Epimorphism surjectivity and the Beth definability property

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Epimorphism surjectivity

Definition

Let K be a class of algebras. A homomorphism $f : \boldsymbol{A} \to \boldsymbol{B}$ in K is an epimorphism if for every pair $g, h : \boldsymbol{B} \rightrightarrows \boldsymbol{C}$ of homomorphisms in K

$$\text{if } g \circ f = h \circ f \text{, then } g = h.$$

- Are epis surjective in a variety?
- Yes: Boolean algebras, Heyting algebras, lattices, semilattices and (Abelian) groups.
- No: distributive lattices, rings with unity and monoids.
- Thus epimorphism surjectivity is not preserved in subvarieties!

Beth property

- Let $\mathcal{L}$ be algebraizable with equivalence formulas $\boldsymbol{\rho}(x, y)$.

Definition

Let Γ be a set of formulas over $X \cup Z$ with $X \cap Z = \emptyset$ and $X \neq \emptyset$.

1. Γ implicitly defines Z in terms of X if

$$\Gamma \cup \sigma(\Gamma) \vdash_{\mathcal{L}} \boldsymbol{\rho}(z, \sigma z) \text{ for every } z \in Z$$

for every substitution σ that fixes X.

2. Γ explicitly defines Z in terms of X if for every $z \in Z$ there is a formula φ_z over X only such that

$$\Gamma \vdash_{\mathcal{L}} \boldsymbol{\rho}(z, \varphi_z).$$

- $\mathcal{L}$ has the (resp. finite) Beth property when 1 (resp. with Z finite) implies 2.

Transfer theorem

Definition

A homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is almost onto if $\boldsymbol{B}$ is generated by $f(A) \cup \{b\}$ for some $b \in B$.

Theorem (Blok and Hoogland)

Let $\mathcal{L}$ be an algebraizable logic.

1. $\mathcal{L}$ has the Beth property iff epis are surjective in $\mathrm{Alg}^*\mathcal{L}$.
2. $\mathcal{L}$ has the finite Beth property iff almost onto epis are surjective in $\mathrm{Alg}^*\mathcal{L}$.

- Blok and Hoogland conjectured that

Beth property $\neq$ finite Beth property.

Why Heyting algebras?

- We want to establish Blok and Hoogland's conjecture by finding a variety (that algebraizes a logic) where:
  1. Almost onto epimorphisms are surjective.
  2. Epimorphisms need not be surjective.

Theorem (Kreisel)

Every axiomatic extension of **IPC** has the finite Beth property.

- This result can be re-stated as follows:

Theorem

In varieties of Heyting algebras almost onto epis are surjective.

- To establish Blok and Hoogland's conjecture, it is enough to find a variety of Heyting algebras where epis need not be surjective.

K-epic subalgebras

Definition

Let K be a quasi-variety and $\boldsymbol{B} \in \mathsf{K}$. A subalgebra $\boldsymbol{A} \leq \boldsymbol{B}$ is K-epic if for every pair of homomorphisms $f, g\colon \boldsymbol{B} \rightrightarrows \boldsymbol{C} \in \mathsf{K}$

$$\text{if } f\restriction_A = g\restriction_A \text{, then } f = g.$$

- Epis are surjective in K iff no $\boldsymbol{B} \in \mathsf{K}$ has a proper K-epic subalgebra.

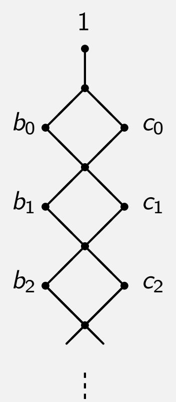
Theorem (Campercholi)

Let K be a quasi-variety and $\boldsymbol{A} \leq \boldsymbol{B} \in \mathsf{K}$. TFAE:

1. $\boldsymbol{A}$ is a K-epic subalgebra of $\boldsymbol{B}$.
2. For every $b \in B$ there is a primitive positive formula $\varphi(\vec{x}, y)$ and $\vec{a} \in A$ such that

$$\mathsf{K} \vDash \forall \vec{x}, y, z((\varphi(\vec{x}, y) \,\&\, \varphi(\vec{x}, z)) \to y \approx z) \text{ and } \boldsymbol{B} \vDash \varphi(\vec{a}, b).$$

- Let $\boldsymbol{A}$ be the Heyting algebra depicted below and $\boldsymbol{B}$ the subalgebra with universe $\{0, b_0, b_1, b_2, \dots, 1\}$.



- We claim that $\boldsymbol{B}$ is a $\mathbb{V}(\boldsymbol{A})$-epic subalgebra of $\boldsymbol{A}$.
- We need to find primitive positive formulas that define partial functions in $\mathbb{V}(\boldsymbol{A})$ and, moreover, construct $\boldsymbol{A}$ out of $\boldsymbol{B}$.

Partial functions

- Consider the conjunction of equations

$$\varphi(x_0, x_1, x_2, y_0, y_1, y_2) := \mathop{\&}_{n\leq 2} (x_n \to y_n \approx y_n \,\&\, y_n \to x_n \approx x_n)$$
$$\mathop{\&}_{n\leq 1} (x_n \wedge y_n \approx x_{n+1} \vee y_{n+1}).$$

- and the primitive positive formula

$$\Phi(x_0, x_1, x_2, y_0) := \exists y_1 y_2 \varphi.$$

- Φ define a partial 3-ary function in $\mathbb{V}(\boldsymbol{A})$: For every $\boldsymbol{C} \in \mathbb{V}(\boldsymbol{A})$ and $a_0, a_1, a_2 \in C$ there is at most one $e \in C$ s.t.

$$\boldsymbol{C} \vDash \Phi(a_0, a_1, a_2, e).$$

- Applying this partial function to $\boldsymbol{B}$ we recover the whole $\boldsymbol{A}$:

$$\boldsymbol{A} \vDash \Phi(b_{n+2}, b_{n+1}, b_n, c_n) \text{ for every } n \in \omega.$$

Two Beth properties

- Epimorphisms need not to be surjective in $\mathbb{V}(\boldsymbol{A})$.
- Observe that $\mathbb{V}(\boldsymbol{A})$ satisfies the weak Pierce law

$$(y \to x) \vee (((x \to y) \to x) \to x) \approx 1.$$

- Then $\mathbb{V}(\boldsymbol{A})$ is locally finite.

Theorem (Blok-Hoogland's conjecture)

1. Epimorphisms need not be surjective in locally finite varieties of Heyting algebras.
2. The Beth property and the finite Beth property are different in locally tabular superintuitionistic logics.

Rieger-Nishimura lattice

Definition

A Heyting algebra $\boldsymbol{A}$ has width n if the largest antichain in principal upsets of $\langle \mathrm{Pr}(\boldsymbol{A}), \subseteq \rangle$ has exactly n elements.

- Let W_n be the class of Heyting algebras of width $\leq n$. It is a variety.
- $\mathbb{V}(\boldsymbol{A})$ has width 2.
- The Rieger-Nishimura lattice has width 2.

Theorem

1. There is a continuum of varieties of Heyting algebras width ≤ 2 where epimorphisms need not be surjective.
2. Among them there is the variety generated by the Rieger-Nishimura lattice.

Finite depth

Definition

A Heyting algebra $\boldsymbol{A}$ has depth n if the longest chain in $\langle \mathrm{Pr}(\boldsymbol{A}), \subseteq \rangle$ has exactly n elements.

Let HA_n be the class of Heyting algebras of depth $\leq n$.

Theorem (Maksimova and Ono)

HA_n is a variety axiomatized by $h_n \approx 1$, where $h_0 = y$ and for $n > 0$

$$h_n := x_n \vee (x_n \to h_{n-1}).$$

- A variety of Heyting algebras has finite depth when its members have finite depth.

Theorem

Let K be a variety of Heyting algberas. If K has finite depth, then epimorphisms are surjective in K.

Consequences

- Finitely generated varieties of Heyting algebras are known to have finite depth.

Corollary

1. Epimorphisms are surjective in finitely generated varieties of Heyting algbreas.
2. Tabular superintuitionistic logics have the Beth property.
3. Superintuitionistic logics, whose theorems include h_n for some $n \in \omega$, have the Beth property.
4. Epimorphisms are surjective in all varieties of Gödel algebras.

Strong epimorphism surjectivity

Definition

A class of algebras K has strong epimorphism surjectivity if whenever $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is homomorphism in K and $b \in B \smallsetminus f(A)$, there are homomorphisms $g, h\colon \boldsymbol{B} \rightrightarrows \boldsymbol{C}$ in K such that

$$g \circ f = h \circ f \text{ and } g(b) \neq h(b).$$

Theorem (Maksimova)

There are finitely many varieties of Heyting algebras with strong epimorphism surjectitiy.

- There is a continuum of varieties of depth ≤ 3.
- Thus there is a continuum of varieties with epimorphism surjectivity but not strong epimorphism surjectivity.

Thanks for coming!