The computational complexity of the Leibniz hierarchy

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Logics

Definition

A logic $\vdash$ is a consequence relation over the set of formulas Fm of an algebraic language,which is substitution invariant in the sense that

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma(\Gamma) \vdash \sigma(\varphi)$$

for all substitutions $\sigma \colon \boldsymbol{Fm} \to \boldsymbol{Fm}$.

- Logics are consequence relations (as opposed to sets of valid formulas).
- Example: **IPC** is the logic defined as follows:

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \text{for every Heyting algebra } \boldsymbol{A} \text{ and } \vec{a} \in A, \text{ if } \Gamma^{\boldsymbol{A}}(\vec{a}) = 1, \text{ then } \varphi^{\boldsymbol{A}}(\vec{a}) = 1.$$

Relative equational consequence

Definition

Let K be a class of similar algebras. Given a set of equations $\Theta \cup \{\varphi \approx \psi\}$, we define

$$\Theta \vDash_{\mathsf{K}} \varphi \approx \psi \iff \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and } \vec{a} \in A, \text{ if } \epsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}) \text{ for all } \epsilon \approx \delta \in \Theta, \text{ then } \varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}).$$

The relation $\vDash_{\mathsf{K}}$ is the equational consequence relative to K.

- Example: If K is the variety of Heyting algebras, then

$$\varphi \approx 1, \varphi \to \psi \approx 1 \vDash_{\mathsf{K}} \psi \approx 1.$$

Algebraizable logics

Example: Consider

$$\mathbf{IPC} = \text{intuitionistic propositional logic}$$
$$\mathsf{HA} = \text{variety of Heyting algebras}$$

- Pick the translations between formulas and equations:

$$\varphi \longmapsto \varphi \approx 1$$
$$\alpha \approx \beta \longmapsto \{\alpha \leftrightarrow \beta\}.$$

- These translations allow to equi-interpret $\vdash_{\mathbf{IPC}}$ and $\vDash_{\mathsf{HA}}$:

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \varphi \approx 1$$
$$\Theta \vDash_{\mathsf{HA}} \varphi \approx \psi \iff \{\alpha \leftrightarrow \beta : \alpha \approx \beta \in \Theta\} \vdash_{\mathbf{IPC}} \{\varphi \leftrightarrow \psi\}.$$

- Moreover, the translations are one inverse to the other:

$$\varphi \approx \psi \mathrel{=\!\!|\!\vDash_{\mathsf{HA}}} \varphi \leftrightarrow \psi \approx 1 \text{ and } \varphi \dashv\vdash_{\mathbf{IPC}} \varphi \leftrightarrow 1.$$

- Hence $\vdash_{\mathbf{IPC}}$ and $\vDash_{\mathsf{HA}}$ are essentially the same.

- Intuitive idea: a logic $\vdash$ is algebraizable when it can be essentially identified with a relative equational consequence $\vDash_{\mathsf{K}}$.

Definition

A logic $\vdash$ is algebraizable when there exists:

1. A class of algebras K (of the same type as $\vdash$);
2. A set of equations $\boldsymbol{\tau}(x)$ in one variable x;
3. A set of formulas $\boldsymbol{\rho}(x, y)$ in two variables x and y

such that $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ equi-interpret $\vdash$ and $\vDash_{\mathsf{K}}$:

$$\Gamma \vdash \varphi \Longleftrightarrow \boldsymbol{\tau}(\Gamma) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi)$$
$$\Theta \vDash_{\mathsf{K}} \varphi \approx \psi \Longleftrightarrow \boldsymbol{\rho}(\Theta) \vdash \boldsymbol{\rho}(\varphi, \psi)$$

and the two interpretations are one inverse to the other:

$$\varphi \approx \psi \dashv\vDash_{\mathsf{K}} \boldsymbol{\tau\rho}(\varphi, \psi) \text{ and } \varphi \dashv\vdash \boldsymbol{\rho\tau}(\varphi).$$

Algebraization Problem

- We study the computational aspects of the following problem:

Algebraization Problem

Given a logic $\vdash$, determine whether $\vdash$ is algebraizable or not.

- Logic can be presented (at least) in two ways:

syntactically = by means of Hilbert calculi
semantically = by means of collections of logical matrices.

Theorem (M. 2015)

The Algebraization Problem for logics presented by finite consistent Hilbert calculi is undecidable.

Semantic Algebraization Problem

Given a finite reduced logical matrix $\langle \boldsymbol{A}, F \rangle$ of finite type, determine whether its induced logic is algebraizable or not.

- There is an easy decision procedure for this problem because:

Theorem

Let $\langle \boldsymbol{A}, F \rangle$ be a finite reduced matrix and $\vdash$ its induced logic. $\vdash$ is algebraizable iff there is a finite set of equations $\boldsymbol{\tau}(x)$ and a finite set of formulas $\boldsymbol{\rho}(x, y)$ such that

$$a = b \Longleftrightarrow \boldsymbol{\rho}(a, b) \subseteq F$$
$$a \in F \Longleftrightarrow \boldsymbol{A} \vDash \boldsymbol{\tau}(a).$$

- Since finitely generated free algebras over $\mathbb{V}(\boldsymbol{A})$ are finite, we can just check the existence of the sets $\boldsymbol{\rho}(x, y)$ and $\boldsymbol{\tau}(x)$.
- Hence the Semantic Algebraization Problem is in **EXPTIME**.

A useful **EXPTIME**-complete problem

- We want to prove that the Semantic Algebraization Problem is complete for **EXPTIME**.
- We need to construct a polynomial-time reduction to such a complete problem.

The Problem Gen-Clo

Given a finite algebra $\boldsymbol{A}$ of finite type and a function $h \colon A^n \to A$, determine whether h belongs to the clone of $\boldsymbol{A}$ or not.

- Gen-Clo$^1_3$ is the same problem, restricted to the case where h is unary and the operations of $\boldsymbol{A}$ are at most ternary.

Theorem (Bergman, Juedes, and Slutzki)

Both Gen-Clo and Gen-Clo$^1_3$ are complete for **EXPTIME**.

- We will construct a polynomial reduction of Gen-Clo$^1_3$ to the Semantic Algebraization Problem.

Reduction

Pick an input $\langle \boldsymbol{A}, h\rangle$ for Gen-Clo$^1_3$. We define a new algebra $\boldsymbol{A}^\flat$ as:

- The universe of $\boldsymbol{A}^\flat$ is eight disjoint copies $A_1, \dots, A_8$ of A: An arbitrary finite set of elements in $A^\flat$ can be denote as

$$\{a_1^{m_1}, \dots, a_n^{m_n}\}$$

  for some $a_1, \dots, a_n \in A$ and $m_1, \dots, m_n \leq 8$.
- The basic operation of $\boldsymbol{A}^\flat$ are as follows:

1. For every n-ary basic f of $\boldsymbol{A}$, we add an operation $\hat{f}$ on $\boldsymbol{A}^\flat$ as

$$\hat{f}(a_1^{m_1} \dots, a_n^{m_n}) := f^{\boldsymbol{A}}(a_1, \dots, a_n)^5.$$

2. Then we add to $\boldsymbol{A}^\flat$ the following operation $\Box$:

$$\Box(a^m) := \begin{cases} a^m & \text{if } m = 1 \text{ or } m = 2 \\ a^{m-1} & \text{if } m \text{ is even and } m \geq 3 \\ a^{m+1} & \text{if } m \text{ is odd and } m \geq 3. \end{cases}$$

3. Finally we add to $\boldsymbol{A}^\flat$ the following operation $\heartsuit$:

$$\heartsuit(a^m, b^n, c^k) := \begin{cases} a^1 & \text{if } a^m = c^k \text{ and } h(a)^5 = b^n \\ & \text{and } m \in \{1,3,4\} \\ a^2 & \text{if } a^m = c^k \\ & \text{and } h(a)^5 = b^n \text{ and } m \in \{2,5,6,7,8\} \\ a^4 & \text{if } m, k \in \{1,3,4\} \\ & \text{and (either } a^m \neq c^k \text{ or } h(a)^5 \neq b^n) \\ a^7 & \text{if } \{m,k\} \cap \{2,5,6,7,8\} \neq \emptyset \text{ and} \\ & \text{(either } a^m \neq c^k \text{ or } h(a)^5 \neq b^n). \end{cases}$$

- Then define $F \subseteq A^\flat$ as follows: $F := A_1 \cup A_2$.
- The pair $\langle \boldsymbol{A}^\flat, F\rangle$ is a finite reduced matrix of finite type, and thus an input for the Semantic Algebraization Problem!

Remark

Since the arity of the operations of $\boldsymbol{A}$ is bounded by 3, the matrix $\langle \boldsymbol{A}^\flat, F\rangle$ can be constructed in polynomial time.

Hardness result

Theorem

There is a polynomial-time reduction of Gen-Clo$^1_3$ to the Semantic Algebraization Problem, i.e. given a finite algebra $\boldsymbol{A}$ of finite type, whose basic operations are at most ternary, and a unary map $h\colon A \to A$, TFAE:

1. h belongs to the clone of $\boldsymbol{A}$.
2. The logic induced by the matrix $\langle \boldsymbol{A}^\flat, F\rangle$ is algebraizable.

Corollary

The Semantic Algebraization Problem is complete for **EXPTIME**.

- Variants of the construction $\boldsymbol{A} \longmapsto \langle \boldsymbol{A}^\flat, F\rangle$ can be used to show that

Theorem

The problem of determining whether the logic of a finite reduced matrix of finite type belongs to any of the following classes

- algebraizable logics
- protoalgebraic logics
- equivalential logics
- truth-equational logics
- order algebraizable logics,

is hard for **EXPTIME**.

- For all the above classes of logics (except the one of truth-equational logics), the problem is complete for **EXPTIME**.

Further questions

- A similar situation appears in the study of Malsetv conditions:

Theorem (Freese and Valeriote)

The problem of determining whether a finite algebra $\boldsymbol{A}$ of finite type generates a congruence distributive (resp. modular) variety is complete for **EXPTIME**.

- However, the above problems become tractable when $\boldsymbol{A}$ is idempotent, i.e when for every operation f of $\boldsymbol{A}$ and $a \in A$

$$f^{\boldsymbol{A}}(a, \dots, a) = a$$

Open Problem

Find tractability conditions for Semantic Algebraization Problem.

- Remark: idempotency will not work here, since no idempotent non-trivial matrix determines an algebraizable logic.

Finally...

...thank you for coming!