VARIETIES OF DE MORGAN MONOIDS: MINIMALITY AND IRREDUCIBLE ALGEBRAS

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Abstract. It is proved that every finitely subdirectly irreducible De Morgan monoid $\boldsymbol{A}$ (with neutral element e) is either (i) a Sugihara chain in which e covers $\neg e$ or (ii) the union of an interval subalgebra $[\neg a, a]$ and two chains of idempotents, $(\neg a]$ and $[a)$, where $a = (\neg e)^2$. In the latter case, the variety generated by $[\neg a, a]$ has no nontrivial idempotent member, and $\boldsymbol{A}/[\neg a)$ is a Sugihara chain in which $\neg e = e$. It is also proved that there are just four minimal varieties of De Morgan monoids. These findings are then used to simplify the proof of a description (due to K. Świrydowicz) of the lower part of the subvariety lattice of relevant algebras. The results throw light on the models and the axiomatic extensions of fundamental relevance logics.

1. Introduction

De Morgan monoids are commutative monoids with a residuated distributive lattice order and a compatible antitone involution $\neg$, where $a \leqslant a^2$ for all elements a. They form a variety, DMM.

The explicit study of residuated lattices goes back to Ward and Dilworth [76] and has older antecedents (see the citations in [9, 24, 29]). Much of the interest in De Morgan monoids stems, however, from their connection with relevance logic, discovered by Dunn [15] and recounted briefly below in Section 4.1 (where further references are supplied). A key fact, for our purposes, is that the axiomatic extensions of Anderson and Belnap's logic $\mathbf{R}^{\mathbf{t}}$

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and the varieties of De Morgan monoids form anti-isomorphic lattices, and the latter are susceptible to the methods of universal algebra.

Slaney [61, 62] showed that the free 0–generated De Morgan monoid is finite, and that there are only seven non-isomorphic subdirectly irreducible 0–generated De Morgan monoids. No similarly comprehensive classification is available in the 1–generated case, however, where the algebras may already be infinite. In 1996, Urquhart [73, p. 263] observed that "[t]he algebraic theory of relevant logics is relatively unexplored, particularly by comparison with the field of algebraic modal logic." Acquiescing in a paper of 2001, Dunn and Restall [17, Sec. 3.5] wrote: "Not as much is known about the algebraic properties of De Morgan monoids as one would like." These remarks pre-date many recent papers on residuated lattices—see the bibliography of [24], for instance. But the latter have concentrated mainly on varieties incomparable with DMM (e.g., Heyting and MV-algebras), larger than DMM (e.g., full Lambek algebras) or smaller (e.g., Sugihara monoids).

A De Morgan monoid $\boldsymbol{A}$, with neutral element e, is said to be *idempotent* or *anti-idempotent* if it satisfies $x^2 = x$ or $x \leqslant (\neg e)^2$, respectively. The idempotent De Morgan monoids are the aforementioned *Sugihara monoids*, and their structure is very well understood. Anti-idempotence is equivalent to the demand that no nontrivial idempotent algebra belongs to the variety generated by $\boldsymbol{A}$ (Corollary 3.6), hence the terminology.

It is well known that a De Morgan monoid is finitely subdirectly irreducible iff the element e is join-prime. The first main result of this paper shows that any such De Morgan monoid $\boldsymbol{A}$ is either (i) a totally ordered Sugihara monoid in which e covers $\neg e$ or (ii) the union of an interval subalgebra $[\neg a, a]$ and two chains of idempotent elements, $(\neg a]$ and $[a)$, where $a = (\neg e)^2$. In the latter case, the anti-idempotent subalgebra is the e–class of a congruence θ such that $\boldsymbol{A}/\theta$ is a totally ordered Sugihara monoid in which $\neg e = e$, and all other θ–classes are singletons. (See Theorem 5.17 and Remark 5.19.)

Subalgebra structure aside, another measure of the complexity of a De Morgan monoid $\boldsymbol{A}$ is the height, within the subvariety lattice of DMM, of the variety generated by $\boldsymbol{A}$. Accordingly, the present paper initiates an analysis of the lattice of varieties of De Morgan monoids. We prove that such a variety consists of Sugihara monoids iff it omits a certain pair of four-element algebras (Theorem 5.21). This implies that DMM has just four minimal subvarieties, all of which are finitely generated (Theorem 6.1). The covers of these atoms are investigated in a sequel paper [51].

For philosophical reasons, relevance logic also emphasizes a system called **R**, which lacks the so-called Ackermann truth constant **t** (corresponding to the neutral element of a De Morgan monoid). The logic **R** is algebraized by the variety RA of *relevant algebras*. Świrydowicz [68] has described the bottom of the subvariety lattice of RA. We simplify the proof of his result (see Theorem 7.8), using our analysis (from Section 5) of the subvarieties of DMM.

These findings have implications for the extension lattices of both $\mathbf{R}$ and $\mathbf{R^t}$. For instance, Świrydowicz's theorem has been applied recently to show that no consistent axiomatic extension of $\mathbf{R}$ is structurally complete, except for classical propositional logic [56]. The situation for $\mathbf{R^t}$ is very different and is the subject of ongoing investigation by the present authors. (Regarding fragments of $\mathbf{R^t}$, see [66] and [53, Thm. 9.7].)

2. Residuated Structures

Definition 2.1. An *involutive (commutative) residuated lattice*, or briefly, an *IRL*, is an algebra $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ comprising a commutative monoid $\langle A; \cdot, e\rangle$, a lattice $\langle A; \wedge, \vee\rangle$ and a function $\neg\colon A \longrightarrow A$, called an *involution*, such that $\boldsymbol{A}$ satisfies the (first order) formulas $\neg\neg x = x$ and

$$x \cdot y \leqslant z \iff \neg z \cdot y \leqslant \neg x, \tag{1}$$

cf. [24].[1] Here, $\leqslant$ denotes the lattice order (i.e., $x \leqslant y$ abbreviates $x \wedge y = x$) and $\neg$ binds more strongly than any other operation; we refer to $\cdot$ as *fusion.*

Setting $y = e$ in (1), we see that $\neg$ is antitone. In fact, De Morgan's laws for $\neg, \wedge, \vee$ hold, so $\neg$ is an anti-automorphism of $\langle A; \wedge, \vee\rangle$. If we define

$$x \to y \;:=\; \neg(x \cdot \neg y) \quad\text{and}\quad f \;:=\; \neg e,$$

then, as is well known, every IRL satisfies

$$x \cdot y \leqslant z \iff y \leqslant x \to z \quad \text{(the law of residuation)}, \tag{2}$$

$$\neg x = x \to f \quad\text{and}\quad x \to y = \neg y \to \neg x \quad\text{and}\quad x \cdot y = \neg(x \to \neg y). \tag{3}$$

Definition 2.2. A *(commutative) residuated lattice*—or an *RL*—is an algebra $\boldsymbol{A} = \langle A; \cdot, \to, \wedge, \vee, e\rangle$ comprising a commutative monoid $\langle A; \cdot, e\rangle$, a lattice $\langle A; \wedge, \vee\rangle$ and a binary operation $\to$, called *residuation*, where $\boldsymbol{A}$ satisfies (2).

Thus, up to term equivalence, every IRL has a reduct that is an RL. Conversely, every RL can be embedded into (the RL-reduct of) an IRL; see [26] and the antecedents cited there. Every RL satisfies the following well known formulas. Here and subsequently, $x \leftrightarrow y$ abbreviates $(x \to y) \wedge (y \to x)$.

$$x \cdot (x \to y) \leqslant y \quad\text{and}\quad x \leqslant (x \to y) \to y \tag{4}$$

$$(x \cdot y) \to z = y \to (x \to z) = x \to (y \to z) \tag{5}$$

$$(x \to y) \cdot (y \to z) \leqslant x \to z \tag{6}$$

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z) \tag{7}$$

$$x \leqslant y \implies ((x \cdot z \leqslant y \cdot z) \;\&\; (z \to x \leqslant z \to y) \;\&\; (y \to z \leqslant x \to z)) \tag{8}$$

$$x \leqslant y \iff e \leqslant x \to y \tag{9}$$

$$x = y \iff e \leqslant x \leftrightarrow y \tag{10}$$

$$e \leqslant x \to x \quad\text{and}\quad e \to x = x \tag{11}$$

[1] The signature in [24] is slightly different, but the definable terms are not affected.

(12) $e \leqslant x \iff x \to x \leqslant x.$

By (10), an RL $\boldsymbol{A}$ is *nontrivial* (i.e., $|A| > 1$) iff e is not its least element, iff e has a strict lower bound. A class of algebras is said to be *nontrivial* if it has a nontrivial member.

Another consequence of (10) is that a non-injective homomorphism h between RLs must satisfy $h(c) = e$ for some $c < e$. (Choose $c = e \wedge (a \leftrightarrow b)$, where $h(a) = h(b)$ but $a \neq b$.)

In an RL, we define $x^0 := e$ and $x^{n+1} := x^n \cdot x$ for $n \in \omega$.

Lemma 2.3. *If a (possibly involutive) RL $\boldsymbol{A}$ has a least element $\bot$, then $\top := \bot \to \bot$ is its greatest element and, for all $a \in A$,*

$$a \cdot \bot = \bot = \top \to \bot \quad \textit{and} \quad \bot \to a = \top = a \to \top = \top^2.$$

In particular, $\{\bot, \top\}$ is a subalgebra of the $\cdot, \to, \wedge, \vee\, (, \neg)$ reduct of $\boldsymbol{A}$.

Proof. See [52, Prop. 5.1], for instance. (We infer $\top = \top^2$ from (8), as $e \leqslant \top$. The lattice anti-automorphism $\neg$, if present, clearly switches $\bot$ and $\top$.) □

If we say that $\bot, \top$ are *extrema* of an RL $\boldsymbol{A}$, we mean that $\bot \leqslant a \leqslant \top$ for all $a \in A$. An RL with extrema is said to be *bounded.* In that case, its extrema need not be *distinguished* elements, so they are not always retained in subalgebras. The next lemma is a straightforward consequence of (2).

Lemma 2.4. *The following conditions on a bounded IRL $\boldsymbol{A}$, with extrema $\bot, \top$, are equivalent.*

(i) $\top \cdot a = \top$ *whenever* $\bot \neq a \in A$.
(ii) $a \to \bot = \bot$ *whenever* $\bot \neq a \in A$.
(iii) $\top \to b = \bot$ *whenever* $\top \neq b \in A$.

Definition 2.5. Following Meyer [47], we say that an IRL is *rigorously compact* if it is bounded and satisfies the equivalent conditions of Lemma 2.4.

Lemma 2.6. *Let $\boldsymbol{A}$ be an IRL, with $a \in A$. Then*

$$e \leqslant a = a^2 \quad \textit{iff} \quad a \cdot \neg a = \neg a \quad \textit{iff} \quad a = a \to a.$$

Proof. The second and third conditions are equivalent, by the definition of $\to$ and involution properties. Also, $a^2 \leqslant a$ and $a \cdot \neg a \leqslant \neg a$ are equivalent, by (1). From $e \leqslant a$ and (8) we infer $\neg a = e \cdot \neg a \leqslant a \cdot \neg a$. Conversely, $a \to a \leqslant a$ and (11) yield $e \leqslant a$, and therefore $a \leqslant a^2$. □

The class of all RLs and that of all IRLs are finitely axiomatized varieties. They are arithmetical (i.e., congruence distributive and congruence permutable) and have the congruence extension property (CEP). These facts can be found, for instance, in [24, Sections 2.2 and 3.6].

3. Square-Increasing IRLs

An RL is said to be *square-increasing* if it satisfies $x \leqslant x^2$. Every square-increasing RL can be embedded into a square-increasing IRL [44]. Moreover, Slaney [65] has shown that if two square-increasing IRLs have the same RL-reduct, then they are equal. The following formulas are valid in all square-increasing IRLs (and not in all IRLs):

$$x \wedge y \leqslant x \cdot y \tag{13}$$

$$(x \leqslant e \ \ \& \ \ y \leqslant e) \implies x \cdot y = x \wedge y \tag{14}$$

$$e \leqslant x \vee \neg x. \tag{15}$$

The lemma below generalizes another result of Slaney [61, T36, p. 491] (where only the case $a = f$ was discussed, and $\boldsymbol{A}$ satisfied an extra postulate).

Lemma 3.1. *Let $\boldsymbol{A}$ be a square-increasing IRL, with $f \leqslant a \in A$. Then $a^3 = a^2$. In particular, $f^3 = f^2$.*

Proof. As $f \leqslant a$, we have $\neg a = a \to f \leqslant a \to a$, by (3) and (8), so

$$a \to \neg a \leqslant a \to (a \to a) = a^2 \to a, \tag{16}$$

by (8) and (5). By the square-increasing law, (16), (8) and (6),

$$a \to \neg a \leqslant (a \to \neg a)^2 \leqslant (a^2 \to a) \cdot (a \to \neg a) \leqslant a^2 \to \neg a.$$

Thus, $\neg(a^2 \to \neg a) \leqslant \neg(a \to \neg a)$, i.e., $a^2 \cdot a \leqslant a \cdot a$ (see (3)), i.e., $a^3 \leqslant a^2$. The reverse inequality follows from the square-increasing law and (8). $\square$

The first assertion of the next theorem has unpublished antecedents in the work of relevance logicians. A corresponding result for 'relevant algebras' is reported in [69, Prop. 5], but the claim and proof below are simpler.

Theorem 3.2. *Every finitely generated square-increasing IRL $\boldsymbol{A}$ is bounded. More precisely, let $\{a_1, \dots, a_n\}$ be a finite set of generators for $\boldsymbol{A}$, with*

$$c = e \vee f \vee \bigvee\nolimits_{i \leq n} (a_i \vee \neg a_i), \quad \textit{and} \quad b = c^2.$$

Then $\neg b \leqslant a \leqslant b$ for all $a \in A$.

Proof. By De Morgan's laws, every element of $\boldsymbol{A}$ has the form $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$ for some term $\varphi(x_1, \dots, x_n)$ in the language $\cdot, \wedge, \neg, e$. The proof of the present theorem is by induction on the complexity $\#\varphi$ of φ. We shall write $\overline{x}$ and $\overline{a}$ for the respective sequences $x_1, \dots, x_n$ and $a_1, \dots, a_n$.

For the case $\#\varphi \leq 1$, note that $e, a_1, \dots, a_n \leqslant c \leqslant b$, by the square-increasing law. Likewise, $f, \neg a_1, \dots, \neg a_n \leqslant c \leqslant b$, so by involution properties, $\neg b \leqslant e, a_1, \dots, a_n$. Now suppose $\#\varphi > 1$ and that $\neg b \leqslant \psi^{\boldsymbol{A}}(\overline{a}) \leqslant b$ for all terms ψ with $\#\psi < \#\varphi$. The desired result, viz.

$$\neg b \leqslant \varphi^{\boldsymbol{A}}(\overline{a}) \leqslant b,$$

follows from the induction hypothesis and basic properties of IRLs if φ has the form $\neg\psi(\overline{x})$ or $\psi_1(\overline{x}) \wedge \psi_2(\overline{x})$. We may therefore assume that φ is $\psi_1(\overline{x}) \cdot \psi_2(\overline{x})$ for some less complex terms $\psi_1(\overline{x}), \psi_2(\overline{x})$.

By the induction hypothesis and (8), $(\neg b)^2 \leqslant \varphi^{\boldsymbol{A}}(\overline{a}) \leqslant b^2$. As $\neg b \leqslant e$, we have $(\neg b)^2 = \neg b$, by (14). And since $f \leqslant c$, Lemma 3.1 gives $c^3 = c^2$, so $b^2 = c^4 = c^2 = b$. Therefore, $\neg b \leqslant \varphi^{\boldsymbol{A}}(\overline{a}) \leqslant b$, as required. $\square$

In a square-increasing IRL, the smallest subalgebra $\boldsymbol{B}$ (generated by $\emptyset$) has top element $(e \vee f)^2 = f^2 \vee e$ (by Theorem 3.2 and (7)). This is a lower bound of $f \to f^2$ (by (2) and Lemma 3.1), so $f^2 \vee e = f \to f^2$. That the extrema of $\boldsymbol{B}$ can be expressed without using $\wedge, \vee$ is implicit in [47, p. 309]. Note also that $e \leftrightarrow f = f \wedge \neg(f^2)$ is the least element of $\boldsymbol{B}$.

An element a of an [I]RL $\boldsymbol{A}$ is said to be *idempotent* if $a^2 = a$. We say that $\boldsymbol{A}$ is *idempotent* if all of its elements are. In the next result, the key implication is (ii) $\Rightarrow$ (iii). A logical analogue of (ii) $\Leftrightarrow$ (iii) is stated without proof in [47, p. 309].

Theorem 3.3. *In a square-increasing IRL $\boldsymbol{A}$, the following are equivalent.*

(i) $f^2 = f$.
(ii) $f \leqslant e$.
(iii) *$\boldsymbol{A}$ is idempotent.*

Consequently, a square-increasing non-idempotent IRL has no idempotent subalgebra (and in particular, no trivial subalgebra).

Proof. In any IRL, (i) $\Rightarrow$ (ii) instantiates (1) (as $\neg f = e$), and (iii) $\Rightarrow$ (i) is trivial.

(ii) $\Rightarrow$ (iii): Suppose $f \leqslant e$, and let $a \in A$. It suffices to show that $a^2 \leqslant a$, or equivalently (by (1)), that $a \cdot \neg a \leqslant \neg a$. Now, by the square-increasing law, (8), the associativity of fusion, (3) and (4),

$$a \cdot \neg a \leqslant a \cdot (\neg a)^2 = (a \cdot (a \to f)) \cdot \neg a \leqslant f \cdot \neg a \leqslant e \cdot \neg a = \neg a. \qquad \square$$

In a partially ordered set, we denote by $[a)$ the set of all upper bounds of an element a (including a itself), and by $(a]$ the set of all lower bounds.

A *deductive filter* of a (possibly involutive) RL $\boldsymbol{A}$ is a lattice filter G of $\langle A; \wedge, \vee\rangle$ that is also a submonoid of $\langle A; \cdot, e\rangle$. Thus, $[e)$ is the smallest deductive filter of $\boldsymbol{A}$. The lattice of deductive filters of $\boldsymbol{A}$ and the congruence lattice $\boldsymbol{Con\,A}$ of $\boldsymbol{A}$ are isomorphic. The isomorphism and its inverse are given by

$$G \mapsto \boldsymbol{\Omega} G := \{\langle a, b\rangle \in A^2 \colon a \to b,\, b \to a \in G\};$$
$$\theta \mapsto \{a \in A \colon \langle a \wedge e, e\rangle \in \theta\}.$$

For a deductive filter G of $\boldsymbol{A}$ and $a, b \in A$, we often abbreviate $\boldsymbol{A}/\boldsymbol{\Omega} G$ as $\boldsymbol{A}/G$, and $a/\boldsymbol{\Omega} G$ as a/G, noting that

$$a \to b \in G \ \text{ iff } \ a/G \leqslant b/G \ \text{ in } \boldsymbol{A}/G. \tag{17}$$

In the square-increasing case, the deductive filters of $\boldsymbol{A}$ are just the lattice filters of $\langle A; \wedge, \vee \rangle$ that contain e, by (13). This yields the following lemma.

Lemma 3.4. *In a square-increasing IRL $\boldsymbol{A}$,*

(i) *if $e \geqslant b \in A$, then $[b)$ is a deductive filter of $\boldsymbol{A}$, e.g.,*
(ii) *$[\neg(f^2))$ is a deductive filter of $\boldsymbol{A}$.*

Here, (ii) follows from (i), because $e \geqslant \neg(f^2)$ follows from $f \leqslant f^2$.

Theorem 3.5. *Let G be a deductive filter of a square-increasing IRL $\boldsymbol{A}$. Then $\boldsymbol{A}/G$ is idempotent iff $\neg(f^2) \in G$. In particular, $\boldsymbol{A}/[\neg(f^2))$ is idempotent.*

Proof. $\boldsymbol{A}/G$ is idempotent iff $f/G \leqslant e/G$ (by Theorem 3.3), iff $f \to e \in G$ (by (17)), iff $\neg(f^2) \in G$ (as $\neg(f^2) = \neg(f \cdot \neg e) = f \to e$). ☐

We say that a square-increasing IRL is *anti-idempotent* if it satisfies $x \leqslant f^2$ (or equivalently, $\neg(f^2) \leqslant x$). This terminology is justified by the corollary below.

Corollary 3.6. *Let K be a variety of square-increasing IRLs. Then K has no nontrivial idempotent member iff it satisfies $x \leqslant f^2$.*

Proof. ($\Rightarrow$): As K is homomorphically closed but lacks nontrivial idempotent members, Theorem 3.5 shows that the deductive filter $[\neg(f^2))$ of any $\boldsymbol{A} \in \mathsf{K}$ coincides with A, i.e., K satisfies $\neg(f^2) \leqslant x$.

($\Leftarrow$): If $\boldsymbol{A} \in \mathsf{K}$ is idempotent, then $f^2 = f \leqslant e = \neg f = \neg(f^2)$, by Theorem 3.3, so by assumption, $\boldsymbol{A}$ is trivial. ☐

Recall that an algebra $\boldsymbol{A}$ is *subdirectly irreducible* (SI) if its identity relation $\mathrm{id}_A = \{\langle a, a \rangle \colon a \in A\}$ is completely meet-irreducible in its congruence lattice; see for instance [5, Thm. 3.23]. If id_A is merely meet-irreducible in $\boldsymbol{Con\,A}$, then $\boldsymbol{A}$ is said to be *finitely subdirectly irreducible* (FSI), whereas $\boldsymbol{A}$ is *simple* if $|\mathit{Con}\,\boldsymbol{A}| = 2$. (Thus, trivial algebras are FSI, but are neither SI nor simple.)

By Birkhoff's Subdirect Decomposition Theorem [5, Thm. 3.24], every algebra is isomorphic to a subdirect product of SI homomorphic images of itself. (Even a trivial algebra is a copy of the direct product of an empty family.) Also, every algebra embeds into an ultraproduct of finitely generated subalgebras of itself [12, Thm. V.2.14]. Consequently, every variety is generated—and thus determined—by its SI finitely generated members, so we need to understand these algebras in the present context. The following result is well known; see [25, Cor. 14] and [52, Thm. 2.4], for instance. Here and subsequently, an RL $\boldsymbol{A}$ is said to be *distributive* if its reduct $\langle A; \wedge, \vee \rangle$ is a distributive lattice.

Lemma 3.7. *Let $\boldsymbol{A}$ be a (possibly involutive) RL.*

(i) *$\boldsymbol{A}$ is FSI iff e is join-irreducible in $\langle A; \wedge, \vee \rangle$. Therefore, subalgebras and ultraproducts of FSI [I]RLs are FSI.*

(ii) *When* $\boldsymbol{A}$ *is distributive, it is FSI iff* e *is join-prime* (*i.e., whenever* $a, b \in A$ *with* $e \leqslant a \vee b$, *then* $e \leqslant a$ *or* $e \leqslant b$).

(iii) *If there is a largest element strictly below* e, *then* $\boldsymbol{A}$ *is SI. The converse holds if* $\boldsymbol{A}$ *is square-increasing.*

(iv) *If* e *has just one strict lower bound, then* $\boldsymbol{A}$ *is simple. The converse holds when* $\boldsymbol{A}$ *is square-increasing.*

An [I]RL is said to be *semilinear* if it is isomorphic to a subdirect product of totally ordered algebras; it is *integral* if e is its greatest element, in which case it satisfies $e = x \to x = x \to e$. A *Brouwerian algebra* is an integral idempotent RL, i.e., an RL satisfying $x \cdot y = x \wedge y$. Such an algebra is determined by its lattice reduct, and is distributive, by (7). The variety of *relative Stone algebras* comprises the semilinear Brouwerian algebras; it is generated by the Brouwerian algebra on the chain of non-negative integers.

4. De Morgan Monoids

Definition 4.1. A *De Morgan monoid* is a distributive square-increasing IRL.[2] The variety of De Morgan monoids shall be denoted by DMM.

The following lemma is well known and should be contrasted with the previous section's concluding remarks about involutionless algebras.

Lemma 4.2. *A De Morgan monoid is integral iff it is a Boolean algebra (in which the operation* $\wedge$ *is duplicated by fusion).*

Proof. Sufficiency is clear. Conversely, by (15) and De Morgan's laws, the fusionless reduct of an integral De Morgan monoid is a complemented (bounded) distributive lattice, i.e., a Boolean algebra, and $\cdot$ is $\wedge$, by (14). $\square$

An algebra is said to be *n–generated* (where n is a cardinal) if it has a generating subset with at most n elements. Thus, an IRL is 0–generated iff it has no proper subalgebra.

Infinite 1–generated De Morgan monoids exist. Indeed, the integer powers of 2, with the usual order and ordinary multiplication as fusion, can be extended to an algebra of this kind. The larger varieties of distributive and of square-increasing IRLs each have infinite 0–generated members as well [64], but Slaney proved that the free 0–generated De Morgan monoid has just 3088 elements [61]. His arguments show that, up to isomorphism, only eight 0–generated De Morgan monoids are FSI; they are exhibited in [62]. As the seven nontrivial 0–generated FSI De Morgan monoids are finite, they are just the 0–generated SI De Morgan monoids.

A theorem of Urquhart [72] implies that the equational theory of DMM is undecidable, whereas results in [10, 36, 48] show that the respective varieties of distributive and of square-increasing IRLs are generated by their finite

[2] But see the first paragraph of Section 7.

members, whence their equational theories are decidable (although, in the square-increasing case, no primitive recursive decision procedure exists [74]).

Recall that a *quasivariety* is a class of similar algebras closed under isomorphic images, subalgebras, direct products and ultraproducts. Equivalently, it is the model class of some set of pure *quasi-equations*

$$(\alpha_1 = \beta_1 \ \& \ \ldots \ \& \ \alpha_n = \beta_n) \implies \alpha = \beta$$

in an algebraic signature. Here $n \in \omega$, i.e., quasi-equations have finite length and encompass equations. Although a quasivariety need not be homomorphically closed (i.e., it need not be a variety), it must contain a trivial algebra, viz. the direct product of its empty subfamily.

4.1. Relevance Logic and De Morgan Monoids.

For present purposes, a *logic* is a substitution-invariant finitary consequence relation $\vdash$ over sentential formulas in an algebraic signature, cf. [8, 13, 19, 20]. The general connections between residuated structures and *substructural logics* are explained in [24]. In the case of De Morgan monoids, the connection is with the older family of *relevance logics* (a.k.a. relevant logics). The monographs and survey articles on this subject include [2, 3, 11, 17, 40, 41, 57, 58, 60]. The correspondence is as follows.

For each subquasivariety K of DMM, there is a logic $\vdash^{\mathsf{K}}$ with the same signature, defined thus: for any set $\Gamma \cup \{\alpha\}$ of formulas, $\Gamma \vdash^{\mathsf{K}} \alpha$ iff there exist $n \in \omega$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that every algebra in K satisfies

$$e \leqslant \gamma_1 \wedge \ldots \wedge \gamma_n \implies e \leqslant \alpha.$$

The elements (also called the *derivable rules*) of $\vdash^{\mathsf{K}}$ are just the pairs Γ/α for which this is true. In particular, the *theorems* of $\vdash^{\mathsf{K}}$ (i.e., the formulas α for which $\emptyset \vdash^{\mathsf{K}} \alpha$) are just the IRL terms that take values in $[e)$ whenever their variables are interpreted in any member of K.

Because DMM satisfies (10), the logic $\vdash^{\mathsf{K}}$ is *algebraizable* in the sense of [8], with K as its unique equivalent quasivariety. The map $\mathsf{K} \mapsto \vdash^{\mathsf{K}}$ is a lattice anti-isomorphism from the subquasivarieties of DMM to the extensions of the relevance logic $\mathbf{R}^{\mathbf{t}}$ of [2], carrying the subvarieties of DMM onto the axiomatic extensions. In particular, $\mathbf{R}^{\mathbf{t}}$ itself is algebraized by DMM.

The relationship between $\mathbf{R}^{\mathbf{t}}$ and DMM was essentially established by Dunn [15] (see his contributions to [2], as well as [49]). Strictly speaking, $\mathbf{R}^{\mathbf{t}}$ denotes a formal system of axioms and inference rules (see [2, pp. 341–343]), not a consequence relation. Here, however, we routinely attribute to a formal system $\mathbf{F}$ the significant properties of its deducibility relation $\vdash_{\mathbf{F}}$. [3]

[3] The general theory of algebraization [8] applies only to consequence relations. This is in contrast with a tradition—prevalent in relevance logic and elsewhere—of identifying a 'logic' with its set of theorems alone, leaving its rules of derivation under-determined in the absence of further qualification. The same tradition privileges *axiomatic* extensions. No serious ambiguity ensues in the case of $\mathbf{R}^{\mathbf{t}}$, as we can recover the whole of $\vdash_{\mathbf{R}^{\mathbf{t}}}$ from its theorems, via the so-called *enthymematic deduction theorem*: $\Gamma, \alpha \vdash_{\mathbf{R}^{\mathbf{t}}} \beta$ iff $\Gamma \vdash_{\mathbf{R}^{\mathbf{t}}} (\alpha \wedge \mathbf{t}) \to \beta$ ([45]).

Although relevance logic has multiple interpretations (see for instance [59, 67, 70, 71, 73]), it was originally intended as a framework in which the so-called paradoxes of material implication could be avoided. These include the *weakening axiom* $p \to (q \to p)$. The unprovability of this postulate in $\mathbf{R^t}$ reflects the fact that De Morgan monoids need not be integral, and Lemma 4.2 says in effect that classical propositional logic is the extension of $\mathbf{R^t}$ by the weakening axiom. Another relevance logic $\mathbf{R}$, and its connection with De Morgan monoids, will be discussed in Section 7.

5. The Structure of De Morgan Monoids

In the relevance logic literature, a De Morgan monoid is said to be *prime* if it is FSI. The reason is Lemma 3.7(ii), but we continue to use 'FSI' here, as it makes sense for arbitrary algebras. The next result is easy and well known, but note that it draws on all the key properties of De Morgan monoids.

Theorem 5.1. *Let $\boldsymbol{A}$ be a De Morgan monoid that is FSI, with $a \in A$. Then $e \leqslant a$ or $a \leqslant f$. Thus, $A = [e) \cup (f\,]$.*

Proof. As $\boldsymbol{A}$ is square-increasing, $e \leqslant a \vee \neg a$, by (15). So, because $\boldsymbol{A}$ is distributive and FSI, $e \leqslant a$ or $e \leqslant \neg a$, by Lemma 3.7(ii). In the latter case, $a \leqslant f$, because $\neg$ is antitone. □

Corollary 5.2. *Let $\boldsymbol{A}$ be a De Morgan monoid that is SI. Let c be the largest element of $\boldsymbol{A}$ strictly below e (which exists, by Lemma 3.7(iii)). Then $c \leqslant f$.*

The following result about bounded De Morgan monoids was essentially proved by Meyer [47, Thm. 3], but his argument assumes that the elements $\bot, \top$ are distinguished, or at least definable in terms of generators. To avoid that presupposition, we give a simpler and more direct proof.

Theorem 5.3. *Let $\boldsymbol{A}$ be a bounded FSI De Morgan monoid. Then $\boldsymbol{A}$ is rigorously compact (see Definition 2.5).*

Proof. Let $\bot \neq a \in A$, where $\bot, \top$ are the extrema of $\boldsymbol{A}$. It suffices to show that $\top \cdot a = \top$. As $e \cdot a \nleqslant \bot$, we have $\top \cdot a \nleqslant f$, by (1), so

$$e \leqslant \top \cdot a, \tag{18}$$

by Theorem 5.1. Recall that $\top^2 = \top$, by Lemma 2.3. Therefore,

$$\top = \top \cdot e \leqslant \top^2 \cdot a \ \text{ (by (18)) } = \top \cdot a \leqslant \top,$$

whence $\top \cdot a = \top$. □

Corollary 5.4. *If a De Morgan monoid is FSI, then its finitely generated subalgebras are rigorously compact.*

Proof. This follows from Lemma 3.7(i) and Theorems 3.2 and 5.3. □

At this point, we need to recall a few concepts and results from universal algebra. The class operator symbols $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathbb{S}}$ and $\mathbb{P}_{\mathbb{U}}$ stand, respectively, for closure under isomorphic and homomorphic images, subalgebras, direct and subdirect products, and ultraproducts. Also, $\mathbb{V}$ and $\mathbb{Q}$ denote varietal and quasivarietal closure, i.e., $\mathbb{V} = \mathbb{HSP}$ and $\mathbb{Q} = \mathbb{ISPP}_{\mathbb{U}}$. We abbreviate $\mathbb{V}(\{\boldsymbol{A}\})$ as $\mathbb{V}(\boldsymbol{A})$, etc.

A variety K is said to be *finitely generated* if $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ for some finite algebra $\boldsymbol{A}$ (or equivalently, $\mathsf{K} = \mathbb{V}(\mathsf{L})$ for some finite set L of finite algebras). Every finitely generated variety is *locally finite*, i.e., its finitely generated members are finite algebras [12, Thm. II.10.16].

Recall that $\mathbb{P}_{\mathbb{U}}(\mathsf{L}) \subseteq \mathbb{I}(\mathsf{L})$ for any finite set L of finite similar algebras. Given a class L of algebras, let us denote by $\mathsf{L}_{\mathrm{FSI}}$ the class of all FSI members of L.

Jónsson's Theorem [33, 35] asserts that, if L is contained in a congruence distributive variety, then $\mathbb{V}(\mathsf{L})_{\mathrm{FSI}} \subseteq \mathbb{HSP}_{\mathbb{U}}(\mathsf{L})$. In particular, if L consists of finitely many finite similar algebras and $\mathbb{V}(\mathsf{L})$ is congruence distributive, then $\mathbb{V}(\mathsf{L})_{\mathrm{FSI}} \subseteq \mathbb{HS}(\mathsf{L})$.

As RLs are congruence distributive, Jónsson's Theorem shows that, whenever L consists of totally ordered [I]RLs, then so does $\mathbb{V}(\mathsf{L})_{\mathrm{FSI}}$, whence $\mathbb{V}(\mathsf{L})$ consists of semilinear algebras. Indeed, since total order is expressible by a universal positive first order sentence, it persists under the operators $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}_{\mathbb{U}}$.

Definition 5.5. A *Sugihara monoid* is an idempotent De Morgan monoid, i.e., an idempotent distributive IRL.

The variety SM of Sugihara monoids is well understood, largely because of Dunn's contributions to [2]; see [16] also. It is locally finite, but not finitely generated. In fact, SM is the smallest variety containing the unique Sugihara monoid

$$\boldsymbol{S}^* = \langle \{a : 0 \neq a \in \mathbb{Z}\}; \cdot, \wedge, \vee, -, 1 \rangle$$

on the set of all nonzero integers such that the lattice order is the usual total order, the involution $-$ is the usual additive inversion, and the term function of $|x| := x \to x$ is the natural absolute value function. In this algebra,

$$a \cdot b \;=\; \begin{cases} \text{the element of } \{a,b\} \text{ with the greater absolute value, if } |a| \neq |b|; \\ a \wedge b \ \text{ if } |a| = |b|, \end{cases}$$

and the residual operation $\to$ is given by

$$a \to b \;=\; \begin{cases} (-a) \vee b & \text{if } a \leqslant b; \\ (-a) \wedge b & \text{if } a \nleqslant b. \end{cases}$$

Note that $e = 1$ and $f = -1$ in $\boldsymbol{S}^*$. The remark before Definition 5.5 yields:

Lemma 5.6. *Every FSI Sugihara monoid is totally ordered. In particular, Sugihara monoids are semilinear.*

Definition 5.7. An IRL $\boldsymbol{A}$ is said to be *odd* if $f = e$ in $\boldsymbol{A}$.

Theorem 5.8. *Every odd De Morgan monoid is a Sugihara monoid.*

Proof. By Theorem 3.3, every square-increasing odd IRL is idempotent. ☐

In the Sugihara monoid $\boldsymbol{S} = \langle \mathbb{Z}; \cdot, \wedge, \vee, -, 0 \rangle$ on the set of *all* integers, the operations are defined like those of $\boldsymbol{S}^*$, except that 0 takes over from 1 as the neutral element for $\cdot$. Both e and f are 0 in $\boldsymbol{S}$, so $\boldsymbol{S}$ is odd. It follows from Theorem 5.8 and Dunn's results in [2, 16] that the variety of all odd Sugihara monoids is the smallest quasivariety containing $\boldsymbol{S}$, and that SM is the smallest quasivariety containing both $\boldsymbol{S}^*$ and $\boldsymbol{S}$.

For each positive integer n, let $\boldsymbol{S}_{2n}$ denote the subalgebra of $\boldsymbol{S}^*$ with universe $\{-n, \dots, -1, 1, \dots, n\}$ and, for $n \in \omega$, let $\boldsymbol{S}_{2n+1}$ be the subalgebra of $\boldsymbol{S}$ with universe $\{-n, \dots, -1, 0, 1, \dots, n\}$. Note that $\boldsymbol{S}_2$ is a Boolean algebra. The results cited above yield:

Theorem 5.9. *Up to isomorphism, the algebras $\boldsymbol{S}_n$ $(1 < n \in \omega)$ are precisely the finitely generated SI Sugihara monoids, whence the algebras $\boldsymbol{S}_{2n+1}$ $(0 < n \in \omega)$ are just the finitely generated SI odd Sugihara monoids.*

We cannot embed $\boldsymbol{S}$ (nor even $\boldsymbol{S}_{2n+1}$) into $\boldsymbol{S}^*$, owing to the involution. Nevertheless, $\boldsymbol{S}$ is a homomorphic image of $\boldsymbol{S}^*$, and $\boldsymbol{S}_{2n+1}$ is a homomorphic image of $\boldsymbol{S}_{2n+2}$, for all $n \in \omega$. In each case, the kernel of the homomorphism identifies -1 with 1; it identifies no other pair of distinct elements. Also, $\boldsymbol{S}_{2n-1}$ is a homomorphic image of $\boldsymbol{S}_{2n+1}$ if $n > 0$; in this case the kernel collapses $-1, 0, 1$ to a point, while isolating all other elements. Thus, $\boldsymbol{S}_3$ is a homomorphic image of $\boldsymbol{S}_n$ for all $n \geq 3$. In particular, every nontrivial variety of Sugihara monoids includes $\boldsymbol{S}_2$ or $\boldsymbol{S}_3$.

Corollary 5.10. *The lattice of varieties of odd Sugihara monoids is the following chain of order type $\omega + 1$:*

$$\mathbb{V}(\boldsymbol{S}_1) \subsetneq \mathbb{V}(\boldsymbol{S}_3) \subsetneq \mathbb{V}(\boldsymbol{S}_5) \subsetneq \dots \subsetneq \mathbb{V}(\boldsymbol{S}_{2n+1}) \subsetneq \dots \subsetneq \mathbb{V}(\boldsymbol{S}).$$

Proof. See [2, Sec. 29.4] or [27, Fact 7.6]. ☐

Odd Sugihara monoids are categorically equivalent to relative Stone algebras [27, Thm. 5.8]. The equivalence sends an odd Sugihara monoid to the set of lower bounds of its neutral element e, redefining residuation as $(x \to y) \wedge e$ and restricting the other RL-operations, as well as all morphisms. An analogous but more complex result for arbitrary Sugihara monoids is proved in [28, Thm. 10.5] and refined in [23, Thm. 2.24]. The subvariety lattice of SM is fully described in [39]. Every quasivariety of odd Sugihara monoids is a variety [27, Thm. 7.3]. (For a stronger result, see [52, Thm. 9.4].)

As the structure of Sugihara monoids is very transparent, we concentrate now on De Morgan monoids that are *not* idempotent.

Lemma 5.11. *Let $\boldsymbol{A}$ be a non-idempotent FSI De Morgan monoid, and let a be an idempotent element of $\boldsymbol{A}$. If $a \geqslant f$, then $a > e$. In particular, $f^2 > e$.*

Proof. Suppose $a^2 = a \geqslant f$. As $\boldsymbol{A}$ is not idempotent, $f^2 \neq f$, by Theorem 3.3, so $a \neq f$. Therefore, $a \nleqslant f$, whence $e \leqslant a$, by Theorem 5.1. As $f \leqslant a$, we cannot have $a = e$, by Theorem 3.3, so $e < a$. The last claim follows because f^2 is an idempotent upper bound of f (by Lemma 3.1). $\square$

Theorem 5.12. *Let G be a deductive filter of a non-idempotent FSI De Morgan monoid $\boldsymbol{A}$, and suppose $\neg(f^2) \in G$. Then $\boldsymbol{A}/G$ is an odd Sugihara monoid.*

Proof. By Theorems 3.5 and 3.3, $\boldsymbol{A}/G$ is idempotent and $f/G \leqslant e/G$. By Lemma 5.11, $f^2 > e$, i.e., $\neg(f^2) < f$, whence $f \in G$, i.e., $e \to f \in G$. Then $e/G \leqslant f/G$ (by (17)), so $e/G = f/G$, as required. $\square$

Lemma 5.13. *Let $\boldsymbol{A}$ be a De Morgan monoid that is FSI, with $f \leqslant a, b \in A$, where a and b are idempotent. Then $a \leqslant b$ or $b \leqslant a$.*

Proof. If $\boldsymbol{A}$ is a Sugihara monoid, the result follows from Lemma 5.6. We may therefore assume that $\boldsymbol{A}$ is not idempotent, so $e < a, b$, by Lemma 5.11. Then $a \cdot \neg a = \neg a$ and $b \cdot \neg b = \neg b$, by Lemma 2.6, so

$$\begin{aligned}(a \cdot \neg b) \wedge (b \cdot \neg a) &\leqslant (a \cdot \neg b) \cdot (b \cdot \neg a) \quad \text{(by (13))}\\ &= (a \cdot \neg a) \cdot (b \cdot \neg b) = \neg a \cdot \neg b \quad \text{(by the above)}\\ &= \neg a \wedge \neg b \quad \text{(by (14), as } \neg a, \neg b \leqslant e).\end{aligned}$$

Therefore, by De Morgan's laws,

$$\begin{aligned}\neg(\neg a \wedge \neg b) &\leqslant \neg((a \cdot \neg b) \wedge (b \cdot \neg a))\\ &= \neg(a \cdot \neg b) \vee \neg(b \cdot \neg a) = (a \to b) \vee (b \to a)\end{aligned}$$

and $e < a \vee b = \neg(\neg a \wedge \neg b)$, so $e < (a \to b) \vee (b \to a)$. Then, since $\boldsymbol{A}$ is FSI, Lemma 3.7(ii) and (9) yield $e \leqslant a \to b$ or $e \leqslant b \to a$, i.e., $a \leqslant b$ or $b \leqslant a$. $\square$

The subalgebra of an algebra $\boldsymbol{A}$ generated by a subset X of A shall be denoted by $\boldsymbol{Sg}^{\boldsymbol{A}}X$.

Lemma 5.14. *Let $\boldsymbol{A}$ be a De Morgan monoid that is FSI, and let $f \leqslant a \in A$, where $a \nless f^2$. Then a is idempotent.*

Proof. By Lemma 3.1, f^2 is idempotent, so assume that $a \neq f^2$. From $f \leqslant f^2$ and $a \nleqslant f^2$, we infer $a \nleqslant f$. Then $e \leqslant a$, by Theorem 5.1, so $e, f \in [\neg a, a] := \{b \in A : \neg a \leqslant b \leqslant a\}$. Therefore, $\neg(a^2) \leqslant x \leqslant a^2$ for all $x \in Sg^{\boldsymbol{A}}\{a\}$, by Theorem 3.2. By Corollary 5.4, $\boldsymbol{Sg}^{\boldsymbol{A}}\{a\}$ is rigorously compact. In particular,

$$a^2 \cdot x = a^2 \text{ whenever } \neg(a^2) < x \in Sg^{\boldsymbol{A}}\{a\}. \tag{19}$$

As $a \leqslant a^2$ and $a \nleqslant f^2$, we have $a^2 \nleqslant f^2$. But a^2 and f^2 are idempotent, by Lemma 3.1, so $f^2 < a^2$, by Lemma 5.13. Thus, $\neg(a^2) < \neg(f^2) \in Sg^{\boldsymbol{A}}\{a\}$, so

$$a^2 = a^2 \cdot \neg(f^2), \tag{20}$$

by (19). As $\boldsymbol{A}/[\neg(f^2))$ is idempotent (by Theorem 3.5), $\neg(f^2) \leqslant a^2 \to a$, i.e., $a^2 \cdot \neg(f^2) \leqslant a$, by (17) and (2). Then (20) gives $a^2 \leqslant a$, and so $a^2 = a$. $\square$

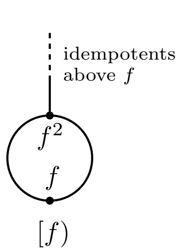
Theorem 5.15. *Let $\boldsymbol{A}$ be a non-idempotent FSI De Morgan monoid, with $f^2 \leqslant a \in A$. Then $\neg a < a$ and the interval $[\neg a, a]$ is a subuniverse (i.e., the universe of a subalgebra) of $\boldsymbol{A}$. In particular, $[\neg(f^2), f^2]$ is a subuniverse of $\boldsymbol{A}$.*

Proof. In $\boldsymbol{A}$, we have $\neg(f^2) \leqslant e$, as noted after Lemma 3.4, while $e < f^2$, by Lemma 5.11. Of course, $\neg a \leqslant \neg(f^2)$, so $\neg a < a$. Thus, $[\neg a, a]$ includes e, and it is obviously closed under $\wedge$, $\vee$ and $\neg$. Closure under fusion follows from (8) and the square-increasing law, because a is idempotent (by Lemma 5.14). $\square$

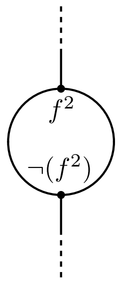
Theorem 5.16. *In any FSI De Morgan monoid, the filter $[f)$ is the union of the interval $[f, f^2]$ and a chain whose least element is f^2. The elements of this chain are just the idempotent upper bounds of f.*

Proof. This follows from Lemma 5.6 when the algebra is idempotent. In the opposite case, the idempotent upper bounds of f are exactly the upper bounds of f^2 (by (8) and Lemma 5.14), and they are comparable with all upper bounds of f (by Lemmas 5.14 and 5.13). $\square$

5.16:



5.17:



Theorem 5.17. *Any non-idempotent FSI De Morgan monoid is the union of the interval subuniverse $[\neg(f^2), f^2]$ and two chains of idempotents, $(\neg(f^2)]$ and $[f^2)$.*

Proof. Let $\boldsymbol{A}$ be a non-idempotent FSI De Morgan monoid. Theorem 5.15 shows that $e, f \in [\neg(f^2), f^2]$ and (with Lemma 2.3) that $\neg(f^2) \cdot f = \neg(f^2)$. Note that $[f^2)$ and $(\neg(f^2)]$ are both chains of idempotents, by Theorem 5.16, involution properties and (14).

Suppose, with a view to contradiction, that there exists $a \in A$ such that $a \notin (\neg(f^2)] \cup [\neg(f^2), f^2] \cup [f^2)$. By Theorem 5.1, $e < a$ or $a < f$. By involutional symmetry, we may assume that $e < a$. Then a is incomparable with f^2 (as $a \notin [\neg(f^2), f^2] \cup [f^2)$), so $f^2 \vee a > f^2$. Also, since $f^2, a \geqslant e$, we have $f^2 \cdot a \geqslant f^2 \vee a$, by (8), so $f^2 \cdot a > f^2$.

Because $a > e$, we have $f \cdot a \geqslant f$. If $f \cdot a \in [\neg(f^2), f^2]$, then

$$f^2 \cdot a \leqslant (f \cdot a)^2 \leqslant f^4 = f^2 \quad \text{(by Lemma 3.1)},$$

a contradiction. So, by Theorem 5.16, $f \cdot a$ is idempotent and $f \cdot a > f^2$. Then $f \cdot a > e, f$, and by Theorem 5.15, $\neg(f \cdot a) < f \cdot a$. This, with Theorem 3.2, shows that $f \cdot a$ is the greatest element of the algebra $\boldsymbol{C} := \boldsymbol{Sg}^{\boldsymbol{A}}\{f \cdot a\}$, and $\neg(f \cdot a)$ is the least element of $\boldsymbol{C}$. Note that $\neg(f \cdot a) < \neg(f^2)$, as $f^2 < f \cdot a$. Now $\boldsymbol{C}$ is rigorously compact, by Corollary 5.4, so $\neg(f^2) \cdot (f \cdot a) = f \cdot a > f^2$. Thus, $\neg(f^2) \cdot (f \cdot a) \nleqslant a$, as $f^2 \nleqslant a$.

Nevertheless, as $\neg(f^2) \cdot f = \neg(f^2)$, we have $(\neg(f^2) \cdot f) \cdot a = \neg(f^2) \cdot a \leqslant a$, because $\neg(f^2) \leqslant e$. This contradicts the associativity of fusion in $\boldsymbol{A}$. Therefore, $A = (\neg(f^2)] \cup [\neg(f^2), f^2] \cup [f^2)$. $\square$

Recall from (14) that fusion and meet coincide on the lower bounds of e in any De Morgan monoid. For the algebras in Theorem 5.17, the behaviour of fusion is further constrained as follows.

Theorem 5.18. *Let $\boldsymbol{A}$ be a non-idempotent FSI De Morgan monoid, and let $f \leqslant a, b \in A$. Then*

$$a \cdot b \;=\; \begin{cases} f^2 & \text{if } a, b \leqslant f^2; \\ \max_{\leqslant}\{a, b\} & \text{otherwise.} \end{cases}$$

If, moreover, $a < b$ and $f^2 \leqslant b$, then $a \cdot \neg b = \neg b = b \cdot \neg b$ and $b \cdot \neg a = b$.

Proof. If $a, b \leqslant f^2$, then $f^2 \leqslant a \cdot b \leqslant f^4 = f^2$, by (8) and Lemma 3.1, so $a \cdot b = f^2$. We may therefore assume (in respect of the first claim) that $a \nleqslant f^2$ or $b \nleqslant f^2$. Then a and b are comparable, by Theorem 5.16. By symmetry, we may assume that $a \leqslant b$ and hence that $b \nleqslant f^2$, so $e < f^2 < b = b^2$, by Theorems 5.15 and 5.16.

If $a = b$, then $a \cdot b = b^2 = b = \max_{\leqslant}\{a, b\}$, so we may assume that $a \neq b$. Thus, $b > a \geqslant f$, and so $\neg b < \neg a \leqslant e < b$.

As b is an idempotent upper bound of $e, f, a, \neg a, \neg b$, Theorem 3.2 shows that b is the greatest element of $\boldsymbol{Sg}^{\boldsymbol{A}}\{a, b\}$, and $\neg b$ is the least element.

By Corollary 5.4, $\boldsymbol{Sg}^{\boldsymbol{A}}\{a, b\}$ is rigorously compact. We shall therefore have $a \cdot b = b = \max_{\leqslant}\{a, b\}$, provided that $\neg b \neq a$. This is indeed the case, as we have seen that $\neg a < b$.

Finally, suppose $a < b$ and $f^2 \leqslant b$. Again, Theorems 5.15 and 5.16 show that $\neg b, b$ are the (idempotent) extrema of the algebra $\boldsymbol{Sg}^{\boldsymbol{A}}\{a, b\}$, whose non-extreme elements include $\neg a, a$, so the remaining claims also follow from the rigorous compactness of $\boldsymbol{Sg}^{\boldsymbol{A}}\{a, b\}$. $\square$

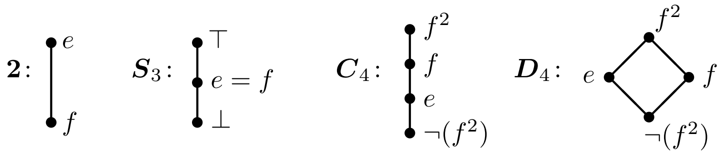
Remark 5.19. The foregoing results imply that, for an FSI De Morgan monoid $\boldsymbol{A}$, there are just two possibilities.

The first is that $f < e$, in which case, by Theorems 3.3 and 5.1 and Lemmas 3.7(iii) and 5.6, $\boldsymbol{A}$ is a totally ordered SI Sugihara monoid whose fusion resembles that of $\boldsymbol{S}^*$, because the latter operation is definable by universal first order sentences, and because $\boldsymbol{A} \in \mathbb{ISP}_{\mathbb{U}}(\boldsymbol{S}^*)$. (See the remarks preceding Definition 5.5 and recall that the absolute value function on $\boldsymbol{S}^*$ is the term function of $x \to x$.) The improvement here on $\boldsymbol{A} \in \mathbb{HSP}_{\mathbb{U}}(\boldsymbol{S}^*)$ is due to the assumption $f < e$. Indeed, a nontrivial congruence on any $\boldsymbol{B} \in \mathbb{SP}_{\mathbb{U}}(\boldsymbol{S}^*)$ must identify f with e, because e covers f in $\boldsymbol{S}^*$, and therefore in $\boldsymbol{B}$.

The second possibility is that $\boldsymbol{A}$ is the 'rigorous extension' of its anti-idempotent subalgebra (on $[\neg(f^2), f^2]$) by an (idempotent) totally ordered odd Sugihara monoid. More precisely, in this case, if $\theta = \boldsymbol{\Omega}[\neg(f^2))$, then $\boldsymbol{A}/\theta$ is a totally ordered odd Sugihara monoid (and is therefore determined by its $e, \leqslant$ reduct), while $[\neg(f^2), f^2]$ is the congruence class e/θ and no two distinct non-elements of $[\neg(f^2), f^2]$ are identified by θ (an easy consequence of Theorem 5.18). Thus, when $\neg(f^2)$ and f^2 are identified in $(\neg(f^2)] \cup [f^2)$, we obtain a copy of $\langle A/\theta; \leqslant\rangle$. Both $\boldsymbol{A}/\theta$ and the algebra on $[\neg(f^2), f^2]$ are FSI, by Lemma 3.7(i), and may be trivial. By the last assertion of Theorem 3.3, $\boldsymbol{A}/\theta$ is not a retract of $\boldsymbol{A}$, unless $\boldsymbol{A}$ is odd (i.e., $\theta = \mathrm{id}_A$). There is no further constraint on $[\neg(f^2), f^2]$, while the $e, \leqslant$ reduct of $\boldsymbol{A}/\theta$ may be any chain with a self-inverting antitone bijection, having a fixed point. In fact, $\boldsymbol{A}$ is a directed union of algebras, each of which results from $[\neg(f^2), f^2]$ by taking a rigorously compact two-point extension finitely many times.

This largely reduces the study of irreducible De Morgan monoids to the anti-idempotent case. $\square$

We depict below the two-element Boolean algebra $\mathbf{2}$ ($= \boldsymbol{S}_2$), the three-element Sugihara monoid $\boldsymbol{S}_3$, and two 0–generated four-element De Morgan monoids, $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$. In each case, the labeled Hasse diagram determines the structure, in view of Lemma 2.3, Theorem 5.3 and the definitions. That $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$ are indeed De Morgan monoids was noted long ago in the relevance logic literature, e.g., [46, 47]. All four algebras are simple, by Lemma 3.7(iv).



The next theorem is implicit in the findings of Slaney [61, 62] mentioned after Lemma 4.2, but it is easier here to give a self-contained proof.

Theorem 5.20. *Let* $\boldsymbol{A}$ *be a simple* 0*–generated De Morgan monoid. Then* $\boldsymbol{A} \cong \mathbf{2}$ *or* $\boldsymbol{A} \cong \boldsymbol{C}_4$ *or* $\boldsymbol{A} \cong \boldsymbol{D}_4$.

Proof. Because $\boldsymbol{A}$ is simple (hence nontrivial) and 0–generated, $\{e\}$ is not a subuniverse of $\boldsymbol{A}$, so $e \neq f$ and e has just one strict lower bound in $\boldsymbol{A}$ (Lemma 3.7(iv)). Suppose $\boldsymbol{A} \ncong \mathbf{2}$. As every simple Boolean algebra is isomorphic to $\mathbf{2}$, Lemma 4.2 shows that $\boldsymbol{A}$ is not integral. Equivalently, f is not the least element of $\boldsymbol{A}$, so $f \nleqslant e$. Then by Theorem 3.3, $\boldsymbol{A}$ is not idempotent and $f < f^2$, hence $\neg(f^2) < e$, so $\neg(f^2)$ is the least element of $\boldsymbol{A}$, i.e., f^2 is the greatest element. Consequently, $a \cdot \neg(f^2) = \neg(f^2)$ for all $a \in A$, by Lemma 2.3, and $a \cdot f^2 = f^2$ whenever $\neg(f^2) \neq a \in A$, by Theorem 5.3.

There are two possibilities for the order: $e < f$ or $e \nleqslant f$. If $e \nleqslant f$, then $e \wedge f < e$, whence $e \wedge f$ is the extremum $\neg(f^2)$ and, by De Morgan's laws, $e \vee f = f^2$. Otherwise, $\neg(f^2) < e < f < f^2$. Either way, $\{\neg(f^2), e, f, f^2\}$ is the universe of a four-element subalgebra of $\boldsymbol{A}$, having no proper subalgebra of its own, so $A = \{\neg(f^2), e, f, f^2\}$, as $\boldsymbol{A}$ is 0–generated. Thus, $\boldsymbol{A} \cong \boldsymbol{C}_4$ if $e < f$, and $\boldsymbol{A} \cong \boldsymbol{D}_4$ if $e \nleqslant f$. □

We remark that both $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$ are categorically equivalent to the variety $\mathbb{V}(\mathbf{2})$ of all Boolean algebras. (Equivalently, $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$ are primal algebras, as they generate arithmetical varieties and are finite, simple and lack proper subalgebras and nontrivial automorphisms; see [22, 32, 42].)

Theorem 5.21. *A variety* K *of De Morgan monoids consists of Sugihara monoids iff it excludes* $\boldsymbol{C}_4$ *and* $\boldsymbol{D}_4$.

Proof. Necessity is clear. Conversely, suppose $\boldsymbol{C}_4, \boldsymbol{D}_4 \notin \mathsf{K}$ and let $\boldsymbol{A} \in \mathsf{K}$ be SI. It suffices to show that $\boldsymbol{A}$ is a Sugihara monoid. Suppose not. Then, by Theorem 5.15 and Remark 5.19, $\neg(f^2) < f^2$ and the subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ on $[\neg(f^2), f^2]$ is nontrivial, whence the 0–generated subalgebra $\boldsymbol{E}$ of $\boldsymbol{A}$ is nontrivial. Recall that every nontrivial finitely generated algebra of finite type has a simple homomorphic image [34, Cor. 4.1.13]. Let $\boldsymbol{G}$ be a simple homomorphic image of $\boldsymbol{E}$, so $\boldsymbol{G} \in \mathsf{K}$. By assumption, neither $\boldsymbol{C}_4$ nor $\boldsymbol{D}_4$ is isomorphic to $\boldsymbol{G}$, but $\boldsymbol{G}$ is 0–generated, so $\mathbf{2} \cong \boldsymbol{G}$, by Theorem 5.20. Thus, $\mathbf{2} \in \mathbb{HS}(\boldsymbol{B})$. Then $\mathbf{2}$ must inherit from $\boldsymbol{B}$ the anti-idempotent identity $x \leqslant f^2$. This is false, however, so $\boldsymbol{A}$ is a Sugihara monoid. □

In what follows, some features of $\boldsymbol{C}_4$ will be important.

Lemma 5.22. *Let* $\boldsymbol{A}$ *be a nontrivial square-increasing IRL.*

(i) *If* $e \leqslant f$ *and* $a \leqslant f^2$ *for all* $a \in A$, *then* $e < f$.

(ii) *If* $e < f$ *in* $\boldsymbol{A}$, *then* $\boldsymbol{C}_4$ *can be embedded into* $\boldsymbol{A}$.

Proof. (i) Suppose $\boldsymbol{A}$ satisfies $e \leqslant f$ and $x \leqslant f^2$. Then $\boldsymbol{A}$ is not idempotent, by Corollary 3.6, so $f \neq e$, by Theorem 3.3, i.e., $e < f$.

(ii) Suppose $e < f$ in $\boldsymbol{A}$. Then $f < f^2$, by Theorem 3.3, i.e., $\neg(f^2) < e$. Thus, $\{\neg(f^2), e, f, f^2\}$ is closed under $\wedge, \vee$ and $\neg$, and $\neg(f^2)$ is idempotent, by (14). By Lemma 3.1, f^2 is an idempotent upper bound of e, so $f^2 \cdot \neg(f^2) = \neg(f^2)$, by Lemma 2.6. Closure of $\{\neg(f^2), e, f, f^2\}$ under fusion follows from these observations and (8), so $\boldsymbol{C}_4$ embeds into $\boldsymbol{A}$. □

Theorem 5.23. (Slaney [62, Thm. 1]) *Let* $h \colon \boldsymbol{A} \to \boldsymbol{B}$ *be a homomorphism, where* $\boldsymbol{A}$ *is an FSI De Morgan monoid, and* $\boldsymbol{B}$ *is nontrivial and* 0*–generated. Then* h *is an isomorphism or* $\boldsymbol{B} \cong \boldsymbol{C}_4$.

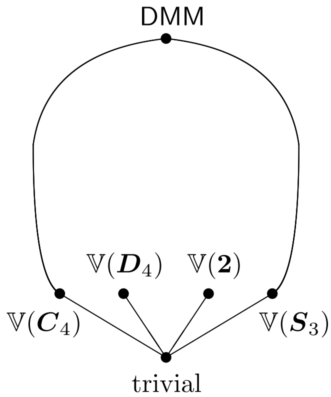
Proof. As $\boldsymbol{B}$ is 0–generated, h is surjective. Suppose h is not an isomorphism. By the remarks preceding Lemma 2.3, $h(a) = e$ for some $a \in A$ with $a < e$. By Theorem 5.1, $a \leqslant f$, so $h(a) \leqslant h(f)$, i.e., $e \leqslant f$ in $\boldsymbol{B}$. As $\boldsymbol{B}$ is 0–generated but not trivial, it cannot satisfy $e = f$, so $e < f$ in $\boldsymbol{B}$. Then $\boldsymbol{C}_4$ embeds into $\boldsymbol{B}$, by Lemma 5.22(ii), so $\boldsymbol{B} \cong \boldsymbol{C}_4$, again because $\boldsymbol{B}$ is 0–generated. □

6. Minimality

A quasivariety is said to be *minimal* if it is nontrivial and has no nontrivial proper subquasivariety. If we say that a variety is *minimal* (without further qualification), we mean that it is nontrivial and has no nontrivial proper subvariety. When we mean instead that it is *minimal as a quasivariety*, we shall say so explicitly, thereby avoiding ambiguity.

Theorem 6.1. *The distinct classes* $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{S}_3)$, $\mathbb{V}(\boldsymbol{C}_4)$ *and* $\mathbb{V}(\boldsymbol{D}_4)$ *are precisely the minimal varieties of De Morgan monoids.*

Proof. Each $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$ is finite and simple, with no proper nontrivial subalgebra, so the nontrivial members of $\mathbb{HS}(\boldsymbol{X})$ are isomorphic to $\boldsymbol{X}$. Thus, the SI members of $\mathbb{V}(\boldsymbol{X})$ belong to $\mathbb{I}(\boldsymbol{X})$, by Jónsson's Theorem, because DMM is a congruence distributive variety. As varieties are determined by their SI members, this shows that $\mathbb{V}(\boldsymbol{X})$ has no proper nontrivial subvariety, and that $\mathbb{V}(\boldsymbol{X}) \neq \mathbb{V}(\boldsymbol{Y})$ for distinct $\boldsymbol{X}, \boldsymbol{Y} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$. As $\mathbb{V}(\mathbf{2})$ and $\mathbb{V}(\boldsymbol{S}_3)$ are the only minimal varieties of Sugihara monoids, Theorem 5.21 shows that they, together with $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$, are the only minimal subvarieties of DMM. □



Bergman and McKenzie [6] showed that every locally finite congruence modular minimal variety is also minimal as a quasivariety. Thus, by Theorem 6.1, $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{S}_3)$, $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$ are minimal as quasivarieties. (In a sequel paper [51], we show that DMM has just 68 minimal subquasivarieties.) With a view to axiomatizing the varieties in Theorem 6.1, consider the following (abbreviated) equations.

$$e \leqslant (x \to y) \vee (y \to x) \tag{21}$$

$$e \leqslant (x \to (y \vee \neg y)) \vee (y \wedge \neg y) \tag{22}$$

$$e \leqslant (f^2 \to x) \vee (x \to e) \vee \neg x \tag{23}$$

$$x \wedge (x \to f) \leqslant (f \to x) \vee (x \to e) \tag{24}$$

$$x \to e \leqslant x \vee (f^2 \to \neg x) \tag{25}$$

It is shown in [30] that an [I]RL $\boldsymbol{A}$ is semilinear (i.e., a subdirect product of chains) iff it is distributive and satisfies (21).

Theorem 6.2.

(i) $\mathbb{V}(\mathbf{2})$ *is axiomatized by adding* $x \leqslant e$ *to the axioms of* DMM;
(ii) $\mathbb{V}(\boldsymbol{S}_3)$ *by adding* $e = f$, (21) *and* (22);
(iii) $\mathbb{V}(\boldsymbol{D}_4)$ *by adding* $x \leqslant f^2$, $x \wedge \neg x \leqslant y$ *and* (23);
(iv) $\mathbb{V}(\boldsymbol{C}_4)$ *by adding* $x \leqslant f^2$, $e \leqslant f$, (21), (24) *and* (25).[4]

Proof. Let $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$. It can be verified mechanically that $\boldsymbol{X}$ satisfies the proposed axioms for $\mathbb{V}(\boldsymbol{X})$. Let $\boldsymbol{A}$ be an SI De Morgan monoid satisfying the same axioms, and let a be the largest element of $\boldsymbol{A}$ strictly below e, which exists by Lemma 3.7(iii). By involution properties, $\neg a$ is the smallest element of $\boldsymbol{A}$ strictly above f. It suffices to show that $\boldsymbol{A} \cong \boldsymbol{X}$.

When $\boldsymbol{X}$ is $\mathbf{2}$, this follows from Lemma 4.2, as every SI Boolean algebra is isomorphic to $\mathbf{2}$.

If $\boldsymbol{X}$ is $\boldsymbol{S}_3$ or $\boldsymbol{C}_4$, then $\boldsymbol{A}$ is totally ordered (because it is semilinear, by (21), and SI).

Suppose that $\boldsymbol{X} = \boldsymbol{S}_3$. In $\boldsymbol{A}$, since $e = f$, we have $a < e < \neg a$, and there is no other element in the interval $[a, \neg a]$. We claim, moreover, that $\neg a$ has no strict upper bound in $\boldsymbol{A}$. Suppose, on the contrary, that $\neg a < b \in A$. By (22) and since e is join-prime (Lemma 3.7(ii)), we have $e \leqslant b \to (a \vee \neg a)$ or $e \leqslant a \wedge \neg a$. But $a \wedge \neg a = a < e$, so by (9), $b \leqslant a \vee \neg a = \neg a$, a contradiction. This vindicates the above claim. By involutional symmetry, a has no strict lower bound in $\boldsymbol{A}$. As $\boldsymbol{A}$ is totally ordered, this shows that $A = \{a, e, \neg a\}$. Then $\boldsymbol{A} \cong \boldsymbol{S}_3$, in view of Lemma 2.3.

We may now assume that $\boldsymbol{X}$ is $\boldsymbol{C}_4$ or $\boldsymbol{D}_4$, so $\boldsymbol{A}$ satisfies $\neg(f^2) \leqslant x \leqslant f^2$ and is therefore rigorously compact (Theorem 5.3) and not idempotent (Corollary 3.6), whence $f < f^2$ and $f \nleqslant e$ in $\boldsymbol{A}$ (Theorem 3.3).

Suppose $\boldsymbol{X} = \boldsymbol{D_4}$. By assumption, $b \wedge \neg b = \neg(f^2)$ for any $b \in A$. If $e < f$, then $e = e \wedge f = \neg(f^2)$, i.e., e is the bottom element of $\boldsymbol{A}$, forcing $\boldsymbol{A}$ to be trivial (see the remarks before Lemma 2.3). This contradiction shows that e and f are incomparable in $\boldsymbol{A}$.

As a is the greatest strict lower bound of e, we now have $a < f$, by Corollary 5.2. Then $a = e \wedge f = \neg(f^2)$ and, by involution properties, no element lies strictly between f and f^2. Suppose $b \in A$, with $\neg(f^2) < b < f$. By (23),

$$e \leqslant (f^2 \to \neg b) \vee (\neg b \to e) \vee b.$$

Since $\boldsymbol{A}$ is rigorously compact and $\neg b \neq f^2$, we have $f^2 \to \neg b = \neg(f^2)$. So, because e is join-prime, $e \leqslant \neg b \to e$ or $e \leqslant b$. The last disjunct is

[4] Of course, (i) is well known. We have not encountered (ii)–(iv) in the literature, but a variant of (ii) could be derived from [16, Cor. 2].

false, for otherwise $e \leqslant b < f$. Therefore, $\neg b \leqslant e$, i.e., $f \leqslant b$, contrary to assumption. Thus, no element of $\boldsymbol{A}$ lies strictly between $\neg(f^2)$ and f and, by involution properties, no element lies strictly between e and f^2. It follows that $A = \{\neg(f^2), e, f, f^2\}$, in view of Theorem 5.1. In this case, $\boldsymbol{A} \cong \boldsymbol{D}_4$.

Lastly, suppose $\boldsymbol{X} = \boldsymbol{C}_4$. Note that $\boldsymbol{C}_4$ embeds into $\boldsymbol{A}$, by Lemma 5.22. As $a < e$, it follows from (25) that

$$e \leqslant a \to e \leqslant a \vee (f^2 \to \neg a),$$

but e is join-prime and $e \nleqslant a$, so $f^2 \leqslant \neg a$, whence $a = \neg(f^2)$. Thus, no element of $\boldsymbol{A}$ lies strictly between $\neg(f^2)$ and e, nor strictly between f and f^2.

Suppose, with a view to contradiction, that $b \in A \setminus \{\neg(f^2), e, f, f^2\}$. By the previous paragraph and since $\boldsymbol{A}$ is totally ordered, $e < b < f$. Then $e \leqslant b \to f$, so by (24),

$$e \leqslant b \wedge (b \to f) \leqslant (f \to b) \vee (b \to e).$$

Now join-primeness of e gives $f \leqslant b$ or $b \leqslant e$, a contradiction, so $\boldsymbol{A} \cong \boldsymbol{C}_4$. $\square$

Theorem 6.1 says, in effect, that for each axiomatic consistent extension $\mathbf{L}$ of $\mathbf{R^t}$, there exists $\boldsymbol{B} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$ such that the theorems of $\mathbf{L}$ all take values $\geqslant e$ on any evaluation of their variables in $\boldsymbol{B}$. Postulates for the four maximal consistent axiomatic extensions of $\mathbf{R^t}$ follow systematically from Theorem 6.2. For example, (21) becomes the axiom $(p \to q) \vee (q \to p)$, while (25) becomes $(p \to \mathbf{t}) \to (p \vee (\mathbf{f}^2 \to \neg p))$.

7. Relevant Algebras

The relevance logic literature is equivocal as to the precise definition of a De Morgan monoid. Our Definition 4.1 conforms with Dunn and Restall [17], Meyer and Routley [50, 59], Slaney [61] and Urquhart [72], yet other papers by some of the same authors entertain a discrepancy. In all sources, the neutral element of a De Morgan monoid $\boldsymbol{A}$ is assumed to exist but, in [62, 63, 64] for instance, it is not distinguished, i.e., the symbol for e (and likewise f) is absent from the signature of $\boldsymbol{A}$. That locally innocuous convention has global effects: it would prevent DMM from being a variety, as it would cease to be closed under subalgebras, and the tight correspondence between axiomatic extensions of $\mathbf{R^t}$ and subvarieties of DMM would disappear.[5]

This may explain why we have found in the literature no analysis of the subvariety lattice of DMM (despite interest in the problem discernable in [46, 47]), and in particular no statement of Theorem 6.1, identifying the only four maximal consistent axiomatic extensions of $\mathbf{R^t}$ (although the algebras defining these extensions were well known to relevance logicians).

[5] The meanings of statements about 'n–generated De Morgan monoids' would also change. For instance, [62, Thm. 5] says that every FSI De Morgan monoid on one idempotent generator is finite, but this is false when e is distinguished, as the proof of [62, Thm. 6] makes clear.

The practice of not distinguishing neutral elements stems from the formal system $\mathbf{R}$ of Anderson and Belnap [2], which differs from $\mathbf{R}^{\mathbf{t}}$ only in that it lacks the sentential constant $\mathbf{t}$ (corresponding to e) and its postulates. The omission of constants from $\mathbf{R}$ produces a desirable *variable sharing principle* for 'relevant' implication:

$$\text{if } \vdash_{\mathbf{R}} \alpha \to \beta, \text{ then } \alpha \text{ and } \beta \text{ have a common variable [4].}$$

The corresponding claim for $\mathbf{R}^{\mathbf{t}}$ is false, e.g.,

$$\vdash_{\mathbf{R}^{\mathbf{t}}} \mathbf{t} \to (p \to p) \text{ and } \vdash_{\mathbf{R}^{\mathbf{t}}} (p \wedge \mathbf{t}) \to (\mathbf{t} \vee q).$$

Definition 7.1. A *relevant algebra* is an algebra $\langle A; \cdot, \wedge, \vee, \neg \rangle$ such that $\langle A; \cdot \rangle$ is a commutative semigroup, $\langle A; \wedge, \vee \rangle$ is a distributive lattice and

$$\begin{aligned}
&\neg\neg a = a \leqslant a \cdot a,\\
&a \leqslant b \text{ iff } \neg b \leqslant \neg a,\\
&a \cdot b \leqslant c \text{ iff } a \cdot \neg c \leqslant \neg b,\\
&a \leqslant a \cdot (\neg(b \cdot \neg b) \wedge \neg(c \cdot \neg c)),
\end{aligned}$$

for all $a, b, c \in A$. The class of all relevant algebras is denoted by RA.

The two defining postulates of RA that are not pure equations can be paraphrased easily as equations, so RA is a variety. It is congruence distributive (since its members have lattice reducts) and congruence permutable (see for instance [75, Prop. 8.3]).

The main motivation for RA is that it algebraizes the logic $\mathbf{R}$. The algebraization process for $\mathbf{R}^{\mathbf{t}}$ and DMM carries over verbatim to $\mathbf{R}$ and RA, provided we use (12) as a formal device for eliminating all mention of e. Further work on relevant algebras can be found in [18, 21, 37, 38, 54, 56, 68, 69].

Because RA *is* closed under subalgebras, its study accommodates the variable sharing principle of relevance logic, without sacrificing the benefits of accurate algebraization. For the algebraist, however, RA has some forbidding features. It lacks the congruence extension property (CEP), for instance, as does its class of finite members (see [14, p. 289]), whereas DMM has the CEP. Also, De Morgan monoids have much in common with abelian groups (residuals being a partial surrogate for multiplicative inverses), but relevant algebras are less intuitive, being semigroup-based, rather than monoid-based.

The following facts are therefore noteworthy.

Theorem 7.2.

(i) RA *coincides with the class of all* e–free subreducts *of De Morgan monoids (i.e., all subalgebras of reducts* $\langle A; \cdot, \wedge, \vee, \neg \rangle$ *of De Morgan monoids* $\boldsymbol{A}$*).*

(ii) *If a relevant algebra is finitely generated, then it is the* e*–free* reduct $\langle A; \cdot, \wedge, \vee, \neg \rangle$ *of a De Morgan monoid* $\boldsymbol{A}$*. In this case, the unique neutral element of* $\boldsymbol{A}$ *is the greatest lower bound of all* $a \to a$*, where* a *ranges over any finite generating set for* $\langle A; \cdot, \wedge, \vee, \neg \rangle$*.*

Here, (ii) is a specialization of [53, Thm. 5.3], but it algebraizes a much older logical result of Anderson and Belnap [2, p. 343], already implicit in the proof of [1, Lem. 2]. We can infer (i) from (ii), as every algebra embeds into an ultraproduct of finitely generated subalgebras of itself (or see [31, Cor. 4.11]).

Theorem 7.2(i) reflects the fact that the $\mathbf{t}$–free fragment of $\vdash_{\mathbf{R^t}}$ is just $\vdash_{\mathbf{R}}$, so there is a smooth passage from either system to the other. In particular, the variable sharing principle holds for the $\mathbf{t}$–free formulas of $\mathbf{R^t}$.

The e–free reduct $\langle A; \cdot, \wedge, \vee, \neg \rangle$ of a De Morgan monoid $\boldsymbol{A}$ shall be denoted by $\boldsymbol{A}^-$. Also, if K is a class of De Morgan monoids, then K^- shall denote the class of e–free reducts of the members of K. In this case, on general grounds,

$$\mathbb{V}(\mathsf{K})^- \subseteq \mathbb{V}(\mathsf{K}^-). \tag{26}$$

Indeed, every equation satisfied by K^- is an e–free identity of K, and therefore of $\mathbb{V}(\mathsf{K})$, and therefore of $\mathbb{V}(\mathsf{K})^-$. Because a De Morgan monoid and its e–free reduct have the same congruence lattice, we also obtain:

Lemma 7.3. *A De Morgan monoid $\boldsymbol{A}$ is SI [resp. FSI; simple] iff the same is true of $\boldsymbol{A}^-$.*

Crucially, however, subalgebras of the e–free reduct of a De Morgan monoid need not contain e, and they need not be reducts of De Morgan monoids themselves, unless they are finitely generated. For instance, the free $\aleph_0$–generated relevant algebra is such a subreduct, and it lacks a neutral element, because the variable sharing principle rules out theorems of $\mathbf{R}$ of the form $\alpha \to (p \to p)$ whenever p is a variable not occurring in the formula α.

Still, because of Theorem 7.2, it is often easiest to obtain a result about relevant algebras indirectly, via a more swiftly established property of De Morgan monoids. This is exemplified below in Corollary 7.4, and more strikingly in Theorem 7.8. (We extend our use of the terms 'bounded' and 'rigorously compact' to relevant algebras in the obvious way, noting that the existence of a neutral element is not needed in the proof of Lemma 2.4.)

Corollary 7.4. ([69]) *Every finitely generated relevant algebra $\boldsymbol{A}$ is bounded.*

Proof. By Theorem 7.2(ii), $\boldsymbol{A}$ is a reduct of a De Morgan monoid with the same finite generating set, so $\boldsymbol{A}$ is bounded, by Theorem 3.2. □

In contrast with this argument, the only published proof of Corollary 7.4, viz. [69, Prop. 5], is quite complicated. (For one generator, the indicated bounds are built up using all six of the inequivalent implicational one-variable formulas of $\mathbf{R}$, determined in [43].) The result is attributed in [69] to Meyer and to Dziobiak (independently).

Corollary 7.5. *Every nontrivial relevant algebra $\boldsymbol{A}$ has a copy of $\mathbf{2}^-$ as a subalgebra.*

Proof. Let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}$ generated by an arbitrary pair of distinct elements of A. By Corollary 7.4, $\boldsymbol{B}$ has (distinct) extrema $\bot, \top$. By Theorem 7.2(ii), $\boldsymbol{B}$ is the e–free reduct of a De Morgan monoid, so by Lemma 2.3, $\{\bot, \top\}$ is the universe of a subalgebra of $\boldsymbol{B}$, isomorphic to $\mathbf{2}^-$. □

Clearly, when a Boolean algebra $\boldsymbol{A}$ is thought of as an integral De Morgan monoid, it has the same term operations as its e–free reduct $\boldsymbol{A}^-$, because e is definable as $x \to x$. Thus, the variety of Boolean algebras can be identified with $\mathbb{V}(\mathbf{2}^-) = \mathbb{Q}(\mathbf{2}^-)$.

Corollary 7.6. *Boolean algebras constitute the smallest nontrivial (quasi) variety of relevant algebras.*

This reconfirms, of course, that classical propositional logic is the largest consistent extension of $\mathbf{R}$. With Theorem 7.2(ii), it also yields the following.

Theorem 7.7. *There is a join-preserving (hence isotone) surjection from the lattice of subvarieties of* DMM *to that of* RA, *defined by* $\mathsf{K} \mapsto \mathbb{V}(\mathsf{K}^-)$.

Moreover, this map remains surjective when its domain is restricted to the varieties that contain $\mathbf{2}$, *together with the trivial variety.*

Proof. Preservation of joins follows from Jónsson's Theorem and the following two facts: (i) an ultraproduct of reducts of members of a class C is the reduct of a (corresponding) ultraproduct of members of C, and (ii) an ultraproduct of members of the join of two varieties belongs to one of the two varieties. To prove surjectivity, let L be a variety of relevant algebras, and L_{FG} its class of finitely generated members. By Theorem 7.2(ii), each $\boldsymbol{A} \in \mathsf{L}_{\mathrm{FG}}$ is the e–free reduct of a unique De Morgan monoid $\boldsymbol{A}^+$. Let $\mathsf{M} = \{\boldsymbol{A}^+ : \boldsymbol{A} \in \mathsf{L}_{\mathrm{FG}}\}$. Then $\mathsf{L}_{\mathrm{FG}} = \mathsf{M}^- \subseteq \mathbb{V}(\mathsf{M})^-$, while (26) shows that $\mathbb{V}(\mathsf{M})^- \subseteq \mathbb{V}(\mathsf{L}_{\mathrm{FG}})$. Thus, $\mathsf{L} = \mathbb{V}(\mathbb{V}(\mathsf{M})^-)$, as varieties are determined by their finitely generated members. If L is nontrivial, then $\mathbf{2}^- \in \mathsf{L}_{\mathrm{FG}}$, by Corollary 7.5, so $\mathbf{2} \in \mathsf{M}$. □

Whereas the above argument about joins would apply in any context where the indicated reduct class is a congruence distributive variety, the surjectivity of the (restricted) function in Theorem 7.7 is a special feature of relevant algebras, reliant on Theorem 7.2(ii). The restricted function is not injective, however. Indeed, Jónsson's Theorem shows that $\mathbb{V}(\mathbf{2}, \boldsymbol{S}_{2n+1}) \subsetneq \mathbb{V}(\boldsymbol{S}_{2n}, \boldsymbol{S}_{2n+1})$ for all integers $n > 1$, but these two varieties have the same image under the map $\mathsf{K} \mapsto \mathbb{V}(\mathsf{K}^-)$, because $\boldsymbol{S}_{2n}^-$ embeds into $\boldsymbol{S}_{2n+1}^-$ (although $\boldsymbol{S}_{2n}$ does not embed into $\boldsymbol{S}_{2n+1}$).[6]

This failure of injectivity limits the usefulness of the above function when we analyse the subvariety lattice of RA. Nevertheless, we can already derive Świrydowicz's description of the lower part of that lattice by a mathematically simpler argument, based on the situation for De Morgan monoids. In

[6] The function sending a subvariety W of RA to the variety generated by the De Morgan monoids whose e–free reducts lie in W is an injective join-preserving right-inverse for the function in Theorem 7.7, but it is not surjective.

particular, we avoid use of the complex ternary relation semantics for **R** (see [59]), employed in [68].

Theorem 7.8. ([68, Thm. 12]) $\mathbb{V}(\boldsymbol{S}_3^-)$, $\mathbb{V}(\boldsymbol{C}_4^-)$ *and* $\mathbb{V}(\boldsymbol{D}_4^-)$ *are exactly the covers of* $\mathbb{V}(\mathbf{2}^-)$ *in the subvariety lattice of* RA.

Proof. Let $\boldsymbol{X} \in \{\boldsymbol{S}_3^-, \boldsymbol{C}_4^-, \boldsymbol{D}_4^-\}$, so $\boldsymbol{X}$ is simple (by Lemma 7.3) and $\boldsymbol{X}$ has just one nontrivial proper subalgebra, which is isomorphic to $\mathbf{2}^-$. Then every SI member of $\mathbb{V}(\boldsymbol{X})$ is isomorphic to $\mathbf{2}^-$ or to $\boldsymbol{X}$, by Jónsson's Theorem (cf. the proof of Theorem 6.1). So, there are no subvarieties of RA strictly between $\mathbb{V}(\mathbf{2}^-)$ and $\mathbb{V}(\boldsymbol{X})$, and $\mathbb{V}(\boldsymbol{X}) \neq \mathbb{V}(\boldsymbol{Y})$ for $\boldsymbol{X} \neq \boldsymbol{Y} \in \{\boldsymbol{S}_3^-, \boldsymbol{C}_4^-, \boldsymbol{D}_4^-\}$.

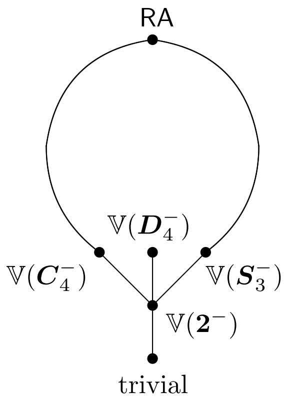
Conversely, let K be a subvariety of RA, not consisting entirely of Boolean algebras. We must show that $\mathbb{V}(\boldsymbol{X}) \subseteq \mathsf{K}$ for some $\boldsymbol{X} \in \{\boldsymbol{S}_3^-, \boldsymbol{C}_4^-, \boldsymbol{D}_4^-\}$.

As $\mathbb{V}(\mathbf{2}^-) \subsetneq \mathsf{K}$ are varieties, there exists a finitely generated SI algebra $\boldsymbol{A} \in \mathsf{K} \setminus \mathbb{V}(\mathbf{2}^-)$. Now $\boldsymbol{A}$ is the e–free reduct of some $\boldsymbol{A}^+ \in \mathsf{DMM}$, by Theorem 7.2(ii), and $\boldsymbol{A}^+$ is SI (by Lemma 7.3) and finitely generated. By (26),

$$\mathbb{V}(\boldsymbol{A}^+)^- \subseteq \mathbb{V}(\boldsymbol{A}) \subseteq \mathsf{K},$$

so it suffices to show that one of $\boldsymbol{S}_3$, $\boldsymbol{C}_4$ or $\boldsymbol{D}_4$ belongs to $\mathbb{V}(\boldsymbol{A}^+)$.

Suppose $\boldsymbol{C}_4, \boldsymbol{D}_4 \notin \mathbb{V}(\boldsymbol{A}^+)$. Then $\boldsymbol{A}^+$ is a Sugihara monoid, by Theorem 5.21. As $\boldsymbol{A}^+$ is SI, finitely generated, and not a Boolean algebra, it is isomorphic to $\boldsymbol{S}_n$ for some $n \geq 3$, by Theorem 5.9. Then $\boldsymbol{S}_3 \in \mathbb{H}(\boldsymbol{A}^+) \subseteq \mathbb{V}(\boldsymbol{A}^+)$ (by the remarks preceding Corollary 5.10), completing the proof. $\square$



This means that the logics algebraized by $\mathbb{V}(\boldsymbol{S}_3^-)$, $\mathbb{V}(\boldsymbol{C}_4^-)$ and $\mathbb{V}(\boldsymbol{D}_4^-)$ are exactly the maximal non-classical axiomatic extensions of **R**, as was observed in [68]. The proof of Theorem 7.8 in [68] relies on a lemma, which says that every bounded SI relevant algebra is rigorously compact [68, Lem. 8]. In [68], the proof of the lemma uses the ternary relation semantics for **R**. As the lemma is itself of some interest, we supply an algebraic justification of it here. The key to the argument is that the subalgebras of FSI relevant algebras are still FSI, but that fact is concealed by the failure of the CEP and the lack of an obvious analogue for Lemma 3.7(i) in RA. One way to circumvent these difficulties is to extend the concept of deductive filters to relevant algebras.

Definition 7.9. A subset F of a relevant algebra $\boldsymbol{A}$ is called a *deductive filter* of $\boldsymbol{A}$ if F is a lattice filter of $\langle A; \wedge, \vee \rangle$ and

$$|a| \,:=\, a \to a \in F \text{ for all } a \in A.$$

Clearly, the set of deductive filters of $\boldsymbol{A}$ is closed under arbitrary intersections and under unions of non-empty directed subfamilies, so it is both an algebraic closure system over A and the universe of an algebraic lattice $\boldsymbol{DFil}\,\boldsymbol{A}$, ordered by inclusion. We denote by $\mathit{DFg}^{\boldsymbol{A}}X$ the smallest deductive filter of $\boldsymbol{A}$ containing X, whenever $X \subseteq A$. Thus, the compact elements of $\boldsymbol{DFil}\,\boldsymbol{A}$ are just the finitely generated deductive filters of $\boldsymbol{A}$, i.e., those of the form $\mathit{DFg}^{\boldsymbol{A}}X$ for some finite $X \subseteq A$.

The deductive filters of a relevant algebra $\boldsymbol{A}$ are just the subsets that contain all $\boldsymbol{A}$–instances of the axioms of $\mathbf{R}$ and are closed under the inference rules—modus ponens and adjunction—of $\mathbf{R}$. (This is easily verified, using (12) and Theorem 7.2(i).) Therefore, by the theory of algebraization [8, Thm. 5.1], and since RA is a variety, we have

$$\boldsymbol{DFil}\,\boldsymbol{A} \cong \boldsymbol{Con}\,\boldsymbol{A}, \text{ for all } \boldsymbol{A} \in \mathsf{RA}.$$

Theorem 7.10. *Let $\boldsymbol{A}$ be a relevant algebra, with $a, b \in A$. Then*

(i) $||a| \wedge |b|| \leqslant |a| \wedge |b|$;

(ii) $\mathit{DFg}^{\boldsymbol{A}}\{a\} = \{c \in A : a \wedge |d| \leqslant c \text{ for some } d \in A\}$;

(iii) $\mathit{DFg}^{\boldsymbol{A}}\{a\} \cap \mathit{DFg}^{\boldsymbol{A}}\{b\} = \mathit{DFg}^{\boldsymbol{A}}\{a \vee b\}$.

Proof. (i) It suffices, by Theorem 7.2(i) and (12), to show that $e \leqslant |a| \wedge |b|$ in any De Morgan monoid that contains $\boldsymbol{A}$ as a subreduct. And this follows from (11).

(ii) Let $F = \{c \in A : a \wedge |d| \leqslant c \text{ for some } d \in A\}$. Then $a \in F$, since $a \wedge |a| \leqslant a$. We claim that F is a deductive filter of $\boldsymbol{A}$. Clearly, F is upward closed. Suppose $c, c' \in F$, so there exist $d, d' \in A$ such that $a \wedge |d| \leqslant c$ and $a \wedge |d'| \leqslant c'$. Then $c \wedge c' \geqslant a \wedge |d| \wedge |d'| \geqslant a \wedge ||d| \wedge |d'||$, by (i), so $c \wedge c' \in F$. Also, for any $d \in A$, we have $a \wedge |d| \leqslant |d|$, so $|d| \in F$. This vindicates the claim. It remains to show that F is the *smallest* deductive filter of $\boldsymbol{A}$ containing a. So, let $G \in \mathit{DFil}\,\boldsymbol{A}$, with $a \in G$, and let $c \in F$. Choose $d \in A$ with $a \wedge |d| \leqslant c$. Since $a, |d| \in G$, we have $a \wedge |d| \in G$, whence $c \in G$, as required.

(iii) Certainly, $a \vee b \in \mathit{DFg}^{\boldsymbol{A}}\{a\} \cap \mathit{DFg}^{\boldsymbol{A}}\{b\}$, as $a, b \leqslant a \vee b$. Now suppose $c \in \mathit{DFg}^{\boldsymbol{A}}\{a\} \cap \mathit{DFg}^{\boldsymbol{A}}\{b\}$. Choose $d, d' \in A$, with $a \wedge |d| \leqslant c$ and $b \wedge |d'| \leqslant c$. Then $c \geqslant a \wedge |d| \wedge |d'|$, $b \wedge |d| \wedge |d'|$, so

$$\begin{aligned} c &\geqslant (a \wedge |d| \wedge |d'|) \vee (b \wedge |d| \wedge |d'|) \\ &= (a \vee b) \wedge (|d| \wedge |d'|) \quad \text{(by distributivity)} \\ &\geqslant (a \vee b) \wedge ||d| \wedge |d'|| \quad \text{(by (i))}. \end{aligned}$$

Thus, $c \in DFg^{\boldsymbol{A}}\{a \vee b\}$ and the result follows. □

Corollary 7.11. *The class of FSI relevant algebras is closed under subalgebras (and ultraproducts).*

Proof. Let $\boldsymbol{A} \in \mathsf{RA}$. Clearly, $DFg^{\boldsymbol{A}}\{a_1, \dots, a_n\} = DFg^{\boldsymbol{A}}\{a_1 \wedge \dots \wedge a_n\}$ for all $a_1, \dots, a_n \in A$, so every finitely generated deductive filter of $\boldsymbol{A}$ is principal. Therefore, by Theorem 7.10(iii), the intersection of any two compact (i.e., finitely generated) elements of $\boldsymbol{DFil\,A}$ is compact. The same applies to the lattice $\boldsymbol{Con\,A}$, as it is isomorphic to $\boldsymbol{DFil\,A}$ (and since lattice isomorphisms between complete lattices preserve compactness). Now the result follows from the well known theorem below. □

Theorem 7.12. ([7]) *In any congruence distributive variety* K, *the following conditions are equivalent.*

(i) *For any* $\boldsymbol{A} \in \mathsf{K}$, *the intersection of any two compact (i.e., finitely generated) congruences of* $\boldsymbol{A}$ *is compact.*
(ii) $\mathsf{K}_{\mathrm{FSI}}$ *is closed under* $\mathbb{S}$ *and* $\mathbb{P}_{\mathbb{U}}$ *(i.e., it is a universal class).*

Finally, as promised, a slight generalization of [68, Lem. 8] follows easily from Corollary 7.11:

Theorem 7.13. *Every bounded FSI relevant algebra* $\boldsymbol{A}$ *is rigorously compact.*

Proof. Let $\bot, \top$ be the extrema of $\boldsymbol{A}$, and consider $\bot \neq a \in A$. We must show that $\top \cdot a = \top$. Observe that $\boldsymbol{B} := \boldsymbol{Sg}^{\boldsymbol{A}}\{\bot, a, \top\}$ is FSI, by Corollary 7.11. As $\boldsymbol{B}$ is finitely generated, it is a reduct of a (bounded) De Morgan monoid $\boldsymbol{B}^{+}$, by Theorem 7.2(ii), which is also FSI, by Lemma 7.3. Now $\boldsymbol{B}^{+}$ is rigorously compact, by Theorem 5.3, so $\top \cdot a = \top$. □

Corollary 7.14. *Every finitely generated subalgebra of an FSI relevant algebra is rigorously compact.*

Proof. Use Corollaries 7.4 and 7.11 and Theorem 7.13. □

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