Unifying the Leibniz and Maltsev hierarchies

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Deductive systems

Definition

A logic is a consequence relation $\vdash$ on the set of formulas of an algebraic language built up with an infinite set of variables s.t.

$$\text{if } \Gamma \vdash \varphi \text{, then } \sigma(\Gamma) \vdash \sigma(\varphi).$$

Let $\vdash$ be a logic, $\boldsymbol{A}$ an algebra and $F \subseteq \boldsymbol{A}$.

1. The Leibniz congruence $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the largest congruence θ of $\boldsymbol{A}$ s.t. F is a union of blocks of θ.
2. The Suszko congruence is
$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\vdash}F := \bigcap\{\boldsymbol{\Omega}^{\boldsymbol{A}}G : G \in \mathcal{F}i_{\vdash}\boldsymbol{A} \text{ and } F \subseteq G\}.$$
3. The Suszko models of $\vdash$ are
$$\text{Mod}^{\text{Su}}(\vdash) := \{\langle \boldsymbol{A}, F\rangle : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\vdash}F = \text{Id}_{\boldsymbol{A}} \text{ and } F \in \mathcal{F}i_{\vdash}\boldsymbol{A}\}.$$

Interpretations between deductive systems

Definition

An interpretation $\boldsymbol{\tau}$ of $\vdash$ into $\vdash'$ is a map that associates every basic n-ary connective $f(x_1, \dots, x_n)$ of $\vdash$ to a term $\varphi(x_1, \dots, x_n)$ of $\vdash'$ in such a way that

$$\text{if } \langle \boldsymbol{A}, F\rangle \in \text{Mod}^{\text{Su}}(\vdash'), \text{ then } \langle \boldsymbol{A}^{\boldsymbol{\tau}}, F\rangle \in \text{Mod}^{\text{Su}}(\vdash)$$

where $\boldsymbol{A}^{\boldsymbol{\tau}} := \langle A, \{\boldsymbol{\tau}(f) : f \text{ is a connective of } \vdash\}\rangle$.

Examples:

- The identity is an interpretation of $\mathcal{IPC}$ in $\mathcal{CPC}$.
- The identity is an interpretation of $\mathcal{CPC}_{\wedge\vee}$ in $\mathcal{CPC}$.

The poset of logics

Definition

We define a pre-order between logics as follows:

$$\vdash \leq \vdash' \Longleftrightarrow \text{there is an interpretation of } \vdash \text{ into } \vdash'.$$

Then we set

$$[\![\vdash]\!] := \{\vdash' : \vdash' \text{ is a logic equi-interpretable with } \vdash\}.$$

Let **Log** be the poset, whose elements are the classes $[\![\vdash]\!]$.

Theorem

Log is a complete meet-semilattice, but it is not a join-semilattice. Moreover, **Log** has no minimum element, it has a maximum and a coatom (that under Vopěnka's Principle is unique).

Term-equivalence and compatible expansions

Definition

Let $\vdash$ and $\vdash'$ be two logics.

1. $\vdash$ and $\vdash'$ are term-equivalent if there are translations $\boldsymbol{\tau}$ of $\vdash$ into $\vdash'$ and $\boldsymbol{\rho}$ in the other direction such that

$$\langle \boldsymbol{A}, F\rangle = \langle \boldsymbol{A}^{\boldsymbol{\tau}\boldsymbol{\rho}}, F\rangle \text{ and } \langle \boldsymbol{B}, G\rangle = \langle \boldsymbol{B}^{\boldsymbol{\rho}\boldsymbol{\tau}}, G\rangle$$

for every $\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}^{\mathrm{Su}}(\vdash')$ and $\langle \boldsymbol{B}, G\rangle \in \mathrm{Mod}^{\mathrm{Su}}(\vdash)$.

2. $\vdash'$ is a compatible expansion of $\vdash$ if the identity is a translation of $\vdash$ into $\vdash'$.

Remark

$\vdash \leq \vdash'$ iff $\vdash'$ is term-equivalent to a compatible expansion of $\vdash$.

Taylorian products of algebras

- Let $\{L_i : i \in I\}$ be a set algebraic languages.
- We define a new language $\otimes_{i\in I} L_i$ by considering as n-ary operations symbols the sequences

$$\langle t_i(x_1, \dots, x_n) : i \in I\rangle$$

where t_i is an n-ary term of L_i in the variables $x_1, \dots, x_n$.

Definition

Let $\{\boldsymbol{A}_i : i \in I\}$ be a set of algebras respectively of language L_i. The Taylorian product of this family is the algebra $\otimes_{i\in I} \boldsymbol{A}_i$ of type $\otimes_{i\in I} L_i$ with universe $\prod_{i\in I} A_i$ and operations defined as

$$\langle t_i : i \in I\rangle(\vec{a}_1, \dots, \vec{a}_n) \coloneqq \langle t_i^{\boldsymbol{A}_i}((\vec{a}_1(i), \dots, \vec{a}_n(i)) : i \in I\rangle.$$

Taylorian products of logics

Definition

Let $\{\vdash_i : i \in I\}$ be a set of logics each of which is formulated in κ_i variables. The Taylorian product of this family is the logic $\otimes_{i\in I} \vdash_i$ formulated in $|I| \cup \bigcup_{i\in I} \kappa_i$ variables induced by the class of matrices

$$\{\langle \bigotimes_{i\in I} \boldsymbol{A}_i, \prod_{i\in I} F_i\rangle : \langle \boldsymbol{A}_i, F_i\rangle \in \mathrm{Mod}^{\mathrm{Su}}(\vdash_i) \text{ for every } i \in I\}.$$

- Observe that Taylorian products of huge families of logics are formulated in huge sets of variables.

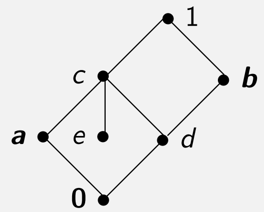
Corollary

Log has infima of families indexed by sets. More precisely,

$$[\![\otimes_{i\in I} \vdash_i]\!] = \bigwedge_{i\in I} [\![\vdash_i]\!].$$

Log has not finite suprema (anecdotally...)

- Let $\boldsymbol{A} = \langle A, \vee, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{0}\rangle$ be the join-semilattice, expanded with constants, depicted below:



- Let $\vdash_\vee$ be the logic determined by the matrix $\langle \boldsymbol{A}, \{1\}\rangle$.
- Let $\vdash_\neg$ be the negation fragment of $\mathcal{CPC}$.
- The supremum of $\vdash_\vee$ and $\vdash_\neg$ in **Log** does not exist.

Theorem

The subposet of **Log** consisting of all equivalential logics is a non-modular complete lattice.

Leibniz conditions and Leibniz classes

Definition

1. A strong Leibniz condition Φ is a logic $\vdash_\Phi$.
2. A logic $\vdash$ satisfies Φ if $\vdash_\Phi \leq \vdash$.
3. A Leibniz condition Φ is a sequence of logics

$$\Psi \coloneqq \{\vdash_\alpha : \alpha \in \mathrm{ORD}\}$$

   such that

$$\text{if } \alpha \leq \beta, \text{ then } \vdash_\beta \leq \vdash_\alpha .$$

4. A logic $\vdash$ satisfies Ψ if $\vdash_\alpha \leq \vdash$ for some $\alpha \in \mathrm{ORD}$.
5. $\mathrm{Mod}(\Psi)$ is the class of logics satisfying Ψ.
6. A class of logics K is a (strong) Leibniz class if $\mathsf{K} = \mathrm{Mod}(\Psi)$ for some (strong) Leibniz condition Ψ.

Equivalentiality is a Leibniz condition

Example

- For every $\alpha \in \mathrm{ORD}$, consider the language $\{\multimap_\beta : \beta \leq \alpha\}$.
- Define

$$\Delta_\alpha(x, y) \coloneqq \{x \multimap_\beta y : \beta \leq \alpha\}.$$

- Let $\vdash_\alpha$ be the logic defined by the rules

$$\emptyset \rhd \Delta_\alpha(x, x)$$
$$x, \Delta_\alpha(x, y) \rhd y$$
$$\Delta_\alpha(x_1, y_1) \cup \Delta_\alpha(x_2, y_2) \rhd \Delta_\alpha(x_1 \multimap_\beta x_2, y_1 \multimap_\beta y_2).$$

- Consider the Leibniz condition

$$\Psi \coloneqq \{\vdash_\alpha : \alpha \in \mathrm{ORD}\}.$$

- $\mathrm{Mod}(\Psi)$ is the class of equivalential logics with theorems.

Semantic description of Leibniz classes

- Leibniz classes can be characterized as follows:

Theorem

Let K be a class of logics. TFAE:

1. K is a Leibniz class.
2. K is closed under term-equivalence, compatible expansions and Taylorian products indexed by sets.
3. There is a complete filter F of **Log** such that

$$\mathsf{K} = \{\vdash : [\![\vdash]\!] \in F\}.$$

- In this picture,

  Strong Leibniz classes = principal filters of **Log**.

The Leibniz hierarchy revisited

- We propose to adopt the following:

Convention

Leibniz hierarchy = poset of Leibniz classes of logics.

Some motivations:

- This perspective subsumes Maltsev conditions.
- Leibniz classes captures the interaction between syntactic conditions and the behaviour of the Leibniz operator.
- Leibniz classes are not too general. They do not include metalogical properties and the Frege hierarchy.

Indecomposable Leibniz classes

- This order-theoretic perspective allows to single out the fundamental bricks of the Leibniz hierarchy:

Definition

A Leibniz class K is indecomposable if it is meet-irreducible among Leibniz classes.

- The class of logics with theorems is idecomposable.

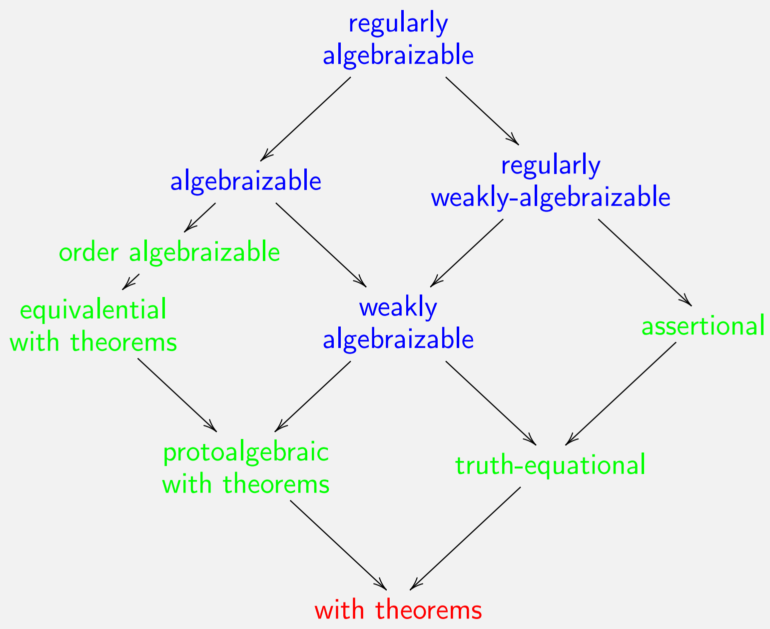
Hopeless Lemma

Let K be a Leibniz class such that:

1. The members of K have theorems.
2. There is a logic with theorems outside K.

Then K is decomposable.

- Almost all reasonable Leibniz classes are decomposable.



Indecomposability among logics with theorems

- We use the following principle independent from GNB.

Vopěnka's Principle

Every prevariety is a generalized quasi-variety.

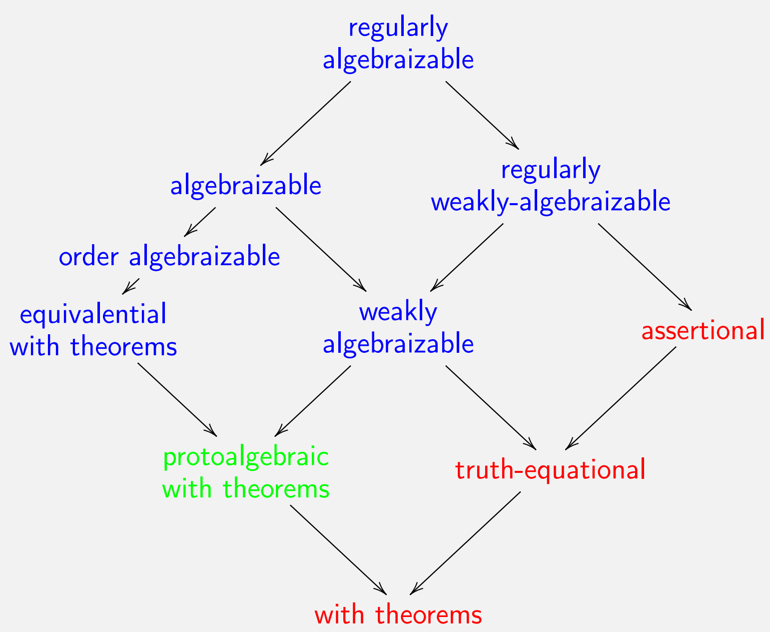
Theorem

Under Vopěnka's Principle, the classes of truth equational and assertional logics are indecomposable among logics with theorems.

Theorem

The classes of order-algebraizable and equivalential logics with theorems are decomposable among logics with theorems.

- It is open whether the class of protoalgebraic logics is decomposable among logics with theorems.



Finitely presentable deductive systems

Definition

1. A logic is finitely presentable if it is finitary, axiomatizable by a finite set of finite rules, and formulated in a finite language.
2. A finitely presentable Leibniz condition is a sequence of finitely presentable and finitely equivalential logics

$$\Psi = \{\vdash_n : n \in \omega\}$$

such that if $n \leq m$, then $\vdash_m \leq \vdash_n$.
3. A class of logics K is a finitely presentable Leibniz class if $\mathsf{K} = \mathrm{Mod}(\Psi)$ for some fin. pres. Leibniz condition Ψ.

Convention

finite companion of the Leibniz hierarchy = poset of finitely presentable Leibniz classes.

The Maltsev hierarchy

Definition

1. A Maltsev condition is a sequence of finitely presentable varieties

$$\Psi = \{\mathsf{V}_n : n \in \omega\}$$

such that

$$\text{if } n \leq m, \text{ then } \mathsf{V}_m \leq \mathsf{V}_n.$$

2. A class of varieties K is a Maltsev class if $\mathsf{K} = \mathrm{Mod}(\Psi)$ for some Maltsev condition Ψ.

Theorem

A class of varieties K is a Maltsev class iff there is a fin. pres. Leibniz class M of 2-deductive systems such that

$$\mathsf{K} = \{\mathsf{V} : \mathsf{V} \text{ is a variety and } \vDash_{\mathsf{V}} \in \mathsf{M}\}.$$

Future directions

- Can we have a suitable version of Taylor terms for logic?
- Can we prove that non-trivial Leibniz conditions implies the validity of some non-trivial (quasi)-equation involving the Leibniz operator?
- Are protoalgebraic logics indecomposable/prime?

Finally...

...thank you for coming!