SM?

- Let $\boldsymbol{A} = \langle A, \wedge, \vee, \to, t, f \rangle \in \mathsf{bRSA}$. We set
$$A^{\rhd\lhd} := \{\langle a, b\rangle \in A^2 \colon a \vee b = t \text{ and } a \wedge b \leq f\}.$$
- For every $\langle a, b\rangle, \langle c, d\rangle \in A^{\rhd\lhd}$,
$$\begin{aligned}\langle a, b\rangle \sqcap \langle c, d\rangle &:= \langle a \wedge c, b \vee d\rangle\\ \langle a, b\rangle \sqcup \langle c, d\rangle &:= \langle a \vee c, b \wedge d\rangle\\ \neg\langle a, b\rangle &:= \langle b, a\rangle\\ \langle a, b\rangle \cdot \langle c, d\rangle &:= \langle s, t\rangle\end{aligned}$$
where
$$\begin{aligned}s &:= ((a \wedge f) \to d) \wedge [((c \wedge f) \to d) \to (a \wedge c)]\\ t &:= ((a \wedge f) \to d) \wedge ((c \wedge f) \to d) \wedge (s \wedge f).\end{aligned}$$
- The twist-product $\boldsymbol{A}^{\rhd\lhd} := \langle A^{\rhd\lhd}, \sqcap, \sqcup, \cdot, \neg, \langle t, f\rangle\rangle$ is a Sugihara monoid.

Theorem (Fussner, Galatos, Přenosil and Raftery)

The maps $\downarrow(\cdot) \colon \mathsf{SM} \longleftrightarrow \mathsf{bRSA} \colon (\cdot)^{\rhd\lhd}$ form a cat. equivalence.

- The category equivalence between SM and bRSA induces an isomorphism between the lattices of subquasi-varieties of SM and bRSA.
- In particular, the easy structure of Var(bRSA) can be used to describe Var(SM).

Varieties of Sugihara monoids

- Recall that $\mathbb{V}(\boldsymbol{Z}^*)$ is the variety SM of Sugihara monoids.
- Let $\boldsymbol{Z}$ be the algebra obtained adding 0 to $\boldsymbol{Z}^*$ and making it be the neutral element of the monoidal operation. $\mathbb{V}(\boldsymbol{Z})$ is the variety OSM of odd Sugihara monoids.
- For every $n, m \in \omega$ with $m \neq 0$, let $\boldsymbol{Z}_{2n+1}$ and $\boldsymbol{Z}_{2m}$ be the subalgebras of $\boldsymbol{Z}$ and $\boldsymbol{Z}^*$, respectively, with universes
$$\{-n, \ldots, -1, 0, 1, \ldots, n\} \text{ and } \{-m, \ldots, -1, 1, \ldots, m\}.$$

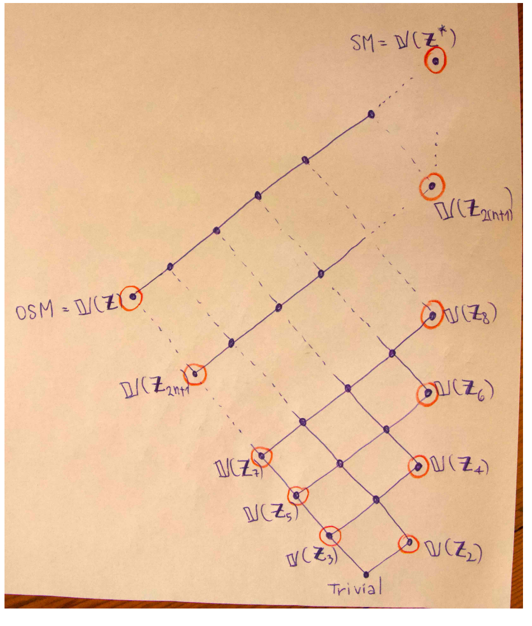


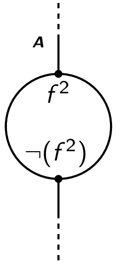
Figure: Varieties of Sugihara monoids

- What about non-idempotent De Morgan monoids?

Theorem

Let $\boldsymbol{A}$ be a non-idempotent FSI De Morgan monoid.

1. The interval $[\neg(f^2), f^2]$ is the universe of a subalgebra of $\boldsymbol{A}$.
2. $\boldsymbol{A}$ is the union of $[\neg(f^2), f^2]$ and two chains of idempotents, $(\neg(f^2)]$ and $[f^2)$.
3. $[\neg(f^2), f^2]$ can be squeezed producing a member of OSM.



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- A non-trivial variety is minimal if it has no proper non-trivial subvarieties.
- Minimal varieties of De Morgan monoids correspond to maximal consistent axiomatic extensions of $\mathbf{R^t}$.
- Our first goal will be to characterize the minimal varieties of De Morgan monoids.

Lemma

Let K be a minimal variety of finite type with a constant symbol t. Then the free 0-generated algebra of K is either trivial or simple.

Idea: Pick K minimal such that the free 0-generated algebra $\boldsymbol{A}$ is non-trivial. Jónsson proved that every non-trivial finitely generated algebra of finite type has a simple homomorphic image. Then pick a simple $\boldsymbol{B} \in \mathbb{H}(\boldsymbol{A})$. The free 0-generated algebra of $\mathbb{V}(\boldsymbol{B})$ is $\boldsymbol{B}$ itself. By minimality of K we have $\mathbb{V}(\boldsymbol{B}) = \mathsf{K}$. Hence $\boldsymbol{A} = \boldsymbol{B}$.

- Then consider a minimal variety K of De Morgan monoids.
- Its free 0-generated algebra $\boldsymbol{A}$ is either trivial or simple.
- If $\boldsymbol{A}$ is trivial, then $\mathsf{K} \vDash t \approx f$. Hence $\mathsf{K} \subseteq \mathsf{OSM}$. Recall that the lattice of subvarieties of OSM was

$$\mathbb{V}(\emptyset) \subsetneq \mathbb{V}(\boldsymbol{Z}_3) \subsetneq \cdots \subsetneq \mathbb{V}(\boldsymbol{Z}_{2n+1}) \subsetneq \ldots \mathbb{V}(\boldsymbol{Z}) = \mathsf{OSM}$$

- By minimality we conclude that $\mathsf{K} = \mathbb{V}(\boldsymbol{Z}_3)$.
- Then suppose that $\boldsymbol{A}$ is simple. Clearly $\boldsymbol{A}$ is a homomorphic image of the free 0-generated De Morgan monoid.

Theorem (Slaney)

The free 0-generated De Morgan monoid $\boldsymbol{B}$ has 3088 elements!

- In particular, $\boldsymbol{B}$ has exactly 3 simple homomorphic images $\boldsymbol{B}_2$, $\boldsymbol{D}_4$ and $\boldsymbol{C}_4$. Then K is the variety generated by one of these algebras.

Theorem

The minimal varieties of De Morgan monoids are $\mathbb{V}(\boldsymbol{Z}_3)$, $\mathbb{V}(\boldsymbol{B}_2)$, $\mathbb{V}(\boldsymbol{D}_4)$ and $\mathbb{V}(\boldsymbol{C}_4)$. Hence there are exactly 4 maximal consistent axiomatic extensions of $\mathbf{R^t}$.

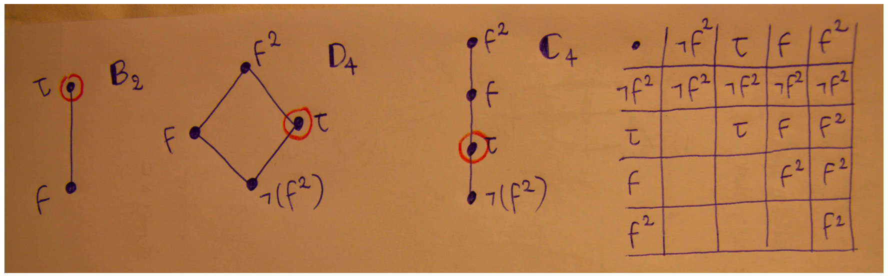

Figure: The algebras $\boldsymbol{B}_2$, $\boldsymbol{D}_4$ and $\boldsymbol{C}_4$

- Numerology of Relevance Logic: There are 68 minimal quasi-varieties of De Morgan monoids.

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Definition

Let $\vdash$ be a logic, i.e. a substitution-invariant consequence relation over formulas in an algebraic signature. A rule $\gamma_1, \dots, \gamma_n \rhd \varphi$ is:

1. Admissible in $\vdash$, if its addition to $\vdash$ produces no new tautologies.
2. Derivable in $\vdash$, if $\gamma_1, \dots, \gamma_n \vdash \varphi$.

The logic $\vdash$ is structurally complete if every admissible rule of $\vdash$ is derivable in $\vdash$.

Theorem (Bergman)

Let $\vdash$ be a finitary logic algebraized by a quasi-variety K. TFAE:

1. $\vdash$ is structurally complete.
2. K is generated as a quasi-variety by $\boldsymbol{Fm}_{\mathsf{K}}(\omega)$.
3. Every finitely generated RSI member of K can be embedded into an ultrapower of $\boldsymbol{Fm}_{\mathsf{K}}(\omega)$.

- Immediate observation: $\mathbf{R^t}$ is not structurally complete.

- Can we say something about admissibility in Relevance Logic thanks to the negative-cone construction?
- Let $\mathbf{R^t_+}$ be the positive, i.e. $\langle \wedge, \vee, \cdot, \to, t \rangle$, fragment of $\mathbf{R^t}$.
- Now, consider a formula φ written in the signature $\langle \wedge, \vee, \to \rangle$.
- We define recursively, the amended version $\varphi^\diamond$ of φ:

$$
\begin{aligned}
x^\diamond &\coloneqq x \wedge e \\
\alpha \wedge \beta &\coloneqq \alpha^\diamond \wedge \beta^\diamond \\
\alpha \vee \beta &\coloneqq \alpha^\diamond \vee \beta^\diamond \\
\alpha \to \beta &\coloneqq (\alpha^\diamond \to \beta^\diamond) \wedge e.
\end{aligned}
$$

Theorem

If $\Gamma \rhd \varphi$ is an admissible rule of **IPC** in the signature $\langle \wedge, \vee, \to \rangle$, then $\Gamma^\diamond \rhd \varphi^\diamond$ is admissible in $\mathbf{R^t_+}$. Moreover, if $\Gamma \rhd \varphi$ is not derivable in **IPC**, then $\Gamma^\diamond \rhd \varphi^\diamond$ is not derivable in $\mathbf{R^t_+}$.

- This trick does not extend to the full-signature $\mathbf{R^t}$, where the amended Mints' rule is not admissible.

- Admissibility can be split in two halves:

Definition

Let $\vdash$ be a logic. A rule $\gamma_1, \dots, \gamma_n \rhd \varphi$ is:

1. Passive in $\vdash$, if there is no substitution σ such that $\sigma(\gamma_1), \dots, \sigma(\gamma_n)$ are theorems.
2. Active in $\vdash$, if it is not passive.

The logic $\vdash$ is passively (resp. actively) structurally complete if every passive (resp. active) admissible rule of $\vdash$ is derivable in $\vdash$.

Theorem (Bergman and Wroński)

Let $\vdash$ be a finitary logic algebraized by a quasi-variety K. Then $\vdash$ is passively structurally complete iff every positive existential sentence either holds in every member of K or in no non-trivial member of K.

- In what follows we focus on passive structural completeness.
- But, is there any interesting consequence of PSC?

- A quasi-variety K has the joint embedding property (JEP) when for every non-trivial $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ there are $\boldsymbol{C} \in \mathsf{K}$ and embeddings $f\colon \boldsymbol{A} \to \boldsymbol{C}$ and $g\colon \boldsymbol{B} \to \boldsymbol{C}$.
- The JEP is a completeness theorem w.r.t. a single algebra:

Theorem (Maltsev)

A quasi-variety has the JEP iff it is generated as a quasi-variety by a single algebra.

- What about the logical meaning of JEP?
- A logic $\vdash$ has the relevance principle (RP) is for every set of formulas $\Gamma \cup \Delta \cup \{\varphi\}$ s.t. $Var(\Gamma \cup \{\varphi\}) \cap Var(\Delta) = \emptyset$,

$$\text{if } \Gamma, \Delta \vdash \varphi \text{, then either } \Delta \text{ is incositent or } \Gamma \vdash \varphi.$$

Theorem

Let $\vdash$ be a finitary logic algebraized by a quasi-variety K. $\vdash$ has the RP iff K has the JEP. Moreover, if $\vdash$ is PSC, then it has the RP.

- Let's turn back to PSC in De Morgan monoids:

Theorem

Let K be a non-trivial variety of De Morgan monoids. $\vdash_{\mathsf{K}}$ is PSC iff one of the following holds:

1. K is a variety of odd Sugihara monoids.
2. K is the variety $\mathbb{V}(\boldsymbol{B}_2)$ of Boolean algebras.
3. $\mathsf{K} = \mathbb{V}(\boldsymbol{D}_4)$.
4. $\boldsymbol{C}_4$ is a retract of all non-trivial members of K.

Theorem (Hu)

If $\boldsymbol{A}$ is a primal algebra, then $\mathbb{V}(\boldsymbol{A})$ is categorically equivalent to the variety of Boolean algebras.

Remark

- Case 1: K is cat. eq. to a variety of relative Stone algebras.
- Case 2-3: K is cat. eq. to the variety of Boolean algebras.
- Case 1-2-3: The logic $\vdash_{\mathsf{K}}$ is structurally complete.

Theorem

Let K be a non-trivial variety of De Morgan monoids. $\vdash_{\mathsf{K}}$ is PSC iff one of the following holds:

1. K is a variety of odd Sugihara monoids.
2. K is the variety $\mathbb{V}(\boldsymbol{B}_2)$ of Boolean algebras.
3. $\mathsf{K} = \mathbb{V}(\boldsymbol{D}_4)$.
4. $\boldsymbol{C}_4$ is a retract of all non-trivial members of K.

- A finer analysis shows that there exists a largest variety M of De Morgan monoids satisfying condition 4.

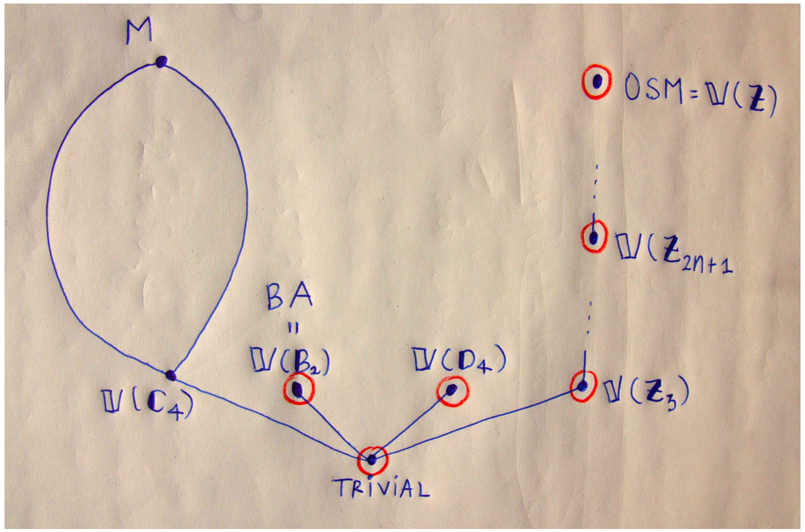

Figure: PSC varieties of De Morgan monoids

- Let us take a closer look at the structure theory of M:

Definition

A Dunn monoid is a distributive RL which satisfies $x \leq x \cdot x$.

- Given a Dunn monoid $\boldsymbol{A}$, its reflection $\mathbb{R}(\boldsymbol{A})$ is the algebra with universe $A \cup A' \cup \{0, 1\}$ defined as:

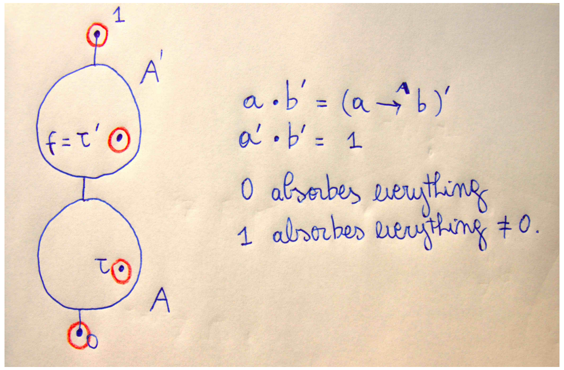

Figure: The reflection $\mathbb{R}(\boldsymbol{A})$ of $\boldsymbol{A}$

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Lemma

If $\boldsymbol{A}$ is a Dunn monoid, then $\mathbb{R}(\boldsymbol{A}) \in \mathsf{M}$.

- Then the variety of Dunn monoid is exactly the class of RL-reducts of M.

Theorem (Urquhart)

The equational theory of Dunn monoids is undecidable.

Corollary

The equational theory of M is undecidable. M is not generated by its finite members (since it has a finite equational basis).

- Given a variety of Dunn monoids K, we set

$$\mathbb{R}(\mathsf{K}) = \mathbb{V}(\{\mathbb{R}(\boldsymbol{A}) : \boldsymbol{A} \in \mathsf{K}\}) \subseteq \mathsf{M}.$$

Lemma

Consider the map $\mathbb{R} \colon \mathrm{Var}(\mathsf{DuM}) \to \mathrm{Var}(\mathsf{M})$.

1. $\mathbb{R}$ is order-reflecting and, therefore, injective.
2. $\mathbb{R}$ preserves structural incompleteness.

- Then M has uncountably many structurally incomplete subvarieties, e.g. apply $\mathbb{R}$ to varieties of Brouwerian algebras.

- In general not every FSI member of M is a reflection of a Dunn monoid.
- We need a more general construction: skew reflections.

Theorem

The FSI members of M coincide with skew reflections of Dunn monoids in which $e \leq f$.

- Unfortunately, the same Dunn monoid may have different skew reflections.
- Developing the study of skew reflections and using combinatorial arguments, we obtain a characterization of the covers of $\mathbb{V}(\boldsymbol{C}_4)$ in M.

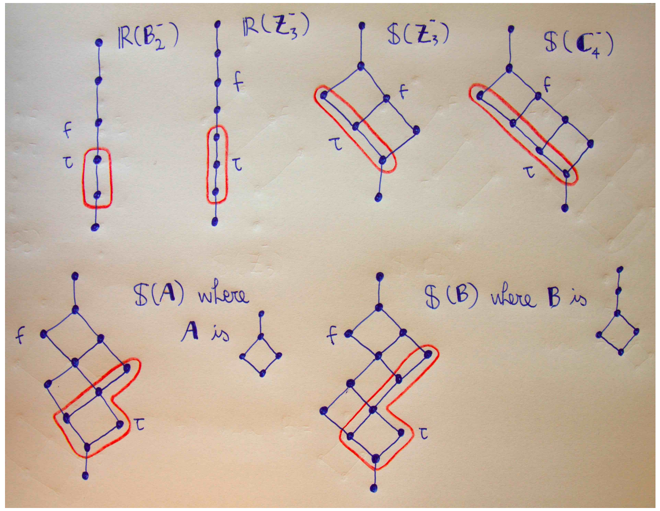

Figure: The covers of $\mathbb{V}(\boldsymbol{C}_4)$ in M

Theorem

There are exactly 6 covers of $\mathbb{V}(\boldsymbol{C}_4)$ in M, all of which are finitely generated.

- There there are exactly 6 PSC axiomatic extensions of $\mathbf{R}^{\mathbf{t}}$ whose unique consistent axiomatic extension is $\vdash_{\mathbb{V}(\boldsymbol{C}_4)}$.

Theorem

Let K be the varietal join of these 6 covers. K is primitive, i.e. every subquasi-variety of K is a variety. Thus all axiomatic extensions of $\vdash_{\mathsf{K}}$ are structurally complete.

- Trick: Use Fleischer's Lemma.

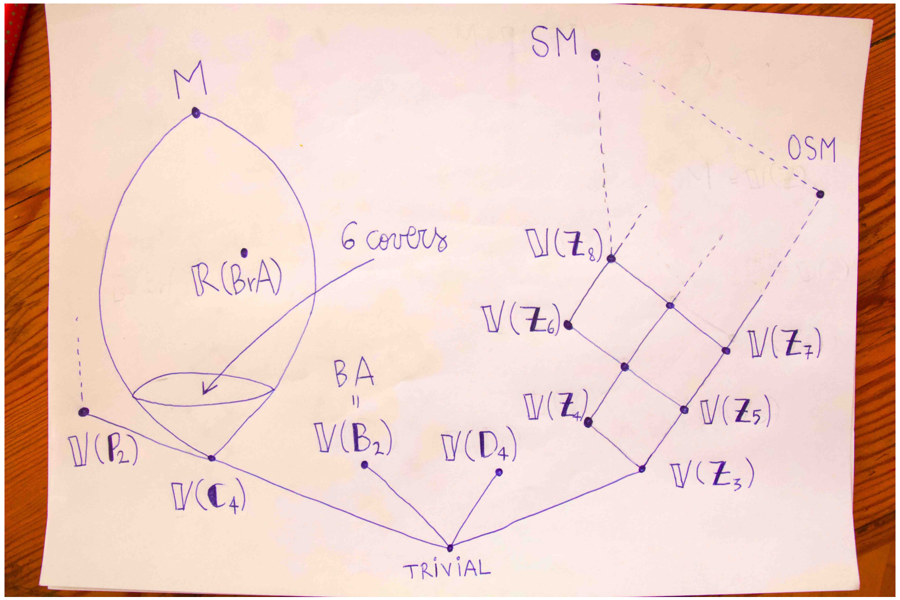

Figure: Some varieties of De Morgan monoids

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Definition

Let K be a class of algebras.

1. A homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ in K is an epimorphism in K if for every pair $g, h\colon \boldsymbol{B} \rightrightarrows \boldsymbol{C}$ of homomorphisms in K

$$\text{if } g \circ f = h \circ f, \text{ then } g = h.$$

2. K has the (epimorphism surjectivity property) ESP if epimorphisms in K are surjective.

- Are epis surjective in a variety?
- Yes: Boolean algebras, Heyting algebras, lattices, semilattices and (Abelian) groups.
- No: distributive lattices, rings with unity and monoids.
- Thus epimorphism surjectivity is not preserved in subvarieties!

Remark: The interest of the the ES property in logic is that it corresponds to the Beth definability property.

Definition

Let K be a quasi-variety and $\boldsymbol{B} \in$ K. A subalgebra $\boldsymbol{A} \leq \boldsymbol{B}$ is K-epic if for every pair of homomorphisms $f, g\colon \boldsymbol{B} \rightrightarrows \boldsymbol{C} \in$ K

$$\text{if } f \restriction_A = g \restriction_A, \text{ then } f = g.$$

- K has the ESP iff no $\boldsymbol{B} \in$ K has a proper K-epic subalgebra.

Theorem (Campercholi)

Let K be a quasi-variety and $\boldsymbol{A} \leq \boldsymbol{B} \in$ K. TFAE:

1. $\boldsymbol{A}$ is a K-epic subalgebra of $\boldsymbol{B}$.
2. For every $b \in B$ there is a primitive positive formula $\varphi(\vec{x}, y)$ and $\vec{a} \in A$ such that

$$\mathsf{K} \vDash \forall \vec{x}, y, z((\varphi(\vec{x}, y) \,\&\, \varphi(\vec{x}, z)) \to y \approx z) \text{ and } \boldsymbol{B} \vDash \varphi(\vec{a}, b).$$

- Can we inherit some knowledge about the ESP from Brouwerian algebras to De Morgan monoids?
- The following observation goes in this direction:

Lemma

A variety of Dunn Monoids K has the ES property iff $\mathbb{R}(\mathsf{K})$ has it.

- Natural question: Are there varieties of Brouwerian algebras lacking the ESP?

Theorem (Bezhanishvili, M. and Raftery)

There is a continuum of varieties of Brouwerian algebras without the ESP.

- There is a continuum of subvarieties of M lacking the ESP.
- Let's see some examples.

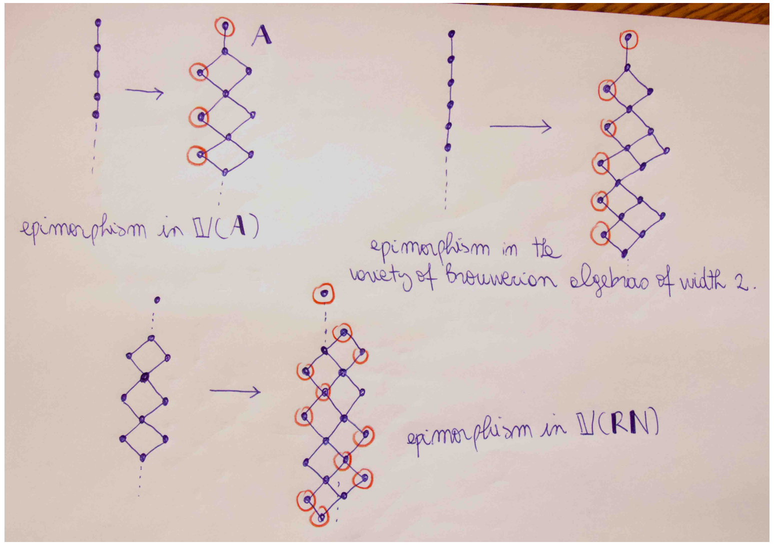

Figure: Varieties of Brouwerian algebras lacking the ES property

- What about positive results on the ESP? Working with Esakia duality, we obtain:

Theorem (Bezhanishvili, M. and Raftery)

Varieties of Brouwerian algebras of bounded depth have the ESP. These include all finitely generated varieties of Brouwerian algebras.

- There is a continuum of subvarieties of M with the ESP.
- Can we generalize the above theorem to the setting of De Morgan monoids?

Theorem

Let K be a variety of De Morgan monoids, whose negative cones have bounded depth and whose FSI members are generated by their negative cones. Then K has the ESP.

- Obs: This result holds indeed for square-increasing [I]RLs.

Finally...

Thank you for your attention!