Adjunctions as translations between relative equational consequences

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Aim of the talk

We will try to relate the following concepts:

- Adjunctions between quasi-varieties.
- Translations between logics:

  Kolmogorov's translations of $\mathcal{CPC}$ into $\mathcal{IPC}$

  Gödel's translation of $\mathcal{IPC}$ into $\mathcal{S}4$.

- Twist constructions:

  Distributive lattices $\longmapsto$ Kleene lattices

  Lattices $\longmapsto$ Bilattices

Adjoint Functors

Definition

A pair of functors $\mathcal{F} \colon \mathsf{X} \longleftrightarrow \mathsf{Y} \colon \mathcal{G}$ is an adjunction if there is a pair of natural transformation $\eta \colon 1_{\mathsf{X}} \to \mathcal{GF}$ and $\epsilon \colon \mathcal{FG} \to 1_{\mathsf{Y}}$ such that

$$1_{\mathcal{G}(\boldsymbol{B})} = \mathcal{G}(\epsilon_{\boldsymbol{B}}) \circ \eta_{\mathcal{G}(\boldsymbol{B})} \text{ and } 1_{\mathcal{F}(\boldsymbol{A})} = \epsilon_{\mathcal{F}(\boldsymbol{A})} \circ \mathcal{F}(\eta_{\boldsymbol{A}}).$$

for every $\boldsymbol{A} \in \mathsf{X}$ and $\boldsymbol{B} \in \mathsf{Y}$.

- In this case $\mathcal{F}$ is left adjoint to $\mathcal{G}$ and $\mathcal{G}$ right adjoint to $\mathcal{F}$.
- Our first goal is to give an algebraic characterization of adjunctions between quasi-varieties:

  right adjoints = generalized twist constructions.

Twist constructions

Well-known example

- A Kleene lattice $\boldsymbol{A} = \langle A, \sqcap, \sqcup, \neg, 0, 1\rangle$ is a De Morgan algebra in which the equation $x \sqcap \neg x \leq y \sqcup \neg y$ holds.
- Given a bounded distributive lattice $\boldsymbol{A}$, the Kleene lattice $\mathcal{G}(\boldsymbol{A})$ has universe

  $$G(A) \coloneqq \{\langle a, b\rangle \in A^2 : a \wedge b = 0\}$$

  and operations defined as

  $$\langle a, b\rangle \sqcap \langle c, d\rangle \coloneqq \langle a \wedge c, b \vee d\rangle$$
  $$\neg\langle a, b\rangle \coloneqq \langle b, a\rangle \quad 1 \coloneqq \langle 1, 0\rangle \quad 0 \coloneqq \langle 0, 1\rangle$$

In general twist constructions involve two steps (given an algebra $\boldsymbol{A}$):

- Do the κ-power of A for some cardinal κ. (above $\kappa = 2$).
- Select in some elements $G(A) \subseteq A^\kappa$ and define new basic operations for $G(\boldsymbol{A})$ which are κ-sequences of operations of $\boldsymbol{A}$.

Matrix Powers with Infinite Exponent

- Let X be a class of similar algebras and $\kappa > 0$ be a cardinal.
- Consider the language $\mathscr{L}_X^\kappa$ whose n-ary operations are the κ-sequences

$$\langle t_i : i < \kappa \rangle \text{ where each } t_i \text{ is a term of X in variables } \vec{x}_1, \ldots, \vec{x}_n.$$

Definition

Consider an algebra $\boldsymbol{A} \in \mathsf{X}$. We let $\boldsymbol{A}^{[\kappa]}$ be the algebra of type $\mathscr{L}_X^\kappa$ with universe A^κ where

$$\langle t_i : i < \kappa \rangle^{\boldsymbol{A}^{[\kappa]}}(\vec{a}_1, \ldots, \vec{a}_n) = \langle t_i^{\boldsymbol{A}}(\vec{a}_1/\vec{x}_1, \ldots, \vec{a}_n/\vec{x}_n) : i < \kappa \rangle.$$

The κ-th matrix power of X is the class

$$\mathsf{X}^{[\kappa]} := \mathbb{I}\{\boldsymbol{A}^{[\kappa]} : \boldsymbol{A} \in \mathsf{X}\}.$$

Compatible Equations

Definition

Let X be a class of algebras of language $\mathscr{L}_X$ and $\mathscr{L} \subseteq \mathscr{L}_X$. A set of equations θ in one variable is compatible with $\mathscr{L}$ in X if for every n-ary operation $\varphi \in \mathscr{L}$ we have that:

$$\theta(x_1) \cup \cdots \cup \theta(x_n) \vDash_X \theta(\varphi(x_1, \ldots, x_n)).$$

- For every $\boldsymbol{A} \in \mathsf{X}$, we let $\boldsymbol{A}(\theta, \mathscr{L})$ be the algebra of type $\mathscr{L}$ with universe

$$A(\theta, \mathscr{L}) = \{a \in A : \boldsymbol{A} \vDash \theta(a)\}$$

  equipped with the restriction of the operations in $\mathscr{L}$.
- We obtain a functor

$$\theta_{\mathscr{L}} \colon \mathsf{X} \to \mathbb{I}\{\boldsymbol{A}(\theta, \mathscr{L}) : \boldsymbol{A} \in \mathsf{X}\}.$$

Generalized twist constructions

- According to the previous abstractions, a generalized twist construction between two quasi-varieties K and V is a functor of the form

$$\theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{K} \to \mathsf{V}$$

  where θ is compatible with $\mathscr{L}$ in $\mathsf{Y}^{[\kappa]}$. The idea is that:
  1. $[\kappa]$ produce powers $\boldsymbol{A}^\kappa$ of algebras in $\boldsymbol{A} \in \mathsf{K}$.
  2. $\theta_{\mathscr{L}}$ selects elements of A^κ and defined new basic operations.

Canonical form

- It turns out that among quasi-varieties

  right adjoints = generalized twist constructions.
- More precisely, we have the following:

Theorem

Let X and Y be quasi-varieties.

1. For every non-trivial right adjoint

$$\mathcal{G} \colon \mathsf{Y} \to \mathsf{X}$$

   there is a (generalized) quasi-variety K and functors

$$[\kappa] \colon \mathsf{Y} \to \mathsf{K} \text{ and } \theta_{\mathscr{L}} \colon \mathsf{K} \to \mathsf{X}$$

   such that $\mathcal{G}$ is naturally isomorphic to $\theta_{\mathscr{L}} \circ [\kappa]$.
2. Every functor of the form $\theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{X}$ is a right adjoint.

Translations Between Languages

Definition

Consider a cardinal $\kappa > 0$. A κ-translation of $\mathscr{L}_X$ into $\mathscr{L}_Y$ is a map $\boldsymbol{\tau}\colon \mathscr{L}_X \to \mathscr{L}_Y^\kappa$ that preserves arities.

- $\boldsymbol{\tau}$ extends to a map from formulas of X to formulas of $Y^{[\kappa]}$
- and lifts to a map from sets of equations of X to sets of equations of Y as follows:

$$\Phi \longmapsto \{\boldsymbol{\tau}(\epsilon)(i) \approx \boldsymbol{\tau}(\delta)(i) : i < \kappa \text{ and } \epsilon \approx \delta \in \Phi\}.$$

Translations Between Relative Equational Consequences

Definition

A translation of $\vDash_X$ into $\vDash_Y$ is a pair $\langle \boldsymbol{\tau}, \Theta \rangle$ where $\boldsymbol{\tau}$ is a κ-translation of $\mathscr{L}_X$ into $\mathscr{L}_Y$ and a set of equations Θ of Y in κ-many variables that satisfies the following conditions:

1. For every set of equations $\Phi \cup \{\epsilon \approx \delta\}$:

$$\text{If } \Phi \vDash_X \epsilon \approx \delta, \text{ then } \boldsymbol{\tau}(\Phi) \cup \bigcup_{x \in Var} \Theta(\vec{x}) \vDash_Y \boldsymbol{\tau}(\epsilon \approx \delta).$$

2. For every n-ary operation $\psi \in \mathscr{L}_X$:

$$\Theta(\boldsymbol{\tau}(x_1)) \cup \dots \cup \Theta(\boldsymbol{\tau}(x_n)) \vDash_Y \Theta(\boldsymbol{\tau}\psi(x_1, \dots, x_n)).$$

Gödel's Translation

- Gödel provided an interpretation of $\mathcal{IPC}$ into global $\mathcal{S}4$.
- Let $\boldsymbol{\tau}$ be the 1-translation of $\mathscr{L}_{HA}$ into $\mathscr{L}_{IA}$ defined as:

$$x \star y \longmapsto x \star y \quad \neg x \longmapsto \Box\neg x \quad x \to y \longmapsto \Box(x \to y)$$

  for $\star \in \{\wedge, \vee\}$.
- Let σ be the substitution sending x to $\Box x$ for every $x \in Var$.
- Then we have:

$$\Gamma \vdash_{\mathcal{IPC}} \varphi \iff \sigma\boldsymbol{\tau}(\Gamma) \vdash_{\mathcal{S}4} \sigma\boldsymbol{\tau}(\varphi)$$

- Define $\Theta(x) = \{x \approx \Box x\}$. Then:

$$\Phi \vDash_{HA} \epsilon \approx \delta \iff \boldsymbol{\tau}(\Phi) \cup \bigcup_{x \in Var} \Theta(x) \vDash_{IA} \boldsymbol{\tau}(\epsilon \approx \delta)$$

- Moreover $\langle \boldsymbol{\tau}, \Theta \rangle$ is a translation of $\vDash_{HA}$ into $\vDash_{IA}$.

From Translations to Right Adjoints

- Let $\langle \boldsymbol{\tau}, \Theta \rangle$ be a κ-translation of $\vDash_X$ into $\vDash_Y$.
- Consider the sublanguage of $Y^{[\kappa]}$:

$$\mathscr{L} = \{\boldsymbol{\tau}(\psi) : \psi \in \mathscr{L}_X\}.$$

- Consider the set of equations of $Y^{[\kappa]}$ in one variable:

$$\theta = \{\vec{\epsilon} \approx \vec{\delta} : \epsilon \approx \delta \in \Theta\}.$$

Lemma

The map $\theta_{\mathscr{L}} \circ [\kappa]\colon Y \to X$ is a right adjoint.

- Gödel's translation induces the functor

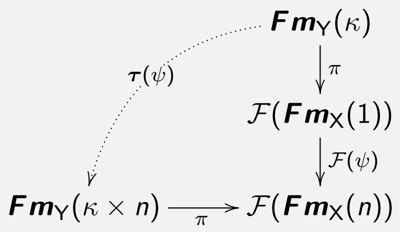
$$\text{Open}\colon \text{IA} \to \text{HA}$$

  and Kolmogorov's translation the functor

$$\text{Regular}\colon \text{HA} \to \text{BA}.$$

From Adjunctions to Translations

- Consider $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ left adjoint.
- We have $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)) = \boldsymbol{Fm}_{\mathsf{Y}}(\kappa)/\theta$ for some κ and θ.
- Consider the homomorphism $\psi\colon \boldsymbol{Fm}_{\mathsf{X}}(1) \to \boldsymbol{Fm}_{\mathsf{X}}(n)$.



Lemma
The pair $\langle \boldsymbol{\tau}, \Theta \rangle$ is a translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$.

Miscellanea

Some applications of these tools:

- Universal Algebra: congruence regularity is not a linear Maltsev condition.
- Abstract Algebraic Logic: every prevariety is categorically equivalent to the equivalent algebraic semantics of an algebraizable logic.
- Computational aspects: the problem of determining whether two finite algebras are related by an adjunction is decidable.

Finally...

...thank you for coming!