Varieties of positive interior algebras: structural completeness

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- Modal algebras form the algebraic semantics of normal modal logics. They are Boolean algebras with a unary operation $\Box$ s.t.

$$\Box 1 \approx 1 \text{ and } \Box(x \wedge y) \approx \Box x \wedge \Box y.$$

- Modal algebras can be presented also as BAs with a unary operation $\Diamond$. The two presentations produce term-equivalent varieties setting $\Box x := \neg\Diamond\neg x$ and $\Diamond x := \neg\Box\neg x$.
- Positive modal algebras are $\langle \wedge, \vee, \Box, \Diamond, 0, 1\rangle$-subreducts of modal algebras. Equivalently...

Definition

An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \Box, \Diamond, 0, 1\rangle$ is a positive modal algebra if $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice s.t. for all $a, b \in A$,

$$\Box(a \wedge b) = \Box a \wedge \Box b \quad \Diamond(a \vee b) = \Diamond a \vee \Diamond b$$
$$\Box a \wedge \Diamond b \leq \Diamond(a \wedge b) \quad \Box(a \vee b) \leq \Box a \vee \Diamond b.$$

- Positive modal algebras form a variety (a.k.a. equational class).

- We focus on two central varieties of (positive) modal algebras:

Recall that...

- The algebraic semantics of $\boldsymbol{K4}$ consists of K4-algebras, i.e. modal algebras validating $\Box x \leq \Box\Box x$ (equiv. $\Diamond\Diamond x \leq \Diamond x$).
- The algebraic semantics of $\boldsymbol{S4}$ consists of interior algebras, i.e. modal algebras validating $\Box\Box x \leq \Box x \leq x$ (equiv. $x \leq \Diamond x \leq \Diamond\Diamond x$).

Definition

Let $\boldsymbol{A}$ be a positive modal algebra.

1. $\boldsymbol{A}$ is a positive K4-algebra if it satisfies $\Box x \leq \Box\Box x$ and $\Diamond\Diamond x \leq \Diamond x$. We denote by PK4 the variety they form.
2. $\boldsymbol{A}$ is a positive interior algebra if it satisfies $\Box\Box x \leq \Box x \leq x$ and $x \leq \Diamond x \leq \Diamond\Diamond x$. We denote by PIA the variety they form.

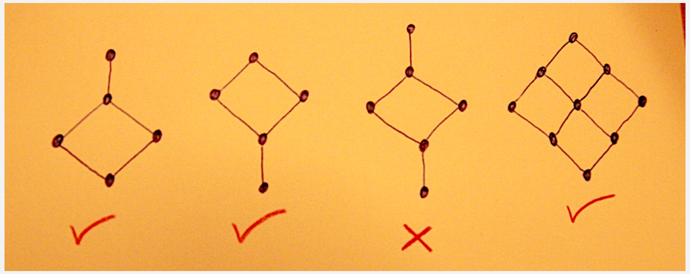
- Positive K4-algebras (resp. interior algebras) are subreducts of K4-algebras (resp. interior algebras).

- It is well the congruences of a modal algebra $\boldsymbol{A}$ are in one-to-one correspondence with its open filters.
- Fundamental application: This correspondence identifies subdirectly irreducible modal algebras as those which have a smallest open filter different from $\{1\}$.
- Problem: The correspondence between congruences and open filters is lost in the positive setting. A useful description of subdirectly irreducible positive modal algebras is unknown.
- A couple of facts are known, e.g. every finitely subdirectly irreducible positive interior algebra $\boldsymbol{A}$ is well-connected, i.e.

$$\text{if } \Box a \vee \Box b = 1, \text{ then } a = 1 \text{ or } b = 1$$
$$\text{if } \Diamond a \wedge \Diamond b = 0, \text{ then } a = 0 \text{ or } b = 0.$$

However, there are well-connected positive interior algebras, which are not finitely subdirectly irreducible.

- Some example of simple positive interior algebras:



- Obs: The positive interior algebras above contain a non-simple and non-trivial subalgebra (e.g. any four-element chain).

Corollary

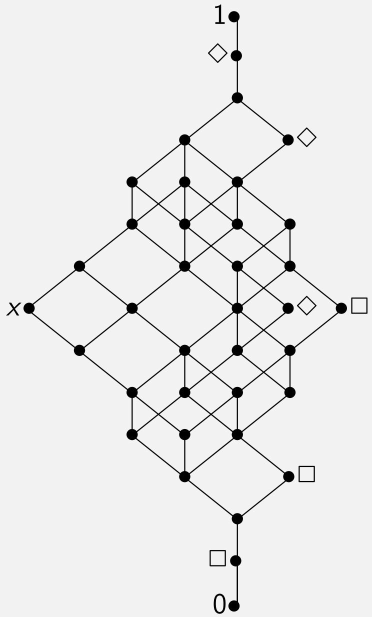
Positive interior algebras do not have the congruence extension property (CEP). Thus they do not have EDPC.

- This contrasts with the full signature case, since interior algebras have EDPC.

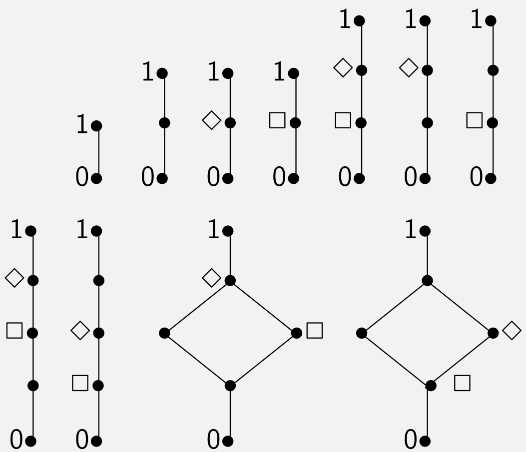
- The free one-generated algebra of PK4 is infinite: it contains an infinite descending chain

$$\{[\![\Box^n x]\!] : n \in \omega\}.$$

- The free two-generated positive interior algebra is infinite.
- While the free one-generated algebra of PIA is finite. This contrasts with the full-signature case.



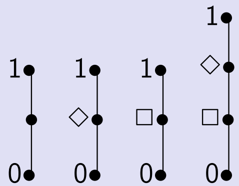
- From the knowledge of the structure of the free one-generated positive interior algebra we can derive all one-generated subdirectly irreducible positive interior algebras:



- Playing with these one-generated subdirectly irreducible algebras, we obtain a description of the bottom part of the subvariety lattice $\Lambda(\mathsf{PIA})$ of positive interior algebras.

Theorem

1. There is a unique minimal subvariety DL of PIA, term-equivalent to that of bounded distributive lattices.
2. The unique covers of DL in $\Lambda(\mathsf{PIA})$ are the ones generated by one of the following algebras $\boldsymbol{D}_3$, $\boldsymbol{C}_3^a$, $\boldsymbol{C}_3^b$ and $\boldsymbol{D}_4$:



3. If K is a subvariety of PIA such that $\mathsf{DL} \subsetneq \mathsf{K}$, then K includes one of the covers of DL.

- To climb higher (from bottom to top) in the lattice $\Lambda(\mathsf{PIA})$, we need the following:

Definition

Let K be a variety. A subdirectly irreducible $\boldsymbol{A} \in \mathsf{K}$ is a splitting algebra in K if there is a largest subvariety of K excluding $\boldsymbol{A}$.

Lemma (McKenzie)

If a congruence distributive variety is generated by its finite members, then its splitting algebras are finite.

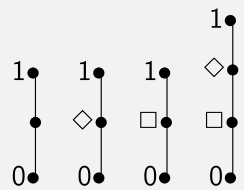
Corollary

Splitting algebras in PK4 and in PIA are finite.

- Problem: Which finite s.i. algebras are splitting in PK4 and in PIA? In the full-signature case the answer is all (by EDPC). In the positive case it is unknown.

Typical application: $\mathbb{V}(\boldsymbol{D}_4)$ has no join-irreducible cover in $\Lambda(\mathsf{PIA})$.

- Consider the positive interior algebras $\boldsymbol{D}_3$, $\boldsymbol{C}_3^a$, $\boldsymbol{C}_3^b$ and $\boldsymbol{D}_4$:



- $\boldsymbol{D}_3$ is splitting in PK4 with splitting identity

$$\Diamond x \wedge \Box\Diamond x \leq x \vee \Box x \vee \Diamond\Box x. \tag{1}$$

- $\boldsymbol{C}_3^a$ and $\boldsymbol{C}_3^b$ are splitting in PIA, respectively with splitting identities

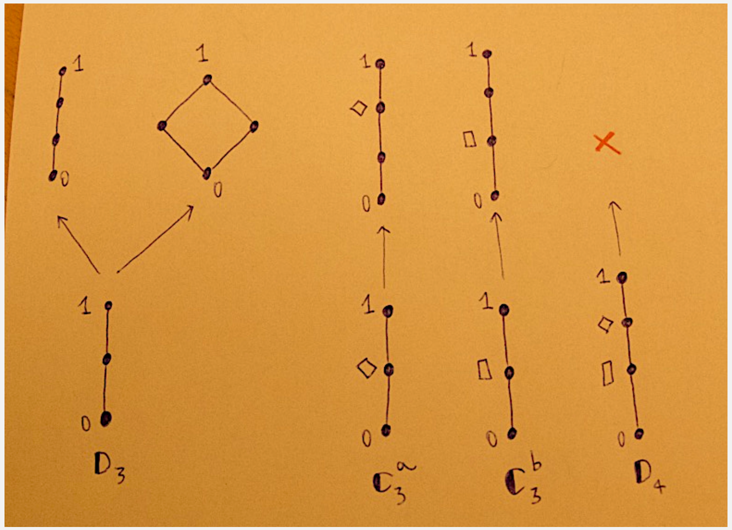
$$\Diamond\Box\Diamond x \approx \Diamond x \text{ and } \Box\Diamond\Box x \approx \Box x. \tag{2}$$

- $\mathbb{V}(\boldsymbol{D}_4)$ is axiomatized by

$$\Box\Diamond x \approx \Box x \text{ and } \Diamond\Box x = \Diamond x. \tag{3}$$

- Since (3) is a consequence of (1, 2), we are done.

- The next figure shows the join-irreducible covers of the varieties generated by $\boldsymbol{D}_3$, $\boldsymbol{C}_3^a$, $\boldsymbol{C}_3^b$ and $\boldsymbol{D}_4$ in $\Lambda(\mathsf{PIA})$:



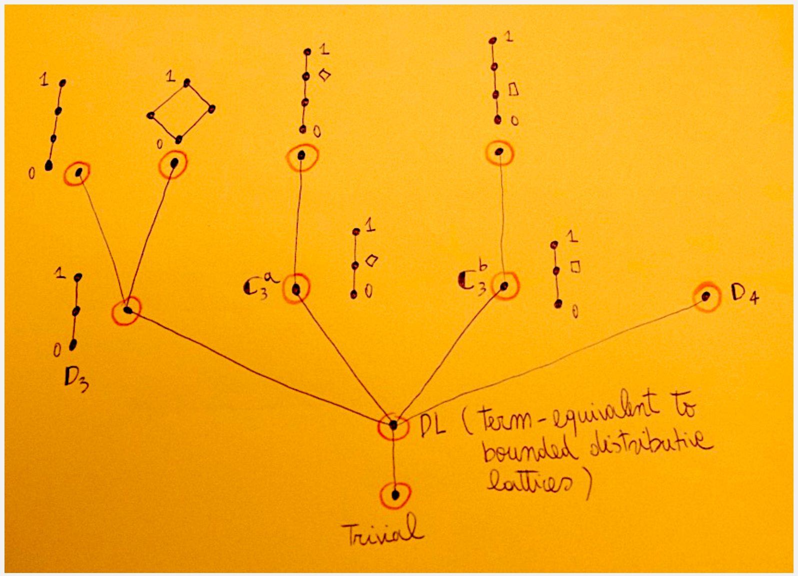


Figure: Join-irreducible varieties of depth ≤ 4 in $\Lambda(\mathsf{PIA})$.

Definition

Let K be a variety and consider a quasi-equation

$$\Phi := \varphi_1 \approx \psi_1 \& \ldots \& \varphi_n \approx \psi_n \to \varphi \approx \psi.$$

1. Φ is active in K when there is a substitution σ such that $\mathsf{K} \vDash \sigma\varphi_i \approx \sigma\psi_i$ for every $i \leq n$.
2. Φ is passive in K if it is not active in K.
3. Φ is admissible in K if for every substitution σ:

$$\text{if } \mathsf{K} \vDash \sigma\varphi_i \approx \sigma\psi_i \text{ for every } i \leq n, \text{ then } \mathsf{K} \vDash \sigma\varphi \approx \sigma\psi.$$

4. Φ is derivable in K if $\mathsf{K} \vDash \Phi$.

- Observe that passive quasi-equations are vacuously admissible.

Definition

Let K be a variety.

1. K is actively structurally complete (ASC) if every active admissible quasi-equation is derivable.
2. K is passively structurally complete (PSC) if every passive (admissible) quasi-equation is derivable.
3. K is structurally complete (SC) if every admissible quasi-equation is derivable.
4. K is hereditarily structurally complete (SHC) if every subvariety of K is SC.

- Clearly, (ASC) + (PSC) = (SC), and (SHC) implies (SC).
- We aim to understand the various structural completeness in subvarieties of PK4.

Theorem

Let K be a SC variety of positive modal algebras. Either $\mathsf{K} = \mathbb{V}(\boldsymbol{B}_2)$ or there are $n, m \geq 1$ such that

$$\mathsf{K} \vDash \Box x \wedge \cdots \wedge \Box^n x \leq x \text{ and } \mathsf{K} \vDash x \leq \Diamond x \vee \cdots \vee \Diamond^m x.$$

Corollary

Let K be a SC variety of positive K4-algebras. Either $\mathsf{K} = \mathbb{V}(\boldsymbol{B}_2)$ or $\mathsf{K} \subseteq \mathsf{PIA}$.

- Hence, while trying to spot SC subvarieties of PK4, we can restrict to subvarieties of PIA.

Theorem

Let K be a non-trivial variety of positive interior algebras. TFAE:

1. K is actively structurally complete.
2. K excludes $\boldsymbol{D}_3$, $\boldsymbol{C}_3^a$ and $\boldsymbol{C}_3^b$.
3. $\mathsf{K} = \mathsf{DL}$ or $\mathsf{K} = \mathbb{V}(\boldsymbol{D}_4)$.
4. K is hereditarily structurally complete.
5. K is structurally complete.
6. K satisfies the equations $\Box\Diamond x \approx \Box x$ and $\Diamond\Box x \approx \Diamond x$.

Corollary

Let K be a non-trivial variety of positive K4-algebras. TFAE:

1. K is structurally complete.
2. $\mathsf{K} = \mathbb{V}(\boldsymbol{B}_2)$ or $\mathsf{K} = \mathsf{DL}$ or $\mathsf{K} = \mathbb{V}(\boldsymbol{D}_4)$.
3. K is hereditarily structurally complete.

- There are only 3 non-trivial SC subvarieties of PK4.

- Problem: What about ASC and PSC subvarieties of PK4?
- For ASC the answer is unknown.
- Let $\boldsymbol{C}_2$ be the two-element positive interior algebra. For PSC we have:

Theorem

Let K be a non-trivial variety of positive K4-algebras. TAFE:

1. K is passively structurally complete.
2. Either $\mathsf{K} = \mathbb{V}(\boldsymbol{B}_2)$ or ($\boldsymbol{Fm}_{\mathsf{K}}(0) = \boldsymbol{C}_2$ and $\boldsymbol{C}_2$ is the unique simple member of K).
3. Either $\mathsf{K} = \mathbb{V}(\boldsymbol{B}_2)$ or ($\boldsymbol{Fm}_{\mathsf{K}}(0) = \boldsymbol{C}_2$ and K excludes $\boldsymbol{D}_3$).
4. Either $\mathsf{K} = \mathbb{V}(\boldsymbol{B}_2)$ or

$$\mathsf{K} \vDash \Diamond 1 \approx 1, \Box 0 \approx 0, \Diamond x \wedge \Box\Diamond x \leq x \vee \Box x \vee \Diamond\Box x.$$

- There are infinitely many PSC subvarieties of PIA.

Finally...

...thank you for coming!