Frames, ordered algebras, and quantifiers for deductive systems

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Some puzzling questions on relational semantics

1. What is a frame? (for an arbitrary algebraic language)
2. What does it mean that a logic has a local relational semantics?
3. Why do most logics have a semantics of ordered algebras?
4. Are there logic-based dualities/completions for ordered algebras?

What is a frame? (for an arbitrary algebraic language)

Definition

1. An order type for an algebraic language $\mathscr{L}$ is an assignment to every symbol $f \in \mathscr{L}$ of a choice of which arguments of f will be treated as increasing and which ones as decreasing.
2. An ordered language is an algebraic language equipped with an order type.
3. Let $\mathscr{L}$ be an ordered language. An $\mathscr{L}$-algebra is a pair $\langle \boldsymbol{A}, \leqslant \rangle$ where $\boldsymbol{A}$ is an algebra, $\leqslant$ a partial order on A, and if $f = f(\vec{x}, \vec{y})$, then $f^{\boldsymbol{A}}$ is incr. on $\vec{x}$ and decr. on $\vec{y}$ w.r.t. $\leqslant$.

Examples:

- Consider the language of FL_e-algebras $\langle \wedge, \vee, \cdot, \to, 1, 0 \rangle$.
- Let $\mathscr{L}$ be the ordered language according to which $\wedge, \vee, \cdot$ are increasing and $\to$ is decreasing on the first argument and increasing on the second.
- Then every FL-algebra (when equipped with the lattice order) is indeed an $\mathscr{L}$-algebra.
- A similar situation holds for Modal Algebras.

Definition

A polarity is a triple $\langle W, J, R\rangle$ such that W and J are non-empty sets and $R \subseteq W \times J$.

- Every polarity $\langle W, J, R\rangle$ induces a Galois connection

$$(\cdot)^{\rhd} : \mathcal{P}(W) \longleftrightarrow \mathcal{P}(J) : (\cdot)^{\lhd}$$

by setting for $A \subseteq W$ and $B \subseteq J$

$$A^{\rhd} := \{j \in J : \langle w, j\rangle \in R \text{ for all } w \in A\}$$
$$B^{\lhd} := \{w \in W : \langle w, j\rangle \in R \text{ for all } j \in B\}.$$

- Indeed, we have that $B \subseteq A^{\rhd} \iff A \subseteq B^{\lhd}$.
- Then $(\cdot)^{\rhd\lhd} : \mathcal{P}(W) \to \mathcal{P}(W)$ is a closure operator on W. We denote its lattice of closed sets by $\mathcal{G}(W, J, R)$.
- We define two preorders $\langle W, \leqslant_W\rangle$ and $\langle J, \leqslant_J\rangle$ as follows:

$$w_1 \leqslant_W w_2 \iff w_2^{\rhd\lhd} \subseteq w_1^{\rhd\lhd}$$
$$j_1 \leqslant_J j_2 \iff j_1^{\lhd} \subseteq j_2^{\lhd}.$$

1. A labeling map for an algebraic language $\mathscr{L}$ is a function $\beta \colon \mathscr{L} \to \{\Box, \Diamond\}$.
2. A labeled language is an algebraic language $\mathscr{L}$ equipped with a labeling map β. Sometimes we write $\mathscr{L}^{\beta}$.

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-preframe is a structure

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

where $\langle W, J, R\rangle$ is a polarity such that $\leqslant_W$ and $\leqslant_J$ are partial orders, and for every operation symbol $f \in \mathscr{L}$ such that $f = f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ we have:

$$\text{if } \beta(f) = \Diamond, \text{ then } T_f \subseteq W^m \times J^n \times W$$
$$\text{if } \beta(f) = \Box, \text{ then } T_f \subseteq J^m \times W^n \times J.$$

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-frame

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

is an $\mathscr{L}$-preframe such that for all connectives $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ s.t. $\beta(f) = \Diamond$:

(a) For every $\vec{w}_1, \vec{w}_2 \in W^m$, $\vec{j}_1, \vec{j}_2 \in J^n$, and $u_1, u_2 \in W$ such that $\vec{w}_2 \leqslant_W \vec{w}_1$, $\vec{j}_2 \leqslant_J \vec{j}_1$ and $u_1 \leqslant_W u_2$,

$$\text{if } \langle \vec{w}_1, \vec{j}_1, u_1\rangle \in T_f, \text{ then } \langle \vec{w}_2, \vec{j}_2, u_2\rangle \in T_f.$$

(b) $T_f(\vec{w}, \vec{j})$ is a closed set of $(\cdot)^{\rhd\lhd}$ for all $\vec{w} \in W^m$ and $\vec{j} \in J^n$.

Connectives f such that $\beta(f) = \Box$ need to satisfy a dual requirement.

We refer to W and J as to the sets of worlds and co-worlds of $\boldsymbol{F}$ respectively.

- A valuation in a $\mathscr{L}$-frame $\boldsymbol{F}$ is a map $v \colon Var \to \mathcal{G}(W, J, R)$.
- We want to define two relations of satisfaction and co-satisfaction of formulas under v, respectively at worlds $w \in W$ and co-worlds $j \in J$, in symbols

$$w, v \Vdash \varphi \text{ and } j, v \succ \varphi.$$

- For every variable $x \in Var$, we set

$$w, v \Vdash x \iff w \in v(x)$$
$$j, v \succ x \iff j \in v(x)^{\rhd}.$$

- Moreover, for every connective $f(\vec{x}; \vec{y})$ s.t. $\beta(f) = \Diamond$ we set:

$$w, v \Vdash f(\vec{\varphi}, \vec{\psi}) \iff w \in \{r \in W : \text{there are } \vec{u} \in W^m \text{ and } \vec{i} \in J^n \text{ s.t. } \langle \vec{u}, \vec{i}, r\rangle \in T_f \text{ and for all } k \leqslant m, t \leqslant n\ u_k, v \Vdash \varphi_k \text{ and } i_t, v \succ \psi_t\}^{\rhd\lhd}$$

$$j, v \succ f(\vec{\varphi}, \vec{\psi}) \iff j \in \{w \in W : w, v \Vdash f(\vec{\varphi}, \vec{\psi})\}^{\rhd}.$$

- A dual definition applied to connectives $f(\vec{x}; \vec{y})$ s.t. $\beta(f) = \Box$.

Frames $\boldsymbol{F}$ can be transformed into algebras $\boldsymbol{F}^+$ as follows:

- The universe of $\boldsymbol{F}^+$ is $\mathcal{G}(W, J, R)$.
- For every connective $f(z_1, \ldots, z_n)$ and $a_1, \ldots, a_n \in F^+$,

$$f^{\boldsymbol{F}^+}(a_1, \ldots, a_n) := \{w \in W : w, v \Vdash f(z_1, \ldots, z_n)\}$$

where v is any valuation in $\boldsymbol{F}$ s.t. $v(z_i) = a_i$.

Definition

Let $\mathscr{L}$ be a labeled ordered language.

1. An $\mathscr{L}$-general frame is a pair $\langle \boldsymbol{F}, A\rangle$ where $\boldsymbol{F}$ is an $\mathscr{L}$-frame and A is the universe of a subalgebra of $\boldsymbol{F}^+$.
2. The complex algebra of a general frame $\langle \boldsymbol{F}, A\rangle$ is

$$\langle \boldsymbol{F}, A\rangle^+ := \langle \boldsymbol{A}, \subseteq\rangle \text{ where } \boldsymbol{A} \leqslant \boldsymbol{F}^+.$$

Remark

If $\langle \boldsymbol{F}, A\rangle$ is an $\mathscr{L}$-general frame, then $\langle \boldsymbol{F}, A\rangle^+$ is an $\mathscr{L}$-algebra.

What does it mean that a logic has a local relational semantics?

Let Fr be a class of $\mathscr{L}$-general frames.

1. The local consequence relation of Fr is:

$$\varGamma \vdash^{l}_{\mathrm{Fr}} \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle \in \mathrm{Fr} \text{ and } w \in W \text{ if } w, v \Vdash \varGamma, \text{ then } w, v \Vdash \varphi.$$

2. The colocal consequence relation of Fr is:

$$\varGamma \vdash^{cl}_{\mathrm{Fr}} \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle \in \mathrm{Fr} \text{ and } j \in J \text{ if } j, v \succ \varGamma, \text{ then } j, v \succ \varphi.$$

Definition

Let $\mathscr{L}$ be a labeled ordered language. A logic $\vdash$ is a $\mathscr{L}$-local (resp. colocal) consequence if it is the local (resp. colocal) consequence of a class of $\mathscr{L}$-general frames.

Remark

A logic is local consequence iff it is a colocal consequence.

Definition

A logic $\vdash$ is monotone if there is an ordered language $\mathscr{L}$ over $\mathscr{L}_{\vdash}$ s.t. every connective $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ is increasing in $\vec{x}$ and decreasing in $\vec{y}$ on $\boldsymbol{Fm}$ w.r.t. $\vdash$, i.e. if for every φ and ψ such that $\varphi \vdash \psi$ we have

$$f(\delta_1, \ldots, \delta_{i-1}, \varphi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon}) \vdash f(\delta_1, \ldots, \delta_{i-1}, \psi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon})$$
$$f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \psi, \epsilon_{j+1}, \ldots, \epsilon_n) \vdash f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \varphi, \epsilon_{j+1}, \ldots, \epsilon_n)$$

for every $\vec{\delta}$ and $\vec{\epsilon}$. In this case, $\vdash$ is $\mathscr{L}$-monotone.

Theorem (Syntactic characterization of local consequences)

Let $\mathscr{L}$ be an ordered language, and β a labeling map. The following conditions are equivalent:

1. $\vdash$ is an $\mathscr{L}$-monotone logic.
2. $\vdash$ is an $\mathscr{L}^{\beta}$-local consequence.

Definition

Let K be a class of ordered algebras. The logic $\vdash_{\mathsf{K}}^{\leqslant}$ preserving degrees of truth of K is defined as follows:

$$\Gamma \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff \text{for all } \langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{K}, \text{ hom } v \colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ and } a \in A$$
$$\text{if } a \leqslant v(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } a \leqslant v(\varphi).$$

Remark

Let K be a class of $\mathscr{L}$-algebras. The logic $\vdash_{\mathsf{K}}^{\leqslant}$ is an $\mathscr{L}^{\beta}$-local consequence (for every β).

Examples of local consequences:

- Local consequences of normal modal logics.
- Superintuitionistic logics.
- Logics preserving degrees of truth of residuated lattices.
- Fragments of local consequences are still local consequences.

Why do most logics have a semantics of ordered algebras?

Definition

Let $\vdash$ be a logic and $\mathscr{L}$ be an ordered language over $\mathscr{L}_{\vdash}$.

1. An $\mathscr{L}$-algebra $\langle \boldsymbol{A}, \leqslant \rangle$ is an $\mathscr{L}$-ordered model of $\vdash$ if for every $a \in A$ the upset $\uparrow a$ is a deductive filter of $\vdash$.
2. Accordingly, we set

$$\mathrm{Alg}_{\mathscr{L}}^{\leqslant}(\vdash) := \{\langle \boldsymbol{A}, \leqslant \rangle : \langle \boldsymbol{A}, \leqslant \rangle \text{ is an } \mathscr{L}\text{-ordered model of } \vdash\}.$$

Remark

$\mathrm{Alg}_{\mathscr{L}}^{\leqslant}(\vdash)$ is closed under $\mathbb{S}$ and $\mathbb{P}$ (and $\mathbb{P}_{\mathrm{u}}$ if $\vdash$ is finitary).

- Non-Mathematical Thesis: $\mathrm{Alg}_{\mathscr{L}}^{\leqslant}(\vdash)$ should be understood as the class of distinguished ordered models of $\vdash$ (from the point of view of the ordered language $\mathscr{L}$).

Theoretic justification of $\mathrm{Alg}_{\mathscr{L}}^{\leqslant}(\vdash)$

Definition

Let $\vdash$ be a logic and $\langle \boldsymbol{F}, A \rangle$ be an $\mathscr{L}$-general frame.

1. $\langle \boldsymbol{F}, A \rangle$ is a model of $\vdash$ if its local consequence extends $\vdash$.
2. $\langle \boldsymbol{F}, A \rangle$ is a co-model of $\vdash$ if its co-local consequence extends $\vdash$.

Theorem

Let $\vdash$ be a logic, $\mathscr{L}$ an ordered lang. over $\mathscr{L}_{\vdash}$, β a labeling map.

$$\mathrm{Alg}_{\mathscr{L}}^{\leqslant}(\vdash) = \{\langle \boldsymbol{F}, A \rangle^{+} : \langle \boldsymbol{F}, A \rangle \text{ is an } \mathscr{L}^{\beta}\text{-general frame and a model of } \vdash\}.$$

In other words, $\mathrm{Alg}_{\mathscr{L}}^{\leqslant}(\vdash)$ is the class of complex algebras of relational models of $\vdash$ (from the point of view of $\mathscr{L}$ and β).

- Rephrasing: Logics may have a semantics of ordered algebras, because they have a local relational semantics.

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: semilattice-based logics

Theorem

Let K be a variety with a semilattice reduct s.t. when ordered under the meet-order is a class of $\mathscr{L}$-algebras. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}}) = \{\langle \boldsymbol{A}, \leqslant\rangle : \boldsymbol{A} \in \mathsf{K} \text{ and } \leqslant \text{ is the meet-order of } \boldsymbol{A}\}.$$

Examples:

- Let K be a variety of modal algebras, and $\vdash$ the local consequence of the normal modal logic associated with K. Then $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is K with the lattice order (for the natural $\mathscr{L}$).
- Let K be a variety of Heyting algebras, and $\vdash$ the superintuitionistic logic associated with K. Then $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is K with the lattice order (for the natural $\mathscr{L}$).

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: intensional fragments

For the natural ordered languages $\mathscr{L}$:

- Let $\mathbf{IPC}_{\to}$ be the $\langle\to\rangle$-fragment of Intuitionistic Logic. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\to}) = \text{ Hilbert algebras } + \text{Hilbert-order}.$$

- Let $\mathbf{InFL}^{\leqslant}_{\mathrm{e}}$ be the $\langle\cdot,\to\rangle$-fragment of the logic preserving degrees of truth of commutative FL-algebras. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}^{\leqslant}_{\mathrm{e}}) = \langle\cdot,\to,\leqslant\rangle\text{-subreducts of commutative FL-algebras}.$$

- Let $\mathbf{InR}^{\leqslant}$ be the $\langle\cdot,\to,\neg\rangle$-fragment of the logic preserving degrees of truth of De Morgan monoids. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InR}^{\leqslant}) = \langle\cdot,\to,\neg,\leqslant\rangle\text{-subreducts of De Morgan monoids}.$$

Are there logic-based dualities/completions for ordered algebras?

Definition

Let $\vdash$ be a logic and $\mathscr{L}$ an ordered language over $\mathscr{L}_{\vdash}$. The $\mathscr{L}$-cologic of $\vdash$ is the logic $\vdash^{\partial}_{\mathscr{L}}$ preserving degrees of truth of

$$\{\langle \boldsymbol{A}, \leqslant^{\partial}\rangle : \langle \boldsymbol{A}, \leqslant\rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)\}.$$

Remark:

- If $\vdash$ is the local cons. of a class of $\mathscr{L}$-general frames, then

$$\varphi \vdash \psi \Longleftrightarrow \psi \vdash^{\partial}_{\mathscr{L}} \varphi.$$

- If $\vdash$ is finitary, then $\vdash^{\partial}_{\mathscr{L}}$ is finitary and is the logic induced by the following class of matrices:

$$\{\langle \boldsymbol{A}, I\rangle : \langle \boldsymbol{A}, \leqslant\rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) \text{ and } I \text{ is a poset ideal of } \langle A, \leqslant\rangle\}.$$

- In known cases the co-logic is the expected dual of $\vdash$.

- We are now ready to introduce a class of distinguished relational models of a logic:

Definition

Let $\vdash$ be a logic, $\mathscr{L}$ an ordered language over $\mathscr{L}_\vdash$, and β a labeling map.

1. A $\mathscr{L}^\beta$-general frame $\langle \boldsymbol{F}, A \rangle$ is a $\mathscr{L}^\beta$-distinguished model of $\vdash$ when it is a model of $\vdash$ and co-model of $\vdash^\partial_\mathscr{L}$.
2. Accordingly, we set

$$\mathsf{Rel}^\beta_\mathscr{L}(\vdash) = \{\langle \boldsymbol{F}, A\rangle : \langle \boldsymbol{F}, A\rangle \text{ is a } \mathscr{L}^\beta\text{-distinguished model of } \vdash\}.$$

Remarks

- Now, $\mathsf{Alg}^\leqslant_\mathscr{L}(\vdash)$ and $\mathsf{Rel}^\beta_\mathscr{L}(\vdash)$ are respectively the classes of distinguished ordered models and relational models of $\vdash$. It is natural to wonder whether they are inter-translatable.

Definition

Let $\vdash$ be a logic, $\mathscr{L}$ an ordered language over $\mathscr{L}_\vdash$, and β a labeling map. For every $\langle \boldsymbol{A}, \leqslant\rangle \in \mathsf{Alg}^\leqslant_\mathscr{L}(\vdash)$ we define:

1. The $\mathscr{L}$-canonical polarity of $\langle \boldsymbol{A}, \leqslant\rangle$ is the polarity

$$\mathsf{Pol}_\mathscr{L}\langle \boldsymbol{A}, \leqslant\rangle \coloneqq \langle W, J, R\rangle$$

where $R \subseteq W \times J$ is the relation of non-empty intersection and

$$W = \{w \subseteq A : w \text{ is both an upset and a } \vdash\text{-filter}\}$$
$$J = \{j \subseteq A : w \text{ is both an upset and a } \vdash^\partial_\mathscr{L}\text{-filter}\}.$$

2. The canonical $\mathscr{L}^\beta$-frame of $\langle \boldsymbol{A}, \leqslant\rangle$ is

$$\langle \boldsymbol{A}, \leqslant\rangle_+ \coloneqq \langle \mathsf{Pol}_\mathscr{L}\langle \boldsymbol{A}, \leqslant\rangle, \{R^{\beta(f)}_f : f \in \mathscr{L}\}\rangle.$$

3. The canonical $\mathscr{L}^\beta$-general frame of $\langle \boldsymbol{A}, \leqslant\rangle$ is

$$\langle \boldsymbol{A}, \leqslant\rangle^g_+ \coloneqq \langle \langle \boldsymbol{A}, \leqslant\rangle_+, \lambda[A]\rangle$$

where $\lambda \colon \langle \boldsymbol{A}, \leqslant\rangle \to (\langle \boldsymbol{A}, \leqslant\rangle_+)^+$ is the embedding

$$\lambda(a) \coloneqq \{w \in W : a \in w\}, \text{ for all } a \in A.$$

- The classes of distinguished ordered models and relational models of $\vdash$ are inter-translatable as follows:

Duality Theorem

Let $\vdash$ be a logic, let $\mathscr{L}$ be an ordered language over $\mathscr{L}_\vdash$, and let β be a labeling map. The following maps are well defined:

$$(\cdot)^g_+ \colon \mathsf{Alg}^\leqslant_\mathscr{L}(\vdash) \longleftrightarrow \mathsf{Rel}^\beta_\mathscr{L}(\vdash) \colon (\cdot)^+$$

Moreover, $\mathsf{Alg}^\leqslant_\mathscr{L}(\vdash)$ is the class of complex algebras of $\mathsf{Rel}^\beta_\mathscr{L}(\vdash)$.

Corollary (Completeness)

Let $\vdash$ be an $\mathscr{L}$-monotone logic. Then $\vdash$ and $\vdash^\partial_\mathscr{L}$ are, respectively, the local and the co-local consequences of $\mathsf{Rel}^\beta_\mathscr{L}(\vdash)$ for every β.

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant\rangle \in \mathsf{Alg}^\leqslant_\mathscr{L}(\vdash^\leqslant_\mathsf{K})$ we have:

1. $\boldsymbol{A} \in \mathsf{K}$ and $\leqslant$ is the lattice order of $\boldsymbol{A}$.
2. $\mathsf{Pol}_\mathscr{L}\langle \boldsymbol{A}, \leqslant\rangle = \langle W, J, R\rangle$ is s.t.

$$W = \text{ lattice filters and } J = \text{ lattice ideals}.$$

Moreover, $(\langle \boldsymbol{A}, \leqslant\rangle_+)^+$ is the canonical extension of $\langle \boldsymbol{A}, \leqslant\rangle$.

Implicative fragment of IPC

For all $\langle \boldsymbol{A}, \leqslant\rangle \in \mathsf{Alg}^\leqslant_\mathscr{L}(\mathbf{IPC}_\to)$ we have:

1. $\langle \boldsymbol{A}, \leqslant\rangle$ is a Hilbert algebra equipped with the Hilbert-order.
2. $\mathsf{Pol}_\mathscr{L}\langle \boldsymbol{A}, \leqslant\rangle = \langle W, J, R\rangle$ is s.t.

$$W = \text{ implicative filters and } J = \text{ downsets}.$$

Moreover, $(\langle \boldsymbol{A}, \leqslant\rangle_+)^+$ is intrinsically a Heyting algebra.

Intensional fragment of $\mathbf{FL}_e^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_e^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \leqslant \rangle$-subreduct of a commutative FL-algebra.
2. $\mathsf{Pol}_{\mathscr{L}} \langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{upsets and } J = \text{downsets}.$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_+)^+$ is intrinsically a commutative FL-algebra.

Intensional fragment of $\mathbf{R}^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InR}^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \neg, \leqslant \rangle$-subreduct of a De Morgan monoid.
2. $\mathsf{Pol}_{\mathscr{L}} \langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{intensional filters and } J = \text{intensional ideals}.$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_+)^+$ is intrinsically a De Morgan monoid.

Frames monotone logics ordered algebras logic-based dualities and completions

A sample of what comes next...

- Substructural logics with weakening can be viewed as global consequences in this spirit, e.g. Łukasiewicz is the global version of the logic preserving degrees of truth of MV-algebras.
- One can give a relational semantics for every logic, inspired by the Routley-Meyer semantics for Relevance Logic.
- We can delete co-worlds from frames in nice cases, e.g. distributive substructural and modal logics.
- This approach suggests a semantic-based of expanding every local consequence to the first-order lever with quantifiers and identity, which is axiomatized very transparently by means of meta-rules.
- This yields a complete alternative relational semantics for all first-order modal and superintuitionistic logics.

...thank you for coming!