A correspondence between logical translations and semantic transformations

Tommaso Moraschini

Institute of Computer Science of the Czech Academy of Sciences

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- Classical logic **CPC** is axiomatized the the following axioms

$$x \to (y \to x)$$
$$(x \to (y \to z)) \to ((x \to y) \to (x \to z))$$
$$(x \wedge y) \to x$$
$$(x \wedge y) \to y$$
$$x \to (y \to (x \wedge y))$$
$$x \to (x \vee y)$$
$$y \to (x \vee y)$$
$$(x \to y) \to ((z \to y) \to ((x \vee z) \to y))$$
$$0 \to x$$
$$\neg\neg x \to x$$

  and the rule of Modus Ponens

$$x, x \to y \vdash y.$$

- Classical logic is the logic of Boolean reasoning.

- At the beginning of the last century non-classical logics arose for mathematical, philosophical, and linguistic motivations.
- Intuitionistic logic **IPC**, motivated by constructivism in mathematics, is obtained removing $\neg\neg x \to x$ from the axiomatization of classical logic:

$$x \to (y \to x)$$
$$(x \to (y \to z)) \to ((x \to y) \to (x \to z))$$
$$(x \wedge y) \to x$$
$$(x \wedge y) \to y$$
$$x \to (y \to (x \wedge y))$$
$$x \to (x \vee y)$$
$$y \to (x \vee y)$$
$$(x \to y) \to ((z \to y) \to ((x \vee z) \to y))$$
$$0 \to x$$

  and the rule of Modus Ponens

$$x, x \to y \vdash y.$$

- Modal logic **K** expands the language of classical logic with a unary connective $\Box$, whose intended to meaning is:

$$\Box\varphi \equiv \text{ it is necessary that } \varphi.$$

- **K** is axiomatized by the axioms and rules of classical logic plus the axiom

$$\Box(x \to y) \to (\Box x \to \Box y)$$

  and the Necessitation rule

$$x \vdash \Box x.$$

- Recap: Several logic flourished in the early 20th century, e.g. intuitionistic logic **IPC**, modal logic **K**, their axiomatic extensions etc. (and of course classical logic **CPC**).
- Our understanding of this increasing variety of logics depends on the possibility of drawing comparisons between them, typically though logical translations.

Kolmogorov's translation of **CPC** into **IPC**.

- In 1925 Kolmogorov defined a double-negation translation of the formulas φ of **CPC** into formulas φ^{K} of **IPC** as follows:

$$\begin{aligned} x^{\mathsf{K}} &:= \neg\neg x \text{ for variables } x \\ 0^{\mathsf{K}} &:= 0 \\ (\alpha \wedge \beta)^{\mathsf{K}} &:= \neg\neg(\alpha^{\mathsf{K}} \wedge \beta^{\mathsf{K}}) \\ (\alpha \vee \beta)^{\mathsf{K}} &:= \neg\neg(\alpha^{\mathsf{K}} \vee \beta^{\mathsf{K}}) \\ (\alpha \to \beta)^{\mathsf{K}} &:= \neg\neg(\alpha^{\mathsf{K}} \to \beta^{\mathsf{K}}) \\ (\neg\alpha)^{\mathsf{K}} &:= \neg(\alpha^{\mathsf{K}}) \end{aligned}$$

  where in **IPC** we define $\neg\varphi := \varphi \to 0$.
- Kolmogorov's translation is logically faithful in the sense that for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \Gamma^{\mathsf{K}} \vdash_{\mathbf{CPC}} \varphi^{\mathsf{K}}.$$

Gödel's translation of **IPC** into **S4**.

- One of the most important axiomatic extension of the modal logic **K** is the system **S4** obtained adding the axioms

$$\begin{aligned} \Box x \to x &\equiv \text{ if } \varphi \text{ is necessary, then it hlolds} \\ \Box x \to \Box\Box x &\equiv \text{ if } \varphi \text{ is necessary, then it is necessarily so.} \end{aligned}$$

- In 1933 Gödel defined a translation of **IPC** into **S4** as follows:

$$\begin{aligned} x^{\mathsf{G}} &:= \neg\Box x \text{ for variables } x \\ 0^{\mathsf{G}} &:= 0 \\ (\alpha \wedge \beta)^{\mathsf{G}} &:= \alpha^{\mathsf{G}} \wedge \beta^{\mathsf{G}} \\ (\alpha \vee \beta)^{\mathsf{G}} &:= \alpha^{\mathsf{G}} \vee \beta^{\mathsf{G}} \\ (\alpha \to \beta)^{\mathsf{G}} &:= \Box(\alpha^{\mathsf{G}} \to \beta^{\mathsf{G}}). \end{aligned}$$

- Gödel's translation is logically faithful:

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \Gamma^{\mathsf{G}} \vdash_{\mathbf{S4}} \varphi^{\mathsf{G}}.$$

Semantic dual of Kolmogorov's translation

- The algebraic semantics of **CPC** are Boolean algebras, i.e. algebras $\boldsymbol{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice such that

$$a \vee \neg a = 1 \text{ and } a \wedge \neg a = 0, \text{ for all } a \in A.$$

- The algebraic semantic of **IPC** are Heyting algebras, i.e. algebras $\boldsymbol{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded (distributive) lattice and

$$a \wedge b \leq c \iff a \leq b \to c, \text{ for all } a, b, c \in A.$$

- Kolmogorov's translation of **IPC** into **CPC** has a semantic dual, i.e. the transformation

$$\begin{aligned} &\mathbf{Reg} \colon \mathsf{HA} \to \mathsf{BA} \\ &\boldsymbol{A} \mapsto \mathbf{Reg}(\boldsymbol{A}) := \langle \mathrm{Reg}(\boldsymbol{A}), \wedge, \sqcup, \neg, 0, 1\rangle \end{aligned}$$

  where $\mathrm{Reg}(\boldsymbol{A}) = \{a \in A : \neg\neg a = 1\}$ and $a \sqcup b := \neg\neg(a \vee b)$.

Semantic dual of Gödel's translation

- The algebraic semantics of **S4** are interior algebras, i.e. algebras $\boldsymbol{A} = \langle A, \wedge, \neg, \vee, \Box, 0, 1\rangle$ such that $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\Box$ is an interior operator such that

$$\Box(a \wedge b) = \Box a \wedge \Box b \text{ and } \Box 1 = 1, \text{ for all } a, b \in A.$$

- Gödel's translation of **IPC** into **S4** has a semantic dual, i.e. the transformation

$$\begin{aligned} &\mathbf{Op} \colon \mathsf{IA} \to \mathsf{HA} \\ &\boldsymbol{A} \mapsto \mathbf{Op}(\boldsymbol{A}) := \langle \mathrm{Op}(\boldsymbol{A}), \wedge, \vee, \multimap, 0, 1\rangle \end{aligned}$$

  where $\mathrm{Op}(\boldsymbol{A}) = \{a \in A : \Box a = a\}$ and $a \multimap b := \Box(a \to b)$.
- Recap: Kolmogorov and Gödel's logic translations correspond to semantics transformations in the reverse direction:

$$\begin{aligned} &(\cdot)^{\mathsf{K}} \colon \mathbf{CPC} \to \mathbf{IPC} \text{ and } \mathbf{Reg} \colon \mathsf{HA} \to \mathsf{BA} \\ &(\cdot)^{\mathsf{G}} \colon \mathbf{IPC} \to \mathbf{S4} \text{ and } \mathbf{Op} \colon \mathsf{IA} \to \mathsf{HA}. \end{aligned}$$

Adjoint Functors

- The semantic transformations, dualizing Kolmogorov and Gödel's translations, are special instances of the following:

Definition

A pair of functors $\mathcal{F}\colon \mathsf{X} \longleftrightarrow \mathsf{Y}\colon \mathcal{G}$ is an adjunction if there is a pair of natural transformation $\eta\colon 1_{\mathsf{X}} \to \mathcal{G}\mathcal{F}$ and $\epsilon\colon \mathcal{F}\mathcal{G} \to 1_{\mathsf{Y}}$ such that

$$1_{\mathcal{G}(\boldsymbol{B})} = \mathcal{G}(\epsilon_{\boldsymbol{B}}) \circ \eta_{\mathcal{G}(\boldsymbol{B})} \text{ and } 1_{\mathcal{F}(\boldsymbol{A})} = \epsilon_{\mathcal{F}(\boldsymbol{A})} \circ \mathcal{F}(\eta_{\boldsymbol{A}}).$$

for every $\boldsymbol{A} \in \mathsf{X}$ and $\boldsymbol{B} \in \mathsf{Y}$.

- In this case $\mathcal{F}$ is left adjoint to $\mathcal{G}$ and $\mathcal{G}$ right adjoint to $\mathcal{F}$.
- Under the identification right adjoints = semantic transformations, proving the equivalence

  logical translations $\equiv$ semantic transformations

  amounts to find a syntactic description of right adjoints.

Matrix powers with infinite exponents

- Let X be a class of similar algebras and $\kappa > 0$ be a cardinal.
- Consider the language $\mathscr{L}_{\mathsf{X}}^{\kappa}$ whose n-ary operations are the κ-sequences

  $\langle t_i : i < \kappa \rangle$ where each t_i is a term of X in variables $\vec{x}_1, \dots, \vec{x}_n$.

Definition

Given $\boldsymbol{A} \in \mathsf{X}$, let $\boldsymbol{A}^{[\kappa]}$ be the $\mathscr{L}_{\mathsf{X}}^{\kappa}$-algebra with universe A^{κ} s.t.

$$\langle t_i : i < \kappa \rangle^{\boldsymbol{A}^{[\kappa]}}(\vec{a}_1, \dots, \vec{a}_n) = \langle t_i^{\boldsymbol{A}}(\vec{a}_1/\vec{x}_1, \dots, \vec{a}_n/\vec{x}_n) : i < \kappa \rangle.$$

The κ-th matrix power of X is the class

$$\mathsf{X}^{[\kappa]} := \mathbb{I}\{\boldsymbol{A}^{[\kappa]} : \boldsymbol{A} \in \mathsf{X}\}.$$

- This construction extends to a functor $[\kappa]\colon \mathsf{X} \to \mathsf{X}^{[\kappa]}$.

Compatible Equations

Definition

Let X be a class of algebras of language $\mathscr{L}_{\mathsf{X}}$ and $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. A set of equations θ in one variable is compatible with $\mathscr{L}$ in X if for every n-ary operation $\varphi \in \mathscr{L}$ we have that:

$$\theta(x_1) \cup \dots \cup \theta(x_n) \vDash_{\mathsf{X}} \theta(\varphi(x_1, \dots, x_n)).$$

- For every $\boldsymbol{A} \in \mathsf{X}$, we let $\boldsymbol{A}(\theta, \mathscr{L})$ be the algebra of type $\mathscr{L}$ with universe

  $$A(\theta, \mathscr{L}) = \{a \in A : \boldsymbol{A} \vDash \theta(a)\}$$

  equipped with the restriction of the operations in $\mathscr{L}$.
- We obtain a functor

  $$\theta_{\mathscr{L}}\colon \mathsf{X} \to \mathbb{I}\{\boldsymbol{A}(\theta, \mathscr{L}) : \boldsymbol{A} \in \mathsf{X}\}.$$

Logical description of right adjoints

- It turns out that, among quasi-varieties,

  right adjoints admit a syntactic/logical description.
- More precisely, we have the following:

Theorem

Let X and Y be quasi-varieties.

1. For every non-trivial right adjoint

   $$\mathcal{G}\colon \mathsf{Y} \to \mathsf{X}$$

   there is a (generalized) quasi-variety K and functors

   $$[\kappa]\colon \mathsf{Y} \to \mathsf{K} \text{ and } \theta_{\mathscr{L}}\colon \mathsf{K} \to \mathsf{X}$$

   such that $\mathcal{G}$ is naturally isomorphic to $\theta_{\mathscr{L}} \circ [\kappa]$.
2. Every functor of the form $\theta_{\mathscr{L}} \circ [\kappa]\colon \mathsf{Y} \to \mathsf{X}$ is a right adjoint.

- This syntactic description of right adjoints (inspired by work of McKenzie and others) allows to establish a precise correspondence

$$\text{right adjoints} \equiv \text{logical translations}$$

where the precise meaning of logical translations come from the syntactic canonical form $\theta_{\mathscr{L}} \circ [\kappa]$ of right adjoints.
- This new notion of logical translation embraces most known examples, e.g. Kolmogorov and Gödel's ones.

Recap:

- One can state a precise correspondence between semantic transformations (understood as right adjoints) and translations between logics (understood as equational consequences).
- This yields an algebraic canonical form for right adjoints.
- Some computational results follows, e.g. the problem of determining whether two finite algebras are related by an adjunction is decidable.

Finally...

...thank you for coming!