An Abstract Approach to Consequence Relations

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Abstract

We generalise the Blok–Jónsson account of structural consequence relations, later developed by Galatos, Tsinakis and other authors, in such a way as to naturally accommodate multiset consequence. While Blok and Jónsson admit, in place of sheer formulas, a wider range of syntactic units to be manipulated in deductions (including sequents or equations), these objects are invariably *aggregated* via set-theoretical union. Our approach is more general in that non-idempotent forms of premiss and conclusion aggregation, including multiset sum and fuzzy set union, are considered. In their abstract form, thus, *deductive relations* are defined as additional compatible preorderings over certain partially ordered monoids. We investigate these relations using categorical methods, and provide analogues of the main results obtained in the general theory of consequence relations. Then we focus on the driving example of multiset deductive relations, providing variations of the methods of matrix semantics and Hilbert systems in Abstract Algebraic Logic.

1 Introduction

1.1 Logical consequence: The Blok–Jónsson approach

Let $\mathcal{L}$ be a propositional language (or, which is the same, an algebraic language). According to the standard textbook definition, a *Tarskian consequence relation* (TCR) on $\mathcal{L}$ is a relation $\vdash \subseteq \wp(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ obeying the following conditions for all $X, Y \subseteq Fm_{\mathcal{L}}$ and $\varphi \in Fm_{\mathcal{L}}$:

- $X \vdash \varphi$ whenever $\varphi \in X$. (Reflexivity)
- If $X \vdash \varphi$ and $X \subseteq Y$, then $Y \vdash \varphi$. (Monotonicity)
- If $Y \vdash \varphi$ and $X \vdash \psi$ for every $\psi \in Y$, then $X \vdash \varphi$. (Cut)

Tarskian consequence relations are the primary object of study of *Abstract Algebraic Logic* (AAL: see e.g. [7, 11, 12, 14, 15]), a discipline that aims at providing general tools for the investigation and comparison of the different brands of propositional logics on the market. Among the reasons why AAL has established itself as a mainstream approach there is its success in effectively

accommodating all the main extensions of, and alternatives to, classical propositional logic: intuitionistic logic, modal logics, relevant logics, quantum logics, and whatnot.

Although TCR's are perfectly adequate for the needs of a wide spectrum of such abstract metalogical enquiries, it gradually emerged that they fail to capture a range of situations where we are still reasoning from given premisses to certain conclusions according to the same three principles of Reflexivity, Monotonicity, and Cut, yet we are not manipulating formulas of a given language, but perhaps *sequents* (as in Gentzen calculi) or *equations* (as in equational consequence relations associated with classes of algebras). Actually, it is not unusual for a logic to be given alternative presentations as a "consequence relation" of sorts over different sets of syntactic units. For example, classical logic can be presented not only as a (syntactically or semantically defined) TCR, but also as the derivability relation of the classical sequent calculus, or as the equational consequence relation of the 2-element Boolean algebra. In order to subsume these generalisations of the concept of propositional logic, core AAL was extended to k-dimensional systems [8] and, subsequently, to Gentzen systems [29, 30, 31, 32, 28, 36]. The proliferation of these extensions of the classical AAL theory suggests that the idea of abstracing away from the specifics of these presentations to pinpoint what is essential to a logic is not without its allure.

In their groundbreaking paper [6] (see also the lecture notes [5] of the course "Algebraic structures for logic" that inspired the paper), Wim Blok and Bjarni Jónsson take their cue from such reflections and suggest to replace the set $Fm_{\mathcal{L}}$ in the definition of TCR by an *arbitrary set* A:

Definition 1. *An* abstract consequence relation (ACR) *on the set A is a relation $\vdash \subseteq \wp(A) \times A$ obeying the following conditions for all $X, Y \subseteq A$ and for all $a \in A$:*

- *$X \vdash a$ whenever $a \in X$.* *(Reflexivity)*
- *If $X \vdash a$ and $X \subseteq Y$, then $Y \vdash a$.* *(Monotonicity)*
- *If $Y \vdash a$ and $X \vdash b$ for every $b \in Y$, then $X \vdash a$.* *(Cut)*

The first conceptual hurdle for this general approach is providing a suitable account of *logical* consequence. It is generally agreed that logical consequence, as opposed to consequence in general, is a matter of logical form. In the standard AAL framework, this requirement is rendered precise by adding to the three Tarskian postulates the further condition of *substitution-invariance*. Accordingly, a TCR $\vdash$ on $\mathcal{L}$ is said to *substitution-invariant* if for all $X \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, whenever $X \vdash \varphi$ we also have that $\sigma(X) \vdash \sigma(\varphi)$, where σ is an arbitrary $\mathcal{L}$-substitution (namely, an endomorphism of the formula algebra $\mathbf{Fm}_{\mathcal{L}}$) and $\sigma(X)$ is defined pointwise. It is clear enough that replacing the algebra $\mathbf{Fm}_{\mathcal{L}}$ by the unstructured set A, which behaves as a sort of "black box" in so far as no notion of endomorphism is applicable to it, calls for a completely different account of substitution-invariance.

Blok and Jónsson's response to this problem is insightful. They observe that the application of substitutions to propositional formulas (or, for that matter, to equations or sequents) is reminiscent of an operation of multiplication by a scalar. In fact, if $\mathcal{L}$ is a language, φ is an $\mathcal{L}$-formula and σ_1, σ_2 are $\mathcal{L}$-substitutions, then $(\sigma_1 \circ \sigma_2)(\varphi) = \sigma_1(\sigma_2(\varphi))$, and if ι is the identity $\mathcal{L}$-substitution, $\iota(\varphi) = \varphi$. Generalising this example, we are led to the following abstract counterpart of substitution-invariance. We say that a monoid $\mathbf{M} = \langle M, \cdot, 1\rangle$ *acts* on a set A if there is a map $\star\colon M \times A \to A$ s.t. for all $m_1, m_2 \in M$ and all $a \in A$, $(m_1 \cdot m_2) \star a = m_1 \star (m_2 \star a)$ and $1 \star a = a$. Thus, an ACR $\vdash$ is said to be $\mathbf{M}$-*action-invariant* if for all $X \cup \{a\} \subseteq A$ and $m \in M$, whenever $X \vdash a$ we also have that $\{m \star x : x \in X\} \vdash m \star a$.

For future reference, we need to observe that if $\mathbf{M}$ acts on A, then $\wp(\mathbf{M}) = \langle \wp(M), \cdot', \{1\}\rangle$ (where $\cdot'$ is complex product) acts on $\wp(A)$ via the induced map

$$N \star' X = \{m \star x : m \in N, x \in X\}.$$

One striking feature of Blok and Jónsson's suggestion is that it allows to reformulate, at a very general level, the classical notion of algebraisability [7], in a way that provides a uniform perspective on the algebraisation of logics and Gentzen systems. We remind the reader that a logic $\vdash$ (i.e., a substitution-invariant TCR) is *algebraisable* [7] if there exist a generalized quasi-variety $\mathcal{K}$ and two maps

$$\tau\colon \wp(Fm_{\mathcal{L}}) \longleftrightarrow \wp(Eq_{\mathcal{L}})\colon \rho$$

which commute with unions and substitutions, such that

$$X \vdash \varphi \Longleftrightarrow \tau(X) \models_{\mathcal{K}} \tau(\varphi)$$
$$x \approx y \dashv\models_{\mathcal{K}} \tau\rho(x \approx y)$$

for every $X \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$. In the displays above, $Eq_{\mathcal{L}}$ is the set of $\mathcal{L}$-equations (formally cast as ordered pairs of $\mathcal{L}$-formulas), and $\models_{\mathcal{K}}$ the equational consequence relation of $\mathcal{K}$. When the above conditions hold, $\mathcal{K}$ is unique and is said to be the *equivalent algebraic semantics* of $\vdash$. Given a logic $\vdash$ and a generalized quasi-variety $\mathcal{K}$, we denote by $C_{\vdash}$ and $C_{\mathcal{K}}$ the closure operators associated to the relations $\vdash$ and $\models_{\mathcal{K}}$ respectively. It is well-known (see e.g. [12, p. 149]) that the notion of algebraisability is captured by the existence of a particular isomorphism:

Theorem 2 (Syntactic Isomorphism Theorem)**.** *Let $\vdash$ be a logic and $\mathcal{K}$ a generalised quasi-variety. Moreover, let $\mathcal{T}h(\vdash)$ and $\mathcal{T}h(\models_{\mathcal{K}})$ be the complete lattices of theories (i.e., such sets that $\varphi \in T$ whenever $T \vdash \varphi$; analogously for $\models_{\mathcal{K}}$) of $\vdash$ and $\models_{\mathcal{K}}$, respectively. Then $\vdash$ is algebraisable with equivalent algebraic semantics $\mathcal{K}$ if and only if there is a lattice isomorphism $\Phi\colon Th(\vdash) \to Th(\models_{\mathcal{K}})$ such that $\Phi \circ C_{\vdash} \circ \sigma = C_{\mathcal{K}} \circ \sigma \circ \Phi$ for every substitution σ.*

Blok and Jónsson introduced the following natural generalisation of the concept of algebraisability, based on the criterion provided by the Syntactic Isomorphism Theorem. Let $\mathbf{M}$ be a monoid acting on two sets A_1 and A_2, respectively

through the actions $\star_1$ and $\star_2$. Moreover, let $\vdash_1$ and $\vdash_2$ be two $\mathbf{M}$-action invariant ACR's, respectively on A_1 and A_2. The ACR's $\vdash_1$ and $\vdash_2$ are said to be *equivalent* if their lattices of closed sets, when suitably enriched with the actions of the monoid $\mathbf{M}$, are isomorphic (see [5, Definition 4.5]). Observe that, in the light of the Syntactic Isomorphism Theorem, a logic $\vdash$ is algebraisable with equivalent algebraic semantics $\mathcal{K}$ if and only if the substitution-invariant ACR's $\vdash$ and $\models_{\mathcal{K}}$ are equivalent according to this definition (where $\mathbf{M}$ is the monoid of substitutions, acting in the natural way on formulas and equations).

1.2 A categorical perspective

Nikolaos Galatos and Constantine Tsinakis [18] follow in Blok and Jónsson's footsteps and take their approach to the next level of generality. In particular, they aim at applying categorical and order-theoretic methods to the study of ACR's. One major hindrance to this accomplishment is the intrinsic *asymmetry* of ACR's, whose relata are, respectively, a subset of the base set and an element thereof. With an eye to mending this flaw, they go on to define *symmetric* versions of ACR's on a given set A as preorder relations $\vdash \subseteq \wp(A) \times \wp(A)$ such that $X \vdash Y$ iff $X \vdash y$ for all $y \in Y$; equivalently, preorder relations containing the supersethood relation such that $X \vdash \bigcup \{Z : X \vdash Z\}$. Although these symmetric ACR's are shown to be in bijective correspondence with standard ACR's, their advantage is that their premiss-sets and conclusion-sets are points in a complete lattice of sets, a circumstance that suggests the following generalisation.

Definition 3. *A* Galatos–Tsinakis consequence relation *(*GTCR*) on a complete lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$, with induced order $\leq$, is a preorder $\vdash$ of L that contains $\geq$ and is such that for all $x \in L$, $x \vdash \bigvee \{y \in L : x \vdash y\}$.*

Can action-invariance be accommodated in this broader setting? To do so, we must first enrich our monoids of actions with additional structure, in such a way as to turn them into complete *residuated lattices* [17, 23]. Now, let $\mathbf{M} = \langle M, \wedge^{\mathbf{M}}, \vee^{\mathbf{M}}, \cdot, \backslash, /, 1 \rangle$ be a complete residuated lattice,[1] $\mathbf{L} = \langle L, \wedge^{\mathbf{L}}, \vee^{\mathbf{L}} \rangle$ be a complete lattice and $\star\colon M \times L \to L$ be a map. We say that $\mathbf{M}$ *acts on* $\mathbf{L}$, or also that $\boldsymbol{L} = \langle L, \wedge^{\mathbf{L}}, \vee^{\mathbf{L}}, \star \rangle$ is an $\mathbf{M}$-*module*,[2] if the monoid reduct of $\mathbf{M}$ acts on L and, moreover, there are maps $/_\star\colon L \times L \to M$ and $\backslash_\star\colon M \times L \to L$ such that, for all $m \in M$ and $x, y \in L$,

$$m \star x \leq^{\boldsymbol{L}} y \text{ iff } m \leq^{\mathbf{M}} y/_\star x \text{ iff } x \leq^{\boldsymbol{L}} m\backslash_\star y.$$

A GTCR $\vdash$ on the $\mathbf{M}$-module $\boldsymbol{L}$ is said to be $\mathbf{M}$-*action-invariant* (or, simply, *action-invariant*) if, for all $x, y \in L$ and $m \in M$, $x \vdash y$ implies $m \star x \vdash m \star y$.

From this point onwards, Galatos and Tsinakis continue their investigation in the category $\mathbf{M}-\mathsf{Mod}$ whose objects are $\mathbf{M}$-modules and whose arrows are

[1] In the motivating context of symmetric $\mathbf{N}$-action invariant ACR's, we take $\mathbf{M}$ to be the natural complete residuated lattice over $\wp(N)$.

[2] Note that for modules we use boldface italic font, whereas for algebras a simple boldface. Also, if $\boldsymbol{K}$ is module then $\mathbf{K}$ is its lattice reduct.

residuated maps $\tau\colon \boldsymbol{L} \to \boldsymbol{L}'$ (called *translators*) such that, if $\mathbf{M}$ acts on $\boldsymbol{L}$ and $\boldsymbol{L}'$ via $\star_1$ and $\star_2$ respectively, we have that for all $x \in L$ and $m \in M$, $\tau(m \star_1 x) = m \star_2 \tau(x)$. A noteworthy feature of $\mathbf{M}-\mathsf{Mod}$ is the fact that GTCR's on its objects can be viewed as bona fide objects in the same category. In fact:

- The GTCR's on an $\mathbf{M}$-module $\boldsymbol{L}$ correspond bijectively to closure operators on its lattice reduct $\mathbf{L}$ via the maps $\vdash_{()}$ and $\gamma_{()}$ defined by

$$x \vdash_{\gamma} y \text{ iff } y \leq^{\mathbf{L}} \gamma(x) \text{ and } \gamma_{\vdash}(x) = \bigvee \{y \in L : x \vdash y\}.$$

- If $\boldsymbol{L}$ is an $\mathbf{M}$-module (via $\star$) and $\vdash$ is a GTCR on its lattice reduct $\mathbf{L}$, then the lattice $\mathbf{L}_{\gamma_\vdash}$ of $\gamma_\vdash$-closed elements of $\mathbf{L}$ is the lattice reduct of an $\mathbf{M}$-module $\boldsymbol{L}_{\gamma_\vdash}$ via the map $\star_{\gamma_\vdash} : M \times L_{\gamma_\vdash} \to L_{\gamma_\vdash}$ defined by

$$m \star_{\gamma_\vdash} x = \gamma_\vdash(m \star x).$$

  Moreover, $\gamma_\vdash$ is a morphism in $\mathbf{M}-\mathsf{Mod}$ from $\boldsymbol{L}$ onto $\boldsymbol{L}_{\gamma_\vdash}$.

If $\vdash$ is a GTCR on $\boldsymbol{L}$, then $\mathbf{L}_{\gamma_\vdash}$ is nothing but the lattice of $\vdash$-*theories* — viz., of all $t \in L$ such that $t \vdash x$ implies $x \leq t$.

It turns out that algebraisability can be generalised to the setting of modules over residuated lattices, again thanks to the criterion provided by the Syntactic Isomorphism Theorem of algebraisable logics (see Theorem 2). More precisely, two GTCR's $\vdash_1$ and $\vdash_2$, respectively on the $\mathbf{M}$-modules $\boldsymbol{L}$ and $\boldsymbol{L}'$, are said to be *equivalent* if there is a module isomorphism $f\colon \boldsymbol{L}_{\gamma_{\vdash_1}} \to \boldsymbol{L}'_{\gamma_{\vdash_2}}$. Moreover, we say that an isomorphism $f\colon \boldsymbol{L}_{\gamma_{\vdash_1}} \to \boldsymbol{L}'_{\gamma_{\vdash_2}}$ is *induced* by the translators $\tau\colon \boldsymbol{L} \to \boldsymbol{L}'$ and $\rho\colon \boldsymbol{L}' \to \boldsymbol{L}$ if $f\gamma_{\vdash_1} = \gamma_{\vdash_2}\tau$ and $f^{-1}\gamma_{\vdash_2} = \gamma_{\vdash_1}\rho$. In this case, the classical definition of algebraisability can be restored, in the sense that for every $x, y \in L$ and $z \in L'$ we have that

$$(x \vdash_1 y \Longleftrightarrow \tau(x) \vdash_2 \tau(y)) \text{ and } z \dashv\vdash_2 \tau(\rho(z)).$$

In this parlance, the Syntactic Isomorphism Theorem states that every equivalence between a logic $\vdash$ and the equational consequence $\models_{\mathcal{K}}$ relative to a generalized quasi-variety $\mathcal{K}$ is induced by a pair of translators. It is natural to ask whether this is true for arbitrary equivalences between GTCR's on $\mathbf{M}$-modules. Unfortunately, it turns out that this is false in general, as shown in [19]. Nevertheless, Galatos and Tsinakis find sufficient and necessary conditions for it to be the case. Recall, that an object $\boldsymbol{R}$ in a category C, whose arrows are set-theoretic functions, is *onto-projective* if for any C-morphisms $f\colon \boldsymbol{S} \to \boldsymbol{T}$ and $g\colon \boldsymbol{R} \to \boldsymbol{T}$ between objects in C with f onto, there is a C-morphism $h\colon \boldsymbol{R} \to \boldsymbol{S}$ such that $f \circ h = g$. Here is the main result in the paper by Galatos and Tsinakis:

Theorem 4. *An* $\mathbf{M}$*-module* $\boldsymbol{L}$ *is onto-projective in* $\mathbf{M}-\mathsf{Mod}$ *iff for any* $\mathbf{M}$*-module* $\boldsymbol{L}'$ *and* GTCR*'s* $\vdash_1$ *and* $\vdash_2$*, respectively on* $\boldsymbol{L}$ *and* $\boldsymbol{L}'$*, every residuated order embedding* f *of* $\boldsymbol{L}_{\gamma\vdash_1}$ *into* $\boldsymbol{L}'_{\gamma\vdash_2}$ *is such that* $f\gamma_{\vdash_1} = \gamma_{\vdash_2}\tau$ *for some translator* $\tau\colon \boldsymbol{L} \to \boldsymbol{L}'$*. In particular, if both* $\boldsymbol{L}$ *and* $\boldsymbol{L}'$ *are onto-projective, then every equivalence between* $\vdash_1$ *and* $\vdash_2$ *is induced by translators.*

Generalizations of the above criterion have been obtained in [16] (see also [34]) and [26], while the structure of onto-projective objects in the Blok–Jónsson framework was described in [13]. Note that in [16] the authors have also proved that all epimorphisms in $\mathbf{M}-\mathsf{Mod}$ are onto, so the specification "onto" in the previous theorem is redundant.

Given a propositional language $\mathcal{L}$, the lattice of sets $\wp(\mathbf{Fm}_{\mathcal{L}})$ — that corresponds to the plain old module of $\mathcal{L}$-formulas — is indeed onto-projective in $\wp(\mathbf{End}(\mathbf{Fm}_{\mathcal{L}}))-\mathsf{Mod}$, and so are the modules of $\mathcal{L}$-equations and $\mathcal{L}$-sequents. Thus, Theorem 4 gives an elegant abstract characterisation of those modules that are just as "well-behaved" as these standard examples in terms of admitting a general version of the Syntactic Isomorphism Theorem.

1.3 The substructural challenge

Substructural logics [27, 17, 23], usually introduced by means of sequent calculi where some or all of the standard structural rules (Weakening, Contraction, Exchange, Cut) are restricted or even deleted, provide an interesting challenge for the Blok–Jónsson approach to consequence, as further developed by Galatos and Tsinakis. By this we do not mean that they lie outside its scope — on the contrary, they can be handled even in traditional AAL. There is, in fact, a canonical way to obtain a TCR out of a given substructural sequent calculus. By way of example, consider the sequent calculus FL_e for *full Lambek calculus with exchange* and the variety $\mathcal{FL}_e$ of pointed commutative residuated lattices. Define:

- $X \vdash_{\mathrm{FL}_e} \varphi$ iff the sequent $\Rightarrow \varphi$ is provable in the calculus obtained by adding to FL_e as axioms the sequents in $\{\Rightarrow \psi : \psi \in X\}$ (this is sometimes called [2] the *external consequence relation* of a sequent calculus).
- $X \vdash'_{\mathrm{FL}_e} \varphi$ iff $\{\psi \wedge 1 \approx 1 : \psi \in X\} \models_{\mathcal{FL}_e} \varphi \wedge 1 \approx 1$.

It is well-known [17, Ch. 2] that for any set of formulas $X \cup \{\varphi\}$, $X \vdash_{\mathrm{FL}_e} \varphi$ holds if and only if $X \vdash'_{\mathrm{FL}_e} \varphi$. The relation defined by any of these two equivalent conditions is a TCR to all intents and purposes.

This approach, however, seems to fly in the face of the motivation underlying substructural logics. TCR's are relations between *sets* of formulas and single formulas, whence they are insensitive to the number of occurrences a formula may have in some collection of premisses. In other words, they automatically validate the *Contraction* and the *Anticontraction* rules: if $X, \varphi, \varphi \vdash \psi$, then $X, \varphi \vdash \psi$, and conversely if $X, \varphi \vdash \psi$, then $X, \varphi, \varphi \vdash \psi$. Yet, some substructural

logics (like *linear logic*, [20]) are commonly employed to formalise a "resource-conscious" notion of inference, according to which sentences are information tokens of a given type and for which the Contraction rule is utterly suspect. Other substructural logics in the relevant family [24, 25] aim at capturing a concept of deduction according to which premisses in an argument should be actually *used* to get the conclusion, something which seems to disqualify Anticontraction (and, even more, Weakening). In other words: AAL can certainly accommodate substructural logics into its framework (in the format of propositional logics or of Gentzen systems), and bestow on them the imposing bulk of general results it has to offer, but only at the cost of tweaking the substructural proof systems in such a way as to produce consequence relations that weaken and contract by *fiat*. AAL, in sum, does not stay true to the spirit of substructural logics.

Now return, for a while, to the sequent calculus FL_e. A more plausible candidate for a formalisation of substructural consequence is its so-called *internal consequence relation* [2], namely, that relation that holds between a finite multiset of formulas Γ and a formula φ just in case $\Gamma \Rightarrow \varphi$ is a provable sequent of FL_e. Investigating relations of this kind, however, means overstepping the Tarskian framework under at least two respects:

1. A consequence relation should be conceived of as a relation between a *finite multiset* of formulas and a formula;

2. The Monotonicity postulate should be dropped and the Reflexivity postulate should be restricted.

This approach, indeed, has been followed by Arnon Avron [3] and, sporadically, by a few others [24, 25, 37, 27], who laid down the fundamentals of a theory of multiset consequence. However, to help the theory to get started and make it easier to reconstruct some of the basic AAL theorems, it also seems wise to follow a middle-of-the-road perspective that shortens the gap with the Tarskian paradigm, adopting finite multisets as collections of premisses but leaving the Monotonicity and Reflexivity postulates untouched. The resulting relations have a built-in Weakening condition, although they do not necessarily contract. This policy, as a matter of fact, faces an insurmountable problem. David Ripley [33] has shown that it is not possible to obtain a bijective correspondence between these "naive" multiset consequence relations and closure operators on finite multisets of formulas — any such relation that arises from a closure operator has to obey Contraction.

In the paper [10], two of us developed a strategy to avoid this problem. They adopted a multiple-conclusion format, studying relations between finite multisets of formulas of a given propositional language. They also suitably modified the notions of closure operator and closure system on finite multisets of formulas so as to recover the traditional lattice isomorphism results that characterise the standard set-theoretical framework. These correspondences were laid down as the embryo of a theory that aimed at eventually obtaining appropriate analogues of the main results available in AAL for Tarskian consequence.

It is worth asking whether something like these multiset consequence relations, which have no direct counterpart in AAL, can still ensconce in some more flexible apparatus based on (not necessarily complete) lattices. After all, finite multisets over a set still form a lattice under the operations

$$(\mathfrak{X} \vee \mathfrak{Y})(a) = \sup\{\mathfrak{X}(a), \mathfrak{Y}(a)\} \quad \text{and} \quad (\mathfrak{X} \wedge \mathfrak{Y})(a) = \inf\{\mathfrak{X}(a), \mathfrak{Y}(a)\}$$

(see below), out of which we could define an $\mathbf{M}$-module of sorts that would land us in known territory. However, it is not hard to see that the crucial operation on multisets is not any of these, but rather *multiset sum*:

$$(\mathfrak{X} \uplus \mathfrak{Y})(a) = \mathfrak{X}(a) + \mathfrak{Y}(a).$$

It is via multiset sum that we aggregate multisets of premisses in substructural logics and formulate sequent rules in substructural sequent calculi. Being a non-idempotent operation, though, it is scarcely pliant to the methods reviewed so far. The GTCR's on complete lattices, therefore, need to be replaced by appropriate relations on *dually integral partially ordered Abelian monoids*, the prime motivating example being the po-monoid

$$\left\langle Fm_{\mathcal{L}}^{\flat}, \leqslant, \uplus, \emptyset \right\rangle,$$

where $Fm_{\mathcal{L}}^{\flat}$ is the set of finite multisets of formulas of the propositional language $\mathcal{L}$, $\uplus$ is multiset sum, $\emptyset$ is the empty multiset and $\leqslant$ is the sub-multisethood relation (all these notions will be rehearsed in Section 2).

Using the multiset framework sketched above, the following notion of multiset deductive relation was introduced in [10]:

Definition 5. *A* multiset deductive relation *(*MDR*) on a propositional language $\mathcal{L}$ is a relation $\vdash$ on $Fm_{\mathcal{L}}^{\flat}$ such that for each $\Gamma, \Delta, \Pi \in Fm_{\mathcal{L}}^{\flat}$:*

- $\Gamma \uplus \Delta \vdash \Gamma$. *(Reflexivity)*
- *If* $\Gamma \vdash \Delta$ *and* $\Delta \vdash \Pi$, *then* $\Gamma \vdash \Pi$. *(Transitivity)*
- *If* $\Gamma \vdash \Delta$, *then* $\Gamma \uplus \Pi \vdash \Delta \uplus \Pi$. *(Compatibility)*

1.4 Overview of the paper

This paper is structured as follows. In Section 2 we provide a short primer on multisets. In Section 3, we introduce the concept of a *deductive relation (*DR*)* on a dually integral Abelian po-monoid. This is a modification of the notion of GTCR on a complete lattice, so as to encompass deductive relations on multisets and other examples that are not directly covered by Galatos and Tsinakis' theory. Since, as we have seen, GTCR's on a lattice are in bijective correspondence with closure operators on the same lattice, it is to be expected — if we are on the right track — that an analogous result holds with respect to DR's and some sort of "operational companions" of such. The fact that aggregation

of premisses and conclusions is abstractly represented by a monoidal operation which, unlike set union, is not necessarily idempotent, implies that it won't do to define these operators in the standard way. Thus, *deductive operators (*DO*'s)* on a dually integral Abelian po-monoid $\mathbf{R} = \langle R, \leq, +, 0\rangle$ are introduced as certain maps $\delta\colon R \to \wp(R)$. Similarly, we propose a notion of *deductive system (*DS*)* that appropriately generalises closure systems associated with closure operators. The main result of the section is:

Theorem A (see Theorem 23) *If* $\mathbf{R}$ *is a dually integral Abelian po-monoid, then the posets* $\langle \mathrm{Rel}(\mathbf{R}), \subseteq\rangle$, $\langle \mathrm{Op}(\mathbf{R}), \preccurlyeq\rangle$ *and* $\langle \mathrm{Sys}(\mathbf{R}), \supseteq\rangle$ *are isomorphic.*

In Section 4, the problem of action-invariance is under scrutiny. We define a category $\mathbf{A}-\mathsf{Mod}$ of modules over dually integral po-semirings (called $\mathbf{A}$-*modules*) that is closely related to the category of Galatos and Tsinakis' $\mathbf{M}$-modules and includes as new examples the modules $\boldsymbol{Mult}_{\mathcal{L}}$ whose underlying po-monoids have the form $\langle Fm^{\flat}_{\mathcal{L}}, \leqslant, \uplus, \emptyset\rangle$ for some language $\mathcal{L}$, and whose scalars are finite multisets of $\mathcal{L}$-substitutions. In this wider framework, we obtain analogues of the main theorems proved by Galatos and Tsinakis. Here is an example:

Theorem B (see Theorem 43) *An* $\mathbf{A}$-module $\boldsymbol{R}$ *is onto-projective in* $\mathbf{A}-\mathsf{Mod}$ *iff, for any other* $\mathbf{A}$-*module* $\mathbf{S}$ *and action-invariant* DO*'s* δ *and* γ *on* $\mathbf{R}$ *and* $\mathbf{S}$ *respectively, every injective and order-reflecting morphism* $\Phi\colon \mathbf{R}_\delta \to \mathbf{S}_\gamma$ *is induced by some morphism.*

In Theorem B, $\mathbf{R}_\delta, \mathbf{S}_\gamma$ are $\mathbf{A}$-modules of sets whose universes are, respectively, the union of all δ-images (resp., γ-images) of elements of R (resp. S). We also show that our motivating multiset-theoretical example is just as well-behaved as the standard examples in Galatos and Tsinakis' theory:

Theorem C (see Theorem 48) *Modules arising from multisets are onto-projective in the appropriate categories.*

In Section 5, we zoom in on *multiset deductive relations (*MDR*'s)*. We introduce two types of matrix semantics for such relations. If $\vdash$ is an MDR on a language $\mathcal{L}$, an $\mathcal{L}$-*hypermatrix* is a pair $\langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra and F is a $\leqslant$-downward closed set of finite multisets over A. We show that the $\mathcal{L}$-hypermatrix models of an MDR $\vdash$ correspond bijectively to the models of a Gentzen relation uniquely associated to $\vdash$. This opens the way for importing into our theory all sorts of tools and results from the abstract theory of Gentzen systems [29, 31, 32], including a workable definition of Leibniz congruence of an $\mathcal{L}$-hypermatrix and a completeness theorem for any substitution-invariant MDR:

Theorem D (see Theorem 64) *Any substitution-invariant* MDR *is complete with respect to the class of its reduced* $\mathcal{L}$*-hypermatrix models.*

An alternative type of matrix semantics for MDR's, on the other hand, involves certain structures made up by an algebra $\mathbf{A}$, a dually integral Abelian po-monoid $\langle D, \leq, +, 0\rangle$, and a po-monoid homomorphism from $\langle A^{\flat}, \leqslant, \uplus, \emptyset\rangle$ to $\langle D, \leq, +, 0\rangle$. These structures subsume ordinary logical matrices, which arise

when **D** is the 2-element join semilattice. We clarify the relationship between these *monoidal matrices* and $\mathcal{L}$-hypermatrices. We also show that, in the most favourable cases, the structure of the former can be simplified to a pair constituted by an algebra and a *fuzzy subset* of its universe. We use these simplified matrices to provide a completeness theorem for a multiset-theoretic companion of infinite-valued Łukasiewicz logic.

Finally, we introduce Hilbert systems suited for multiset consequence and prove that their derivability relations are substitution-invariant MDR's; conversely, every substitution-invariant MDR is shown to arise as the derivability relation of some such Hilbert system. As an example, we provide a Hilbert-style axiomatisation of the above-mentioned multiset-theoretic companion of infinite-valued Łukasiewicz logic. The main result is:

Theorem E (see Theorem 78) *Every substitution-invariant* MDR *coincides with the derivability relation of some axiomatic system* AS.

2 Preliminaries on multisets

In this short section we review some basic notions about multisets only to such an extent as it is needed for the purposes of the present paper. For a more comprehensive account, the reader can consult e.g. [4, 35].

By a multiset over a set A we mean a function $\mathfrak{X}$ from A to the set $\mathbb{N}$ of natural numbers.[3] By $\wp^M(A)$ we denote the set of all multisets over A.

The *root set* of the multiset $\mathfrak{X}$ is the set

$$|\mathfrak{X}| = \{a \in A : \mathfrak{X}(a) > 0\}.$$

If $a \in |\mathfrak{X}|$, we say that a is an *element* of $\mathfrak{X}$ of *multiplicity* $\mathfrak{X}(a)$. A multiset $\mathfrak{X}$ is *finite* if $|\mathfrak{X}|$ is finite. By $A^\flat$ we denote the set of all finite multisets over A.

The empty multiset, i.e. the constant function 0, will be denoted by the same symbol $\emptyset$ used for the empty set — the context will always be sufficient to resolve ambiguities.

The set of all multisets over a set A inherits the ordering of $\mathbb{N}$ in the following way:

$$\mathfrak{Y} \leqslant \mathfrak{X} \quad \iff \quad \mathfrak{Y}(a) \leq \mathfrak{X}(a), \text{ for all } a \in A,$$

and, with respect to this ordering, it forms a lattice with joins and meets defined as

$$(\mathfrak{X} \vee \mathfrak{Y})(a) = \sup\{\mathfrak{X}(a), \mathfrak{Y}(a)\} \quad \text{and} \quad (\mathfrak{X} \wedge \mathfrak{Y})(a) = \inf\{\mathfrak{X}(a), \mathfrak{Y}(a)\},$$

for all $a \in A$. The operation $\vee$ is a kind of "union", and, true to form, if we consider the subsets of A as multisets whose elements have multiplicity 1, and

[3] We use letters $\Gamma, \Delta, \Pi, \ldots$ for multisets of *formulas*, while $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ are used for general multisets.

$\mathfrak{X}$ and $\mathfrak{Y}$ are subsets of A, then $\mathfrak{X} \vee \mathfrak{Y} = \mathfrak{X} \cup \mathfrak{Y}$. There is another "union-like" operation of *sum* between multisets, defined as follows:

$$(\mathfrak{X} \uplus \mathfrak{Y})(a) = \mathfrak{X}(a) + \mathfrak{Y}(a), \text{ for all } a \in A.$$

The next proposition shows that the set of finite multisets over a set A can be seen as the universe of a dually integral Abelian po-monoid which we will denote as $\mathbf{A}^\flat$.

Proposition 6. *For any set A the structure*

$$\mathbf{A}^\flat = \left\langle A^\flat, \leqslant, \uplus, \emptyset \right\rangle.$$

is a dually integral Abelian po-monoid, i.e., a structure where:

1. $\left\langle A^\flat, \uplus, \emptyset \right\rangle$ *is Abelian monoid.*
2. $\leqslant$ *is a partial order compatible with* $\uplus$*, i.e.,*

$$\text{if } \mathfrak{X} \leqslant \mathfrak{Y}, \text{ then } \mathfrak{X} \uplus \mathfrak{Z} \leqslant \mathfrak{Y} \uplus \mathfrak{Z}.$$

3. $\emptyset$ *is the bottom element of* $\leqslant$*.*

We will also have occasion to use the operation $\mathfrak{X} \backslash \mathfrak{Y}$ defined by

$$(\mathfrak{X} \backslash \mathfrak{Y})(a) = \max\left(\mathfrak{X}(a) - \mathfrak{Y}(a), 0\right), \text{ for all } a \in A,$$

relying on the context to disambiguate between this operation and standard set-theoretic subtraction.

As it is customary to do, we use square brackets for multiset abstraction; so, for example, $[a, a, b, c]$ will denote the multiset $\mathfrak{X}$ s.t. $\mathfrak{X}(a) = 2, \mathfrak{X}(b) = \mathfrak{X}(c) = 1$, and $\mathfrak{X}(d) = 0$ for any $d \notin \{a, b, c\}$.

Every map $f\colon A \to B$ can be extended to a morphism from $\mathbf{A}^\flat$ to $\mathbf{B}^\flat$ (for which we retain the same symbol) via

$$f(\mathfrak{X}) = [f(x_1), \ldots, f(x_n)]$$

for every $\mathfrak{X} = [x_1, \ldots, x_n]$. We will use this notation without special mention.

3 Deductive relations

3.1 Basic definitions and facts

We emphasised in our introduction that we need to consider more general relations than Galatos and Tsinakis' GTCR's if we want to properly account for multiset consequence relations introduced in Definition 5. By Proposition 6, the set of finite multisets of formulas of a given language can be equipped with the structure of a dually integral Abelian po-monoid. This leads us to the next definition.

Definition 7. *A* deductive relation (DR) *on a dually integral Abelian po-monoid* $\mathbf{R} = \langle R, \leq, +, 0\rangle$ *is a relation* $\vdash$ *on* R *such that for every* $a, b, c \in R$*:*

- *If* $a \leq b$*, then* $b \vdash a$. *(Generalised Reflexivity)*
- *If* $a \vdash b$ *and* $b \vdash c$*, then* $a \vdash c$. *(Transitivity)*
- *If* $a \vdash b$*, then* $a + c \vdash b + c$. *(Compatibility)*

A DR *is* finitary *if for each compact*[4] *element* b *such that* $a \vdash b$ *there is a compact element* $a' \leq a$ *such that* $a' \vdash b$.

Observe that $\vdash$ is a compatible preordering of R and 0 is a $\vdash$-maximum. Also observe that, using properties of compatible preorderings on Abelian monoids and the dual integrality of $\mathbf{R}$, we have:

Lemma 8. *Let* $\vdash$ *be a* DR *on* $\mathbf{R} = \langle R, \leq, +, 0\rangle$*. For all* $a, b, c, d \in R$*:*

1. $a \vdash b$ *and* $c + b \vdash d$ *imply* $c + a \vdash d$. *(Cut)*
2. $a \vdash b$ *implies* $c + a \vdash b$. *(Monotonicity)*

Some examples of DR's follow hereafter. Our prime motivating example, the *multiset deductive relations*, will be thoroughly studied in Section 5, where appropriate particular examples will be given. It is easy to observe that:

Example 9. *Let* $\mathcal{L}$ *be a propositional language. Multiset deductive relations on* $\mathcal{L}$ *are exactly the* DR*'s on* $\mathbf{Fm}^{\flat}_{\mathcal{L}}$.

Of course, standard ACR's (hence, in particular, TCR's) give rise to instances of deductive relations:

Example 10. *Let* $\vdash$ *be an* ACR *on the set* A *(see Definition 1) and let*

$$\mathbf{R}^{\wp(A)} = \langle \wp(A), \subseteq, \cup, \emptyset\rangle .$$

Then $\vdash'$*, where, for all* $X, Y \subseteq A$*,* $X \vdash' Y$ *iff* $X \vdash a$ *for all* $a \in Y$*, is a* DR *on* $\mathbf{R}^{\wp(A)}$.

In view of the previous example, the reader will be curious to figure out whether DR's also generalise GTCR's (see Definition 3). The answer, here, is less straightforward.

Proposition 11. *Let* $\mathbf{L} = \langle L, \wedge, \vee\rangle$ *be a complete lattice with induced order* $\leq$ *and bottom element* 0*. Then the structure*

$$\mathbf{L}' = \langle L, \leq, \vee, 0\rangle$$

is a dually integral Abelian po-monoid and any (finitary) GTCR *on* $\mathbf{L}$ *is a (finitary)* DR *on* $\mathbf{L}'$.

[4] An element $a \in R$ is *compact* if for each directed set $D \subseteq R$ which has a supremum $\sup(D) \geq a$ we have $d \geq a$ for some $d \in D$.

Proof. The only non-trivial part is Compatibility. Assume that $a \vdash b$. Thus, $a \vee c \vdash b$ and $a \vee c \vdash c$ and so $\bigvee\{x : a \vee c \vdash x\} \vdash b \vee c$. By the defining condition on GTCR's, it follows that $a \vee c \vdash b \vee c$. The finitarity part is obvious. □

The converse direction does not hold in general. The next proposition provides a class of explicit counterexamples, indeed, a very wide one, because any substitution-invariant TCR $\vdash$ with a theorem (i.e., an element a such $\emptyset \vdash a$) containing a variable has infinitely many theorems.

Proposition 12. *Let $\vdash$ be a finitary* ACR *with infinitely many theorems*[5] *and $\mathbf{R}^{\wp(A)}$ and let $\vdash'$ be defined as in Example 10. Then the relation defined by:*

$$X \Vdash Y \text{ if there is a finite } Y' \subseteq Y \text{ such that } Y \setminus Y' \subseteq X \text{ and } X \vdash' Y'$$

is a DR *on $\mathbf{R}^{\wp(A)}$ but it is not a* GTCR *on $\wp(A)$.*

Proof. For our first claim, the only nontrivial condition to check is Transitivity. Assume that $X \Vdash Y$ and $Y \Vdash Z$, and let Y' and Z' be finite sets with the required properties. We claim that the finite set $\bar{Z} = Z' \cup (Y' \cap Z)$ witnesses $X \Vdash Z$. In fact:

- Clearly $\bar{Z}$ is a finite subset of Z.
- $X \vdash' \bar{Z}$, because $\Vdash \subseteq \vdash'$ implies that $X \vdash' Z$.
- The final condition is obtained by the following chain:

$$\begin{aligned} Z \setminus (Z' \cup (Y' \cap Z)) &= (Z \setminus Z') \cap (Z \setminus (Y' \cap Z)) = (Z \setminus Z') \cap (Z \setminus Y') \\ &\subseteq Y \cap (Z \setminus Y') \subseteq Y \setminus Y' \subseteq X. \end{aligned}$$

To conclude the proof, it is easy to observe that for each theorem a we have $\emptyset \Vdash a$, yet clearly $\emptyset \nVdash \{a : \emptyset \Vdash a\}$. □

The above result is not that surprising, as our definition of DR only employs finitary operations, as opposed to the use of infinite suprema in GTCR's. Our approach has the advantage that our background structure could be much smaller, as the following proposition shows.

Proposition 13. *Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a complete algebraic lattice with induced order $\leq$ and bottom element 0. Then the structure*

$$K(\mathbf{L}) = \langle K(L), \leq, \vee, 0 \rangle ,$$

where $K(L)$ is the set of compact elements of $\mathbf{L}$, is a dually integral Abelian po-monoid. Moreover, there is bijective correspondence between finitary GTCR*'s on $\mathbf{L}$ and* DR*'s on $K(\mathbf{L})$.*[6]

[5] Actually, any ACR where there is a finite set with infinitely many consequences would do the job.

[6] Note that all DR's on $K(\mathbf{L})$ are finitary.

Proof. For a start, note that finitary GTCR's on complete algebraic lattices are fully determined by their subrelations between compact elements. In fact, for $x \in L$, let

$$C_x = \{c \in K(L) : c \leq x\}.$$

Thus $x = \bigvee C_x$. Now, take $x, y \in L$. We have that $x \vdash y$ iff for each $c \in C_y$ there is $x_c \in C_x$ such that $x_c \vdash c$. As in Example 11, it is possible to show that the restriction of any GTCR on $\mathbf{L}$ is a DR on $K(\mathbf{L})$, and the previous observation entails that if two GTCR's differ they also differ on compact elements.

Conversely, assume that $\vdash$ is a DR and define:

$$x \vdash' y \text{ iff for each } c \in C_y \text{ there is } x_c \in C_x \text{ such that } x_c \vdash c.$$

The only condition in need of a proof is that $x \vdash' p$, where $p = \bigvee\{a \in L : x \vdash' a\}$. Let $C = \{c \in K(L) : x \vdash' c\}$. If we show that $C = C_p$, we are done. For the nontrivial inclusion, assume that $c \in K(L)$ and $c \leq p$. Then there are $c_1, \ldots, c_n$ such that $c \leq c_1 \vee \ldots \vee c_n$ and $x \vdash' c_j$ for all $j \leq n$. This means that for all $j \leq n$ there is $x_j \leq x$ such that $x_j \vdash c_j$. Using properties of DR's we obtain $x_1 \vee \ldots \vee x_n \vdash c_1 \vee \ldots \vee c_n$ and so $x \vdash' c$, i.e., $c \in C$. This mapping is clearly one-one and an inverse to the previous one. □

The next example identifies a deductive relation on fuzzy sets. It is introduced to underscore the generality of our framework, but it will not be further discussed in the remainder of this paper.

Example 14. *Let $\mathcal{L}$ be the language of infinite-valued Łukasiewicz logic Ł. The relation $\vdash \subseteq [0,1]^{Fm_\mathcal{L}} \times [0,1]^{Fm_\mathcal{L}}$ defined as:*

$$\Gamma \vdash \Delta \quad \textit{iff for each } [0,1]\textit{-valued evaluation } e \textit{ we have: if } e(\psi) \geq \Gamma(\psi) \textit{ for each } \psi \in Fm_\mathcal{L}\textit{, then } e(\psi) \geq \Delta(\psi) \textit{ for each } \psi \in Fm_\mathcal{L}$$

is a DR *on*

$$\mathbf{R}^{[0,1]^{Fm_\mathcal{L}}} = \left\langle [0,1]^{Fm_\mathcal{L}}, \leq, \vee, \emptyset \right\rangle,$$

where $\emptyset(\varphi) = 0$ for all $\varphi \in Fm_\mathcal{L}$ and $\vee$ is pointwise supremum.

Among the basic notions of algebraic logic that need to be redefined in our new framework, one certainly finds the concepts of *theory* and *theorem*. Here, we must stray away to a certain extent from the received orthodoxy. In view of Example 10, given a DR $\vdash$ on $\mathbf{R} = \langle R, \leq, +, 0\rangle$, one would expect a $\vdash$-theory to be an *element* of R with certain properties. In AAL, in fact, a theory is a deductively closed set of formulas. In particular, the theory generated by a set of formulas X is the smallest deductively closed set of formulas that includes X — or else, the largest Y such that $X \vdash Y$ — and has the property that its subsets are exactly the consequences of X. However, in the case of MDR's (Definition 5), this would not work. Such a "largest consequence" need not always exist, because it could happen, for instance, that $\Gamma \vdash \Delta$ and $\Gamma \vdash \Pi$ but $\Gamma \nvdash \Delta \vee \Pi$. Nevertheless, it makes sense to collect all the consequences of a given multiset Γ of formulas into a set and view *the set itself* as the deductive closure of Γ. Abstracting away from this particular example, we are led to the following definition (recall that every DR is a preorder on R).

Definition 15. *Let $\vdash$ be a* DR *on* $\mathbf{R}$*. A $\vdash$-theory (or simply a theory, when $\vdash$ is understood) is a $\vdash$-upset T of R. By* $\mathrm{Th}(\vdash)$ *we denote the family of all $\vdash$-theories.*

Proposition 16. *Let $\vdash$ be a* DR *on* $\mathbf{R}$*. Then* $\mathrm{Th}(\vdash)$ *is a closure system on R.*

Proof. Clearly, R is a theory. Suppose that $\{T_i\}_{i\in I}$ is a nonempty family of theories, $a \in \bigcap\{T_i\}_{i\in I}$ and $a \vdash b$. Given an arbitrary T_j $(j \in I)$, $a \in T_j$, whence $b \in T_j$. It follows that $b \in \bigcap\{T_i\}_{i\in I}$. □

Observe that the $\vdash$-theory generated by $a \in R$ is just the principal $\vdash$-upset generated by a. We denote it by $\mathrm{Th}_\vdash(a)$, and we denote by $\mathrm{Th}^p(\vdash)$ the set of all *principal* members of $\mathrm{Th}(\vdash)$, i.e. theories that are generated by a single element. The subscripts in $\mathrm{Th}_\vdash$ will be dropped when the preordering is clear from the context.

Let us note the each $\vdash$-theory is a union of principal theories, a quite unusual feature from the point of view of the general theory of closure systems. Actually we can prove even more:

Proposition 17. *Let $\vdash$ be a* DR *on* $\mathbf{R}$*. Then*

$$\mathrm{Th}_\vdash(X) = \bigcup\{\mathrm{Th}_\vdash(x) : x \in X\}.$$

Proof. Clearly if $x \in X$ then $\mathrm{Th}_\vdash(x) \subseteq \mathrm{Th}_\vdash(X)$ and so one inclusion follows. To prove the converse one it suffices to show that $T = \bigcup\{\mathrm{Th}_\vdash(x) : x \in X\}$ is a $\vdash$-theory. Assume that $a \in T$ and $a \vdash b$. Thus we know that $a \in \mathrm{Th}_\vdash(x)$ for some $x \in X$ and so Transitivity completes the proof. □

Definition 18. *Let $\vdash$ be a* DR *on* $\mathbf{R} = \langle R, \leq, +, 0\rangle$*. The element $b \in R$ is a $\vdash$-theorem if $0 \vdash b$.*

Observe that 0 is always a theorem[7] and the theorems are exactly the $\vdash$-maximal elements of R, since for any $a \in R$ we have $a \vdash 0$. Note that $\geq$, which can be seen as the least DR on $\mathbf{R}$, has 0 as the only theorem.

The reader will recall that one of the main advantages of the notion of GTCR is the fact that the collection of theories of a given GTCR $\vdash$ over a complete lattice is itself a complete lattice, which is furthermore determined by $\vdash$. We prove an analogous results for DR's and *principal* theories.

Theorem 19. *Let $\vdash$ be a* DR *on* $\mathbf{R} = \langle R, \leq, +, 0\rangle$*. Let us define $+^\vdash$ on* $\mathrm{Th}^p(\vdash)$ *as*

$$\mathrm{Th}_\vdash(x) +^\vdash \mathrm{Th}_\vdash(y) = \mathrm{Th}_\vdash(x + y).$$

Then

$$\mathbf{Th}_\vdash = \langle \mathrm{Th}^p(\vdash), \subseteq, +^\vdash, \mathrm{Th}(0)\rangle$$

is a dually integral po-monoid and the mapping $\mathrm{Th}_\vdash : R \to \mathrm{Th}^p(\vdash)$ *is a surjective morphism.*

[7] There are DR's for which 0 is the *only* theorem — for example, any DR that stems from a *theoremless* ACR as in Example 10 has this property.

Proof. The relation $\subseteq$ is clearly a partial order on $\mathrm{Th}^p(\vdash)$ with $\mathrm{Th}(0)$ as a bottom element. Moreover, Th is order preserving: in fact, by Generalised Reflexivity $a \in \mathrm{Th}(a)$ and so, by Transitivity, $a \vdash b$ iff $\mathrm{Th}(b) \subseteq \mathrm{Th}(a)$. Thus, if $a \leq b$ then, by Generalized Reflexivity, $\mathrm{Th}(a) \subseteq \mathrm{Th}(b)$.

We now sketch the proof of the fact that the operation $+^{\vdash}$ is well-defined. To this end, consider $a, b, c, d \in R$ such that $\mathrm{Th}(a) = \mathrm{Th}(b)$ and $\mathrm{Th}(c) = \mathrm{Th}(d)$. In particular, $b \vdash a$ and $d \vdash c$ and so by Compatibility and Transitivity $d+b \vdash a+c$, whence $\mathrm{Th}(a+c) \subseteq \mathrm{Th}(b+d)$. The other inclusion is proved analogously. The fact that $\langle \mathrm{Th}^p(\vdash), +^{\vdash}, \mathrm{Th}(0) \rangle$ is an Abelian monoid is obvious, so we only need to check that $+^{\vdash}$ is compatible with the order. Let $a, b, c \in R$ be such that $\mathrm{Th}(a) \subseteq \mathrm{Th}(b)$. Thus $b \vdash a$ and so $b + c \vdash a + c$. As a consequence,

$$\mathrm{Th}(a) +^{\vdash} \mathrm{Th}(c) = \mathrm{Th}(a + c) \subseteq \mathrm{Th}(b + c) = \mathrm{Th}(b) +^{\vdash} \mathrm{Th}(c).$$

The fact that Th is a surjective morphism is again obvious. □

3.2 Deductive operators and systems

In AAL, propositional logics can be introduced in three different but equivalent ways: via consequence relations, via closure operators, and via closure systems. The same is true of the approach we have taken. While deductive relations are abstract counterparts of TCR's, our next goal is to define suitable abstract notions of deductive operator and deductive system, in such a way as to generalise the lattice isomorphisms between the lattices of consequence relations, of closure operators, and of closure systems that are available in the traditional theory of AAL. Analogues of the classical concepts of closure operator and closure system can be defined as follows.

Definition 20. *A* deductive operator (DO) *on a dually integral Abelian po-monoid* $\mathbf{R} = \langle R, \leq, +, 0 \rangle$ *is a map* $\delta \colon R \to \wp(R)$ *such that for every* $a, b, c \in R$*:*

- $a \in \delta(a)$. *(Enlargement)*
- *If* $a \leq b$*, then* $\delta(a) \subseteq \delta(b)$. *(Order Preservation)*
- *If* $a \in \delta(b)$*, then* $\delta(a) \subseteq \delta(b)$. *(Idempotency)*
- *If* $a \in \delta(b)$*, then* $a + c \in \delta(b + c)$. *(Compatibility)*

Observe that, in full analogy with Definition 15, a DO is a map from elements of R to *subsets* of R.

Next we define the notion of a deductive system; recall that closure systems are systems of theories of some ACR, but as we have seen in the previous subsection the *principal* theories are the crucial ones in our framework (as they can be seen as universes of dually integral Abelian po-monoids). This leads to the following definition:

Definition 21. *A* deductive system (DS) *on a dually integral Abelian po-monoid* $\mathbf{R} = \langle R, \leq, +, 0\rangle$ *is a family* $\mathcal{C} \subseteq \wp(R)$ *of* $\leq$*-downsets of* R *such that for the mapping* $\delta_{\mathcal{C}} \colon R \to \wp(R)$ *defined by* $\delta_{\mathcal{C}}(x) = \bigcap\{C \in \mathcal{C} : x \in C\}$ *we have* $\delta_{\mathcal{C}}(R) = \mathcal{C}$ *and if* $\delta_{\mathcal{C}}(x) \subseteq \delta_{\mathcal{C}}(y)$*, then* $\delta_{\mathcal{C}}(x + z) \subseteq \delta_{\mathcal{C}}(y + z)$ *for all* $x, y, z \in R$*.*

The subscript, or superscript, $\mathcal{C}$ will be omitted whenever it is not needed to clarify potential confusions.

Proposition 22. *Let* $\mathcal{C}$ *be a* DS *on a dually integral Abelian po-monoid* $\mathbf{R}$*. Then the mapping* $\delta_{\mathcal{C}}$ *is a* DO *on* $\mathbf{R}$*.*

Proof. Enlargement is obvious. For Idempotency, assume that $a \in \delta(b)$, $c \in \delta(a)$ and we have $X \in \mathcal{C}$ such that $b \in X$. Then $a \in X$ (due to the first assumption) and so $c \in X$ (due to the second assumption), i.e., $c \in \delta(b)$. If $a \leq b$, then (as each X is $\leq$-downset) $a \in \delta(b)$ and so by Idempotency $\delta(a) \subseteq \delta(b)$, which takes care of Order Preservation. Finally, we prove Compatibility: using the conditions we already established, if $a \in \delta(b)$, then $\delta(a) \subseteq \delta(b)$ and so $\delta(a+c) \subseteq \delta(b + c)$, whence $a + c \in \delta(b + c)$. □

Given a dually integral Abelian po-monoid $\mathbf{R} = \langle R, \leq, +, 0\rangle$, we respectively denote by $\mathrm{Re\,l}(\mathbf{R}), \mathrm{Op}(\mathbf{R})$ and $\mathrm{Sys}(\mathbf{R})$ the sets of deductive relations, deductive operators and deductive systems on $\mathbf{R}$. We next define partial orders on these sets. $\mathrm{Re\,l}(\mathbf{R})$ will be viewed as partially ordered by set inclusion, and $\mathrm{Sys}(\mathbf{R})$ by supersethood. We define an order on $\mathrm{Op}(\mathbf{R})$ as follows: given $\delta, \gamma \in \mathrm{Op}(\mathbf{R})$, we set $\delta \preccurlyeq \gamma$ iff $\delta(a) \subseteq \gamma(a)$ for every $a \in R$.

Theorem 23. *If* $\mathbf{R} = \langle R, \leq, +, 0\rangle$ *is a dually integral Abelian po-monoid, then the posets* $\langle \mathrm{Re\,l}(\mathbf{R}), \subseteq\rangle$*,* $\langle \mathrm{Op}(\mathbf{R}), \preccurlyeq\rangle$ *and* $\langle \mathrm{Sys}(\mathbf{R}), \supseteq\rangle$ *are isomorphic.*

Proof. We define maps $\delta_{(\,)} \colon \mathrm{Re\,l}(\mathbf{R}) \to \mathrm{Op}(\mathbf{R})$ and $\vdash_{(\,)} \colon \mathrm{Op}(\mathbf{R}) \to \mathrm{Re\,l}(\mathbf{R})$ as follows:

$$\begin{aligned} \vdash_{\delta} &= \{\langle a, b\rangle : b \in \delta(a)\} \\ \delta_{\vdash}(a) &= \mathrm{Th}_{\vdash}(a). \end{aligned}$$

It is easy to show that they are well-defined and monotone. We now prove they are mutually inverse. Consider $\vdash \in \mathrm{Rel}(\mathbf{R})$ and observe that $x \vdash y$ iff $y \in \delta_{\vdash}(x)$ iff $x \vdash_{\delta_{\vdash}} y$. On the other hand, for any $\delta \in \mathrm{Op}(\mathbf{R})$:

$$x \in \delta(y) \text{ iff } y \vdash_{\delta} x \text{ iff } x \in \delta_{\vdash_{\delta}}(y).$$

Next, we handle the slightly more complex case of DR's and DS's. We define maps $\mathcal{C}_{(\,)} \colon \mathrm{Rel}(\mathbf{R}) \to \mathrm{Sys}(\mathbf{R})$ and $\vdash_{(\,)} \colon \mathrm{Sys}(\mathbf{R}) \to \mathrm{Rel}(\mathbf{R})$ as follows:

$$\begin{aligned} \mathcal{C}_{\vdash} &= \{\mathrm{Th}_{\vdash}(a) : a \in R\}; \\ \vdash_{\mathcal{C}} &= \{\langle a, b\rangle : b \in \delta_{\mathcal{C}}(a)\}. \end{aligned}$$

Using Theorem 19, we conclude that $\mathcal{C}_{\vdash}$ is indeed a DS, while Proposition 22 guarantees that $\delta_{\mathcal{C}}$ is a DO. Monotonicity of both functions is obvious and so

is the fact that they are mutually inverse. For the sake of completeness, we observe that $\mathcal{C}_\delta = \{\delta(a) : a \in R\}$ maps $\mathrm{Op}(\mathbf{R})$ to $\mathrm{Sys}(\mathbf{R})$, and that such a mapping is inverted by the mapping that sends any DS $\mathcal{C}$ to the already defined deductive operator $\delta_{\mathcal{C}}$. □

Corollary 24. *$\langle \mathrm{Rel}(\mathbf{R}), \subseteq\rangle$, $\langle \mathrm{Op}(\mathbf{R}), \preccurlyeq\rangle$ and $\langle \mathrm{Sys}(\mathbf{R}), \supseteq\rangle$ are complete lattices.*

Proof. By Theorem 23, it will suffice to prove our claim for any one of these posets, say $\langle \mathrm{Op}(\mathbf{R}), \preccurlyeq\rangle$. It is obvious that δ_1, defined by

$$\delta_1(a) = R \text{ for all } a \in R$$

is a DO and that it is the top element w.r.t. $\preccurlyeq$. In order to see that DO's on $\mathbf{R}$ form a complete lattice, it is enough to see that for every family of DO's $\{\delta_i : i \in I\}$, the map $\bigwedge_{i\in I} \delta_i$ defined by

$$\bigwedge_{i\in I} \delta_i(a) = \bigcap\{\delta_i(a) : i \in I\}$$

is again a DO. Enlargement is clear. For Order Preservation, suppose that $a \leq b$ and $c \in \delta_i(a)$ for all $i \in I$. Then for all $i \in I$ we have that $\delta_i(a) \subseteq \delta_i(b)$, whence $c \in \bigcap\{\delta_i(b) : i \in I\}$. As to Idempotency, suppose $a \in \delta_i(b)$ for all $i \in I$, and $c \in \delta_i(a)$ for all $i \in I$. Then for all $i \in I$ we have that $\delta_i(a) \subseteq \delta_i(b)$ and so again $c \in \bigcap\{\delta_i(b) : i \in I\}$. Compatibility, once more, is clear. □

3.3 Blok–Jónsson companions of deductive relations

Deductive relations, deductive operators and deductive systems respectively give rise to special kinds of ACR's (Definition 1), closure operators and closure systems. In the present subsection, we point out the fact that there is a significant transfer of information from the original relations, operators and systems to these "Blok–Jónsson companions", which we now proceed to define.

Given a DR $\vdash$ on $\mathbf{R} = \langle R, \leq, +, 0\rangle$, its *Blok–Jónsson companion* is the relation $\vdash^{BJ} \subseteq \wp(R) \times R$ defined as follows for every $X \subseteq R$ and every $a \in R$:

$$X \vdash^{BJ} a \text{ iff there is } y \in X \text{ s.t. } y \vdash a.$$

Lemma 25. *If $\vdash$ is a DR on $\mathbf{R} = \langle R, \leq, +, 0\rangle$, then $\vdash^{BJ}$ is an ACR on R.*

Proof. For Reflexivity, suppose $x \in X$; we want to show that $X \vdash^{BJ} x$, i.e. that there is $y \in X$ s.t. $y \vdash x$. However, by the reflexivity of $\vdash$, x itself fits the bill.

Monotonicity is straightforward from the definition of $\vdash^{BJ}$.

For Cut, we want to show that $X \vdash^{BJ} y$ and $Z \vdash^{BJ} x$ for all $x \in X$ imply that $Z \vdash^{BJ} y$. In fact, suppose there is $x \in X$ s.t. $x \vdash y$. This implies that there is $z \in Z$ s.t. $z \vdash x$, whence by transitivity of $\vdash$, $z \vdash y$, which in turns entails that $Z \vdash^{BJ} y$. □

Along the same lines, deductive operators and deductive systems on $\mathbf{R}$ can be lifted to closure operators and closure systems, respectively, on the base set R.

Lemma 26. *Given a* DO δ *on* $\mathbf{R} = \langle R, \leq, +, 0\rangle$*, the map* $\delta^{BJ} \colon \wp(R) \to \wp(R)$ *defined as*

$$\delta^{BJ}(X) = \bigcup \{\delta(x) : x \in X\}$$

is a closure operator.

Proof. First, if $a \in X \subseteq R$, then $a \in \delta(a) \subseteq \delta^{BJ}(X)$, and therefore $X \subseteq \delta^{BJ}(X)$. If $X \subseteq Y$, then obviously $\delta^{BJ}(X) \subseteq \delta^{BJ}(Y)$. And finally, assume that $z \in \delta^{BJ}\left(\delta^{BJ}(X)\right)$, i.e., there are y, x such that $x \in X, y \in \delta(x), z \in \delta(y)$. By Idempotency and Generalised Reflexivity, $z \in \delta(z) \subseteq \delta(y) \subseteq \delta(x)$, which means $z \in \delta^{BJ}(X)$. □

Lemma 27. *Given a* DS $\mathcal{C}$ *on* $\mathbf{R} = \langle R, \leq, +, 0\rangle$*, the family* $\mathcal{C}^{BJ} = \{\bigcup \mathcal{Y} : \mathcal{Y} \subseteq \mathcal{C}\}$ *is a closure system.*

Proof. Recall that $\mathcal{C} = \{\delta_{\mathcal{C}}(x) : x \in R\}$ and as $x \in \delta_{\mathcal{C}}(x)$ then $\bigcup \mathcal{C} = R$. Now we only have to prove that for $\mathcal{X} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{X} \in \mathcal{C}$. If $x \in \bigcap \mathcal{X}$ then for each $X \in \mathcal{X}$ there is c_X such that $x \in \delta_{\mathcal{C}}(c_X) \subseteq X$, thus also $\delta_{\mathcal{C}}(x) \subseteq \delta_{\mathcal{C}}(c_X) \subseteq X$ and so $\delta_{\mathcal{C}}(x) \subseteq \bigcap \mathcal{X}$. To conclude the proof just observe that

$$\bigcap \mathcal{X} = \bigcup \{\delta_{\mathcal{C}}(x) : x \in \bigcap \mathcal{X}\} \in \mathcal{C}^{BJ}.$$ □

In full analogy with the above, we call δ^{BJ} and $\mathcal{C}^{BJ}$ the *Blok–Jónsson companions* of δ and $\mathcal{C}$, respectively. Observe that:

Lemma 28. *Let* $\vdash$ *be a* DR *on* $\mathbf{R} = \langle R, \leq, +, 0\rangle$*. The* $\vdash$*-theories are the theories (in the sense of Blok–Jónsson) of* $\vdash^{BJ}$*. Namely, for* $T \subseteq R$*, t.f.a.e.:*

1. *T is a $\vdash$-upset of R.*
2. *$T \vdash^{BJ} x$ implies $x \in T$.*

Proof. (1) implies (2). Suppose T is a $\vdash$-upset of R and there is $y \in T$ s.t. $y \vdash x$. Then $x \in T$.

(2) implies (1). Suppose that for any x, $T \vdash^{BJ} x$ implies $x \in T$, that $y \in T$, and that $y \vdash z$. So $T \vdash^{BJ} z$, whence $z \in T$. □

Let us continue to use the notations $\vdash_{\delta}, \vdash_{\mathcal{C}}, \mathcal{C}_{\vdash}, \mathcal{C}_{\delta}, \delta_{\vdash}$, and $\delta_{\mathcal{C}}$ for the correspondences between DR's, DO's and DS's on $\mathbf{R}$ spelt out in Theorem 23. With an innocent notational abuse, we employ the same symbols for the standard correspondences between the sets $\mathrm{Acr}(R)$ of ACR's, closure operators $\mathrm{Clop}(R)$ and closure systems $\mathrm{Clos}(R)$, all on R. We now prove that the relation of "taking the Blok–Jónsson companion" commutes with these functions.

Theorem 29. *The following diagrams*

$$\begin{array}{ccc} \mathrm{Rel}(\mathbf{R}) & \xrightarrow{\delta_{\vdash}} & \mathrm{Op}(\mathbf{R}) \\ \downarrow{\scriptstyle \vdash^{BJ}} & & \downarrow{\scriptstyle \delta^{BJ}} \\ \mathrm{Acr}(R) & \xrightarrow{\delta_{\vdash}} & \mathrm{Clop}(R) \end{array} \qquad \begin{array}{ccc} \mathrm{Rel}(\mathbf{R}) & \xrightarrow{\mathcal{C}_{\vdash}} & \mathrm{Sys}(\mathbf{R}) \\ \downarrow{\scriptstyle \vdash^{BJ}} & & \downarrow{\scriptstyle \mathcal{C}^{BJ}} \\ \mathrm{Acr}(R) & \xrightarrow{\mathcal{C}_{\vdash}} & \mathrm{Clos}(R) \end{array}$$

$$\begin{array}{ccc} \mathrm{Op}(\mathbf{R}) & \xrightarrow{\delta_{\mathcal{C}}} & \mathrm{Sys}(\mathbf{R}) \\ \downarrow{\scriptstyle \delta^{BJ}} & & \downarrow{\scriptstyle \mathcal{C}^{BJ}} \\ \mathrm{Clop}(R) & \xrightarrow{\delta_{\mathcal{C}}} & \mathrm{Clos}(R) \end{array}$$

as well as the ones we obtain by reversing the above correspondences, are all commutative.

Proof. The correspondences are well-defined by Theorem 23 and Lemmas 25, 26, and 27. We now take care of some of the commutations. We show that $(\vdash_{\delta})^{BJ} = \vdash_{\delta^{BJ}}$; the other commutations are established similarly. In fact, $\langle X, a\rangle \in (\vdash_{\delta})^{BJ}$ iff there exists $x \in X$ s.t. $x \vdash_{\delta} a$, which in turn holds iff there exists $x \in X$ s.t. $a \in \delta(x)$. But this just means that $a \in \bigcup_{x\in X} \delta(x)$, which amounts to $X \vdash_{\delta^{BJ}} a$. Similarly, $(\delta_{\vdash})^{BJ} = \delta_{\vdash^{BJ}}$. In fact,

$$\begin{aligned} (\delta_{\vdash})^{BJ}(X) &= \bigcup\nolimits_{a\in X} \mathrm{Th}_{\vdash}(a) \\ &= \{b \in R : a \vdash b \text{ for some } a \in X\} \\ &= \left\{b \in R : X \vdash^{BJ} b\right\} \\ &= \delta_{\vdash^{BJ}}(X). \end{aligned}$$

□

This theorem implies, in particular, the following corollary:

Corollary 30. *Let* $\mathbf{R} = \langle R, \leq, +, 0\rangle$ *be a dually integral Abelian po-monoid. The complete lattices of Blok–Jónsson companions of* DR*'s,* DO*'s, and* DS*'s on* $\mathbf{R}$ *are isomorphic.*

4 Action-invariance

One of the remarkable achievements of Blok and Jónsson's treatment of logical consequence is its purely abstract account of substitution-invariance. Resorting to appropriate monoidal actions, Blok and Jónsson effectively sidestep the problem brought about by their use of sets with no structure whatsoever to be preserved. As we have seen, Galatos and Tsinakis turn this insight into the starting point for their categorical foundation of the whole subject. It would be highly desirable, then, to lay down a comparable treatment of action-invariance in our framework. This will be done by equipping our Abelian po-monoids with appropriate monoidal actions.

Our guiding example will again be given by multiset deductive relations, i.e., DR's on $\mathbf{Fm}_{\mathcal{L}}^{\flat}$, as we can naturally call an MDR $\vdash$ *substitution-invariant* if for every $\mathcal{L}$-substitution σ and for every $\Gamma, \Delta \in Fm_{\mathcal{L}}^{\flat}$:

$$\text{if } \Gamma \vdash \Delta, \text{ then } \sigma(\Gamma) \vdash \sigma(\Delta).$$

4.1 A categorical setting

For a start, let us recall the notion of partially ordered semiring [22, Ch. 3].

Definition 31. *A* partially ordered semiring *(po-semiring) is a structure* $\mathbf{A} = \langle A, \leq, +, \cdot, 0, 1\rangle$ *where:*

1. $\langle A, \cdot, 1\rangle$ *is a monoid.*
2. $\langle A, \leq, +, 0\rangle$ *is an Abelian po-monoid.*
3. $\sigma \cdot 0 = 0 \cdot \sigma = 0$ *for all* $\sigma \in A$*;*
4. *For every* $\sigma, \pi, \varepsilon \in A$*, we have*

$$\pi \cdot (\sigma + \varepsilon) = (\pi \cdot \sigma) + (\pi \cdot \varepsilon) \text{ and } (\sigma + \varepsilon) \cdot \pi = (\sigma \cdot \pi) + (\varepsilon \cdot \pi).$$

5. *If* $\sigma \leq \pi$ *and* $0 \leq \varepsilon$*, then* $\sigma \cdot \varepsilon \leq \pi \cdot \varepsilon$ *and* $\varepsilon \cdot \sigma \leq \varepsilon \cdot \pi$*.*

A po-semiring $\mathbf{A} = \langle A, \leq, +, \cdot, 0, 1\rangle$ is *dually integral* iff $\langle A, \leq, +, 0\rangle$ is dually integral as a po-monoid. Of course, the dual integrality condition "kills" many among the interesting examples of po-semirings, including all nontrivial po-rings.

Our chief example of dually integral po-semiring will be the semiring of finite multisets of substitutions on formulas of a propositional language $\mathcal{L}$. The role it will play here is analogous to the role played in Galatos and Tsinakis' theory by the complete residuated lattice of sets of $\mathcal{L}$-substitutions.

Example 32. *Let* $\mathcal{L}$ *be a propositional language, and let* $\mathrm{End}(\mathbf{Fm}_{\mathcal{L}})$ *be the set of substitutions of* $\mathbf{Fm}_{\mathcal{L}}$*. The structure*

$$\mathbf{\Sigma}_{\mathcal{L}} = \langle \mathrm{End}(\mathbf{Fm}_{\mathcal{L}})^{\flat}, \leqslant, \uplus, \cdot, \emptyset, [id_{Fm_{\mathcal{L}}}]\rangle,$$

where, for $\mathfrak{X} = [\sigma_1, \ldots, \sigma_n]$, $\mathfrak{Y} = [\pi_1, \ldots, \pi_m] \in \mathrm{End}(\mathbf{Fm}_{\mathcal{L}})^{\flat}$,

$$\mathfrak{X} \cdot \mathfrak{Y} = [\sigma_1 \circ \pi_1, \ldots, \sigma_1 \circ \pi_m, \ldots, \sigma_n \circ \pi_1, \ldots, \sigma_n \circ \pi_m],$$

is a dually integral po-semiring.

With this notion in our quiver, in order to get going we only need to endow our dually integral Abelian po-monoids from the previous section with a suitable operation of multiplication by a scalar.

Definition 33. *Let* $\mathbf{A} = \langle A, \leq^{\mathbf{A}}, +^{\mathbf{A}}, \cdot^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ *be a dually integral po-semiring. An* $\mathbf{A}$*-module is a structure* $\boldsymbol{R} = \langle R, \leq^{\boldsymbol{R}}, +^{\boldsymbol{R}}, 0^{\boldsymbol{R}}, *^{\boldsymbol{R}} \rangle$ *where* $\langle R, \leq^{\boldsymbol{R}}, +^{\boldsymbol{R}}, 0^{\boldsymbol{R}} \rangle$ *is a dually integral Abelian po-monoid and* $*^{\boldsymbol{R}} \colon A \times R \to R$ *is an action of* $\langle A, \cdot^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ *on* R *that is order-preserving in both coordinates and distributes over* $+^{\boldsymbol{R}}$*. In symbols:*

1. $(\sigma \cdot^{\mathbf{A}} \pi) *^{\boldsymbol{R}} a = \sigma *^{\boldsymbol{R}} (\pi *^{\boldsymbol{R}} a)$
2. $1^{\boldsymbol{A}} *^{\boldsymbol{R}} a = a$
3. $0^{\mathbf{A}} *^{\boldsymbol{R}} a = 0^{\mathbf{R}}$
4. $(\sigma *^{\boldsymbol{R}} a) +^{\boldsymbol{R}} (\sigma *^{\boldsymbol{R}} b) = \sigma *^{\boldsymbol{R}} (a +^{\boldsymbol{R}} b)$
5. $(\sigma +^{\mathbf{A}} \pi) *^{\boldsymbol{R}} a = (\sigma *^{\boldsymbol{R}} a) +^{\boldsymbol{R}} (\pi *^{\boldsymbol{R}} a)$
6. *If* $\sigma \leq^{\mathbf{A}} \pi$*, then* $\sigma *^{\boldsymbol{R}} a \leq^{\boldsymbol{R}} \pi *^{\boldsymbol{R}} a$
7. *If* $a \leq^{\mathbf{R}} b$*, then* $\sigma *^{\boldsymbol{R}} a \leq^{\boldsymbol{R}} \sigma *^{\boldsymbol{R}} b$.

Example 34. *Consider the po-semiring* $\boldsymbol{\Sigma}_{\mathcal{L}}$ *defined in Example 32, and let* $\boldsymbol{Mult}_{\mathcal{L}} = \langle Fm^{\flat}_{\mathcal{L}}, \leqslant, \uplus, \emptyset, * \rangle$*, where for*

$$\mathfrak{X} = [\sigma_1, \ldots, \sigma_n] \in \mathrm{End}(\mathbf{Fm}_{\mathcal{L}})^{\flat} \text{ and } \Gamma \in Fm^{\flat}_{\mathcal{L}},$$

we set, resorting to our usual notational conventions,

$$\mathfrak{X} * \Gamma = \sigma_1(\Gamma) \uplus \cdots \uplus \sigma_n(\Gamma).$$

Then $\boldsymbol{Mult}_{\mathcal{L}}$ *is a* $\boldsymbol{\Sigma}_{\mathcal{L}}$*-module.*

Modules over a dually integral po-semiring can be naturally equipped with arrows as follows:

Definition 35. *Let* $\mathbf{A}$ *be a dually integral po-semiring, and* $\boldsymbol{R}$ *and* $\boldsymbol{S}$ *be a pair of* $\mathbf{A}$*-modules. A* morphism $\tau \colon \boldsymbol{R} \to \boldsymbol{S}$ *is a po-monoid homomorphism (i.e., an order-preserving monoid homomorphism) such that* $\tau(\sigma *^{\boldsymbol{R}} a) = \sigma *^{\boldsymbol{S}} \tau(a)$ *for every* $\sigma \in A$ *and* $a \in R$*.*

Given a dually integral po-semiring $\mathbf{A}$, the collection of $\mathbf{A}$-modules with morphisms between them forms a category in which composition and identity arrows are, respectively, standard composition of functions and identity functions. We denote this category by $\mathbf{A}-\mathsf{Mod}$. Isomorphisms in the category $\mathbf{A}-\mathsf{Mod}$ are precisely bijective morphisms.

From now on we will assume that $\mathbf{A}$ is a fixed, but otherwise arbitrary, dually integral po-semiring.

Example 36. *It is expedient to remark that the setting of modules over complete residuated lattices can subsumed under the present one as follows. Recall that every complete residuated lattice* $\mathbf{M} = \langle M, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ *can be naturally turned*

into a dually integral po-semiring $\mathcal{U}(\mathbf{M}) = \langle M, \leq, \vee, \cdot, 0, 1\rangle$ *where* $\leq$ *and* 0 *are respectively the order and the bottom element of the lattice reduct of* $\mathbf{M}$*. Then observe that every* $\mathbf{M}$*-module* $\boldsymbol{L} = \langle L, \wedge, \vee, \star\rangle$ *gives rise to a* $\mathcal{U}(\mathbf{M})$*-module* $\mathcal{U}(\boldsymbol{L}) = \langle L, \leq, \vee, 0, \star\rangle$*, where* $\leq$ *and* 0 *are respectively the order and the bottom element of the lattice reduct of* $\boldsymbol{L}$*. Finally, every translator* $f\colon \boldsymbol{L}_1 \to \boldsymbol{L}_2$ *between* $\mathbf{M}$*-modules induces a morphism* $\mathcal{U}(f)\colon \mathcal{U}(\boldsymbol{L}_1) \to \mathcal{U}(\boldsymbol{L}_2)$ *of* $\mathcal{U}(\mathbf{M})$*-modules by setting* $\mathcal{U}(f)(a) = f(a)$ *for every* $a \in L_1$*. Summing up, the application* $\mathcal{U}(\cdot)$ *can be regarded as a forgetful functor from* $\mathbf{M}-\mathsf{Mod}$ *to* $\mathcal{U}(\mathbf{M})-\mathsf{Mod}$*, which reduces modules over a complete residuated lattice to modules over a dually integral po-semiring.*

We are now ready to give an abstract formulation of action-invariant DR's. Against the backdrop of Theorem 23, these deductive relations can be presented equivalently as deductive operators or as deductive systems. As a matter of fact, it turns out that working with DO's is more convenient, although similar definitions and results can be obtained by putting the other two concepts to the forefront.

Definition 37. *An* action-invariant DO *on an* $\mathbf{A}$*-module* $\boldsymbol{R}$ *is a* DO δ *on its po-monoid reduct* $\langle R, \leq, +, 0\rangle$ *such that for every* $\sigma \in A$ *and* $a, b \in R$*:*

$$\text{if } a \in \delta(b), \text{ then } \sigma * a \in \delta(\sigma * b).$$

To exemplify this concept, we point out that substitution-invariant MDR's give rise to deductive operators that are action-invariant according to the definition just given.

Proposition 38. *Let* $\mathcal{L}$ *be a propositional language. Then an* MDR $\vdash$ *on* $\mathcal{L}$ *is substitution-invariant iff* $\delta_{\vdash}$ *is an action-invariant* DO *on the* $\boldsymbol{\Sigma}_{\mathcal{L}}$*-module* $\boldsymbol{Mult}_{\mathcal{L}}$*. Similarly,* δ *is an action-invariant* DO *on the* $\boldsymbol{\Sigma}_{\mathcal{L}}$*-module* $\boldsymbol{Mult}_{\mathcal{L}}$ *iff* $\vdash_{\delta}$ *is a substitution-invariant* MDR *on* $\mathcal{L}$*.*

Proof. First consider an MDR $\vdash$ on $\mathcal{L}$. Suppose that $\vdash$ is substitution-invariant. As $\mathbf{Fm}^{\flat}_{\mathcal{L}}$ is a reduct of the $\boldsymbol{\Sigma}_{\mathcal{L}}$-module $\boldsymbol{Mult}_{\mathcal{L}}$, by Theorem 23 we only have to show that $\delta_{\vdash}$ is action-invariant. Assume that $\Delta \in \delta_{\vdash}(\Gamma)$ and consider $\mathfrak{X} = [\sigma_1, \ldots, \sigma_k] \in \mathrm{End}(\mathbf{Fm}_{\mathcal{L}})^{\flat}$. From the definition of $\delta_{\vdash}$ and substitution-invariance of $\vdash$ it follows that, for every $i \leq k$, we have that $\sigma_i(\Gamma) \vdash \sigma_i(\Delta)$. Now, applying Compatibility several times, we obtain that

$$\biguplus_{i\leq k} \sigma_i(\Gamma) \vdash \biguplus_{i\leq k} \sigma_i(\Delta).$$

The above display amounts exactly to the fact that $\mathfrak{X} * \Delta \in \delta_{\vdash}(\mathfrak{X} * \Gamma)$. Hence, we conclude that $\delta_{\vdash}$ is action-invariant according to Definition 37.

Conversely, suppose that $\delta_{\vdash}$ is action-invariant and that $\Gamma \vdash \Delta$. Consider a substitution σ. First observe that $\Delta \in \delta(\Gamma)$. Since $\delta_{\vdash}$ is action-invariant and $[\sigma] \in \mathrm{End}(\mathbf{Fm}_{\mathcal{L}})^{\flat}$, we have that $[\sigma] * \Delta \in \delta_{\vdash}(\sigma * \Gamma)$, which means exactly $\sigma(\Gamma) \vdash \sigma(\Delta)$. Hence we conclude that $\vdash$ is substitution-invariant.

The second claim follows from the first one, together with the fact that $\delta_{\vdash_{\delta}} = \delta$ (Theorem 23). □

Given an action-invariant DO δ on $\boldsymbol{R}$, we define a structure

$$\boldsymbol{R}_\delta = \langle \delta(R), \subseteq, +^\delta, \delta\left(0\right), *^\delta \rangle,$$

where for every $\sigma \in A$ and $a, b \in R$:

$$\delta(a) +^\delta \delta(b) = \delta(a + b) \text{ and } \sigma *^\delta \delta(a) = \delta(\sigma * a).$$

Lemma 39. *Let δ be an action-invariant* DO *on the* **A***-module $\boldsymbol{R}$. Then $\boldsymbol{R}_\delta$ is a well-defined* **A***-module and the map $\delta \colon \boldsymbol{R} \to \boldsymbol{R}_\delta$ is a morphism.*

Proof. Using Theorems 23 and 19, we know that

$$\boldsymbol{R}_\delta = \langle \delta(R), \subseteq, +^\delta, \delta(0) \rangle$$

is a well-defined dually integral po-monoid. Now we show that the action $*^\delta$ is well-defined too. Consider $\sigma \in A$ and $a, b \in R$ such that $\delta(a) = \delta(b)$. In particular, we have that $a \in \delta(b)$. By the action-invariance of δ, we obtain that $\sigma * a \in \delta(\sigma * b)$. Thus we conclude that $\delta(\sigma * a) \subseteq \delta(\sigma * b)$. The other inclusion is proved analogously.

Next, we turn to prove that $\boldsymbol{R}_\delta$ is an **A**-module. It only remains to establish the conditions regarding the action $*^\delta$. It is clear that $*^\delta$ is order-preserving on the first coordinate. We prove that the same holds for the second one. Consider $\sigma \in A$ and $a, b \in R$ such that $\delta(a) \leq^\delta \delta(b)$. From the action-invariance of δ it follows that $\sigma * a \in \delta(\sigma * b)$ and, therefore, that $\delta(\sigma * a) \leq^\delta \delta(\sigma * b)$. We conclude that

$$\sigma *^\delta \delta(a) = \delta(\sigma * a) \leq^\delta \delta(\sigma * b) = \sigma *^\delta \delta(b).$$

The fact that $*^\delta$ is a monoidal action and the distributivity conditions are easy exercises.

Finaly, we prove that the map $\delta \colon \boldsymbol{R} \to \boldsymbol{R}_\delta$ is a morphism. Due to Theorem 19, it remains to show that δ respects the monoidal action, which follows directly from the definition of $*^\delta$. □

We conclude this subsection be defining two maps which will play an important role in the next subsection.

Lemma 40. *Let $f \colon \boldsymbol{R} \to \boldsymbol{S}$ be a morphism between* **A***-modules.*

1. *The map $f^* \colon R \to \wp(R)$ defined as:*

$$f^*\left(a\right) = f^{-1}\left(\{x : x \leq f(a)\}\right)$$

   is an action-invariant DO *on $\boldsymbol{R}$.*

2. *The map $\hat{f} \colon \boldsymbol{R}_{f^*} \to f[\boldsymbol{R}]$ defined as:*

$$\hat{f}\left(f^*(a)\right) = f(a)$$

   is a well-defined isomorphism.

Proof. 1. The only condition in Definition 20 that stands in need of a check is Compatibility. Let $a, b, c \in R$ and suppose that $a \in f^*(b)$. This means that $f(a) \leq f(b)$. In particular, we have that

$$f(a +^{\boldsymbol{R}} c) = f(a) +^{\boldsymbol{S}} f(c) \leq^{\boldsymbol{S}} f(b) +^{\boldsymbol{S}} f(c) = f(b +^{\boldsymbol{R}} c).$$

Hence we conclude that $a + c \in f^*(b + c)$. This shows that f^* is a do. It remains to be shown that it is action-invariant. Consider $\sigma \in A$ and suppose that $a \in f^*(b)$. Then

$$f(\sigma *^{\boldsymbol{R}} a) = \sigma *^{\boldsymbol{S}} f(a) \leq^{\boldsymbol{S}} \sigma *^{\boldsymbol{S}} f(b) = f(\sigma *^{\boldsymbol{R}} b).$$

Thus $\sigma *^{\boldsymbol{R}} a \in f^*(\sigma *^{\boldsymbol{R}} b)$, whence our conclusion follows.

2. Observe that the map $\hat{f}$ is well-defined, since $\leq^{\boldsymbol{S}}$ is antisymmetric. It is clear that $\hat{f}$ is a bijection. Since isomorphisms in $\mathbf{A}-\mathsf{Mod}$ are bijective morphisms, it suffices to prove that $\hat{f}$ is a morphism. But this is an exercise, using the definition of $\boldsymbol{R}_f$ and the fact that f is a morphism. □

4.2 Action-invariant representations

The main result in [18], reproduced above as Theorem 4, is an elegant and purely categorical characterisation of the modules over a complete residuated lattice for which an analogue of the Syntactic Isomorphism Theorem (Theorem 2) for algebraisable logics holds. The aim of this subsection is to obtain a similar result in the setting of modules over a dually integral po-semiring.

Definition 41. *Let δ and γ be action-invariant* DO*'s on the $\mathbf{A}$-modules $\boldsymbol{R}$ and $\boldsymbol{S}$, respectively.*

1. *An* action-invariant representation *of δ into γ is an injective morphism $\Phi\colon \boldsymbol{R}_\delta \to \boldsymbol{S}_\gamma$ that reflects the order.*
2. *A representation Φ of δ into γ is* induced *if there is a morphism $\tau\colon \boldsymbol{R} \to \boldsymbol{S}$ that makes the following diagram commute:*

$$\begin{array}{ccc} \boldsymbol{R} & \xrightarrow{\tau} & \boldsymbol{S} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \boldsymbol{R}_\delta & \xrightarrow[\Phi]{} & \boldsymbol{S}_\gamma \end{array}$$

3. *δ and γ are* equivalent *if the $\mathbf{A}$-modules $\boldsymbol{R}_\delta$ and $\boldsymbol{S}_\gamma$ are isomorphic.*

Definition 42. *An $\mathbf{A}$-module $\boldsymbol{R}$ has the representation property (REP) if for any other $\mathbf{A}$-module $\boldsymbol{S}$ and action-invariant* DO*'s δ and γ on $\boldsymbol{R}$ and $\boldsymbol{S}$ respectively, every action-invariant representation of δ into γ is induced.*

We are now ready to provide a characterisation of $\mathbf{A}$-modules with the REP in the spirit of Theorem 4.

Theorem 43. *An $\mathbf{A}$-module has the REP iff it is onto-projective in $\mathbf{A}-\mathsf{Mod}$.*

Proof. The backbone of our argument is essentially the same as in [18, Lemma 8.1]. It is clear that every projective $\mathbf{A}$-module has the REP. Now, let $\boldsymbol{R}$ be an $\mathbf{A}$-module with the REP, and consider two morphisms $f\colon \boldsymbol{S} \to \boldsymbol{T}$ and $g\colon \boldsymbol{R} \to \boldsymbol{T}$ with f onto. By Lemma 40, the derived maps f^* and g^* are action-invariant DO's on $\boldsymbol{R}$ and $\boldsymbol{S}$, respectively. Observe that $g(R)$ is the universe of a submodule $g(\boldsymbol{R})$ of $\boldsymbol{T}$. By Lemma 40 the following maps are isomorphisms:

$$\widehat{f}\colon \boldsymbol{S}_{f^*} \to \boldsymbol{T} \text{ and } \widehat{g}\colon \boldsymbol{R}_{g^*} \to g(\boldsymbol{R}).$$

Let $i\colon g(\boldsymbol{R}) \to \boldsymbol{T}$ be the morphism given by the inclusion relation. Clearly the composition

$$\hat{f}^{-1} \circ i \circ \hat{g}\colon \boldsymbol{R}_{g^*} \to \boldsymbol{S}_{f^*}$$

is a representation of g^* into f^*. Thus we can apply the fact that $\boldsymbol{R}$ has the REP, obtaining a morphism $h\colon \boldsymbol{R} \to \boldsymbol{S}$ such that

$$\hat{f}^{-1} \circ i \circ \hat{g} \circ g^* = f^* \circ h.$$

Hence for every $a \in R$, we have that

$$\begin{aligned} f \circ h(a) &= (\hat{f} \circ f^*) \circ h(a) = \hat{f} \circ (f^* \circ h)(a) = \hat{f} \circ (\hat{f}^{-1} \circ i \circ \hat{g} \circ g^*)(a) \\ &= (\hat{f} \circ \hat{f}^{-1}) \circ i \circ \hat{g} \circ g^*(a) = i \circ \hat{g} \circ g^*(a) = i \circ g(a) = g(a). \end{aligned}$$

Hence we conclude that $f \circ h = g$. Therefore $\boldsymbol{R}$ is onto-projective. □

In order to show that our abstract framework is well-behaved, we are committed to proving that every equivalence between two substitution-invariant MDR's is induced by a pair of endomoprhisms on the $\boldsymbol{\Sigma}_{\mathcal{L}}$-module $\boldsymbol{Mult}_{\mathcal{L}}$ (note that this claim can bee seen as a variant of Isomorphism Theorem in the setting of MDR's). In other words, we want to show that $\boldsymbol{Mult}_{\mathcal{L}}$ has the REP (that is, it is onto-projective) in the category of $\boldsymbol{\Sigma}_{\mathcal{L}}$-modules. Instead of proving this directly, we will take a brief detour and prove some more general results. First, we make a note of the following definition.

Definition 44. *An $\mathbf{A}$-module $\boldsymbol{R}$ is* cyclic *if there is $a \in R$ such that $R = \{\sigma * a : \sigma \in A\}$.*

Observe that every dually integral po-semiring $\mathbf{A} = \langle A, \leq, +, \cdot, 0, 1\rangle$ can be seen as a degenerate instance of $\mathbf{A}$-module if we drop $\cdot$ and 1 from the signature and set $* = \cdot$. Keeping this in mind, we obtain the following:

Lemma 45. *Any dually integral po-semiring $\mathbf{A}$, viewed as an $\mathbf{A}$-module, is cyclic and onto-projective.*

Proof. Clearly $\mathbf{A}$ is cyclic, since $A = \{\sigma \cdot 1 : \sigma \in A\}$. Let $f\colon \boldsymbol{R} \to \boldsymbol{S}$ and $g\colon \mathbf{A} \to \boldsymbol{S}$ be two morphisms, where f is onto. Then fix any $a \in R$ such that $f(a) = g(1)$. We define a map $h\colon A \to R$ via $h(\sigma) = \sigma *^{\boldsymbol{R}} a$. It is easy to see that $h\colon \mathbf{A} \to \boldsymbol{R}$ is a morphism. Moreover, given $\sigma \in A$, we have that

$$g(\sigma) = g(\sigma \cdot 1) = \sigma \cdot g(1) = \sigma \cdot f(a) = f(\sigma * a) = f \circ h(\sigma)).$$

Hence we conclude that $\mathbf{A}$ is onto-projective. □

Cyclic modules can be described in an arrow-theoretic way as follows:

Lemma 46. *An* $\mathbf{A}$*-module* $\boldsymbol{R}$ *is cyclic if and only if there is an onto morphism* $f\colon \mathbf{A} \to \boldsymbol{R}$*.*

Proof. If there is an onto morphism $f\colon \mathbf{A} \to \boldsymbol{R}$ and $x \in R$, then for some $\sigma \in A$ we have that $x = f(\sigma) = \sigma * f(1)$. To prove the converse, it is enough to check that if $R = \{\sigma * v : \sigma \in A\}$, then the map $f\colon \mathbf{A} \to \boldsymbol{R}$ defined by $f(\sigma) = \sigma * v$ is a morphism. □

We are now ready to prove the following characterisation of cyclic and onto-projective objects in $\mathbf{A}-\mathsf{Mod}$.

Theorem 47. *Let* $\boldsymbol{R}$ *be an* $\mathbf{A}$*-module. The following conditions are equivalent:*

1. $\boldsymbol{R}$ *is cyclic and onto-projective.*
2. *There is a retraction* $f\colon \mathbf{A} \to \boldsymbol{R}$*.*
3. *There are* $\mu \in A$ *and* $v \in R$ *such that* $\mu * v = v$ *and* $A * \{v\} = R$ *and for every* $\sigma, \pi \in A$*: if* $\sigma * v \leq \pi * v$*, then* $\sigma \cdot \mu \leq \pi \cdot \mu$*.*

Proof. (1)⇒(2): From Lemma 46 we know that there is a surjective morphism $f\colon \mathbf{A} \to \boldsymbol{R}$. Applying the projectivity of $\boldsymbol{R}$ to the diagram given by f and the identity map $id_{\boldsymbol{R}}$, we conclude that f is a retraction.

(2)⇒(1): From Lemma 46 we know that $\boldsymbol{R}$ is cyclic. Moreover, $\boldsymbol{R}$ is a retract of an onto-projective object by Lemma 45. Thus we conclude that $\boldsymbol{R}$ is onto-projective too.

(2)⇒(3): By assumption, there is an injective morphism $g\colon \boldsymbol{R} \to \mathbf{A}$ such that $1_{\boldsymbol{R}} = f \circ g$. Then we define $v = f(1)$ and $\mu = g(v)$. Since f is onto, we have that $A * \{v\} = R$. Moreover:

$$\mu * v = \mu * f(1) = f(\mu * 1) = f(\mu) = f(g(v)) = v.$$

Finally, consider $\sigma, \pi \in A$ such that $\sigma * v \leq \pi * v$. We have that $\sigma \cdot \mu = \sigma \cdot g(v) = g(\sigma * v) \leq g(\pi * v) = \pi \cdot g(v) = \pi \cdot \mu$.

(3)⇒(2): Since $A * \{v\} = R$, we know that the map $f\colon \mathbf{A} \to \boldsymbol{R}$ defined as $f(\sigma) = \sigma * v$ is an onto morphism. Then let $g\colon \boldsymbol{R} \to \mathbf{A}$ be defined via $g(\sigma * v) = \sigma \cdot \mu$. Using the assumption, it is not difficult to see that g is well-defined and order-preserving. Also, it can be routinely established that g preserves the action and is a monoid homomorphism. Thus, g is a morphism. In order to prove that $f \circ g = 1_{\boldsymbol{R}}$ we consider a generic element $\sigma * v \in R$ and show that:

$$f \circ g(\sigma * v) = \sigma * (f \circ g(v)) = \sigma * f(\mu) = \sigma * (\mu * v) = \sigma * v. \qquad \square$$

Theorem 48. *The* $\boldsymbol{\Sigma}_{\mathcal{L}}$*-module* $\boldsymbol{Mult}_{\mathcal{L}}$ *is cyclic and onto-projective. In particular, this implies that it has the REP.*

Proof. Let x be a designated $\mathcal{L}$-variable, and let $v = [x]$. Moreover, let σ be the $\mathcal{L}$-substitution defined by $\sigma(y) = x$ for all $\mathcal{L}$-variables y, and fix $\mu = [\sigma]$. It is not difficult to see that v and μ satisfy the conditions of Item (3) in Theorem 47. □

5 Multiset deductive relations

Recall that what prompted us to extend Blok and Jónsson's theory was the motivating example of *multiset deductive relations* (MDR's), defined in Definition 5. It turns out that our general theory has interesting offshoots once we focus on this special case — and the whole of the present section will be devoted to buttressing this claim.

For a start, we list some prototypical instances of MDR's.

Example 49. *Recall that an algebra* $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 1\rangle$ *of language* $\mathcal{L}_0 = \langle 2, 2, 2, 2, 0\rangle$ *is a* commutative and integral residuated lattice *(see e.g. [23]) if* $\langle A, \wedge, \vee\rangle$ *is a lattice,* $\langle A, \cdot, 1\rangle$ *is a commutative monoid,* 1 *is the top element w.r.t. the induced order* $\leq$ *of* $\langle A, \wedge, \vee\rangle$*, and the following residuation law holds for every* $a, b, c \in A$*:*

$$a \cdot b \leq c \Longleftrightarrow a \leq b \to c.$$

Given a class $\mathcal{K}$ *of commutative and integral residuated lattices, let the relation* $\vdash_{\mathcal{K}}$ *be defined as follows for all* $\Gamma = [\varphi_1, \ldots, \varphi_n], \Delta = [\psi_1, \ldots, \psi_m] \in Fm^{\flat}_{\mathcal{L}_0}$*:*[8]

$$\Gamma \vdash_{\mathcal{K}} \Delta \Longleftrightarrow \mathcal{K} \models \varphi_1 \cdot \ldots \cdot \varphi_n \leq \psi_1 \cdot \ldots \cdot \psi_m. \tag{1}$$

It can be checked that $\vdash_{\mathcal{K}}$ *is indeed a substitution-invariant* MDR *in the sense of Definition 5.*

Example 49 identifies, for every substructural logic whose equivalent algebraic semantics is a quasi-variety of commutative and integral residuated lattices, a "multiset-theoretic" companion of such that best suits the resource interpretation at which we hinted in our introduction. One particular such logic will play some role in what follows. The multiset companion $\vdash_{\mathcal{MV}}$ of infinite-valued Łukasiewicz logic $\vdash_{\text{Ł}}$ is obtained when the class $\mathcal{K}$ is the variety $\mathcal{MV}$ of *MV-algebras* [9], formulated in the language $\mathcal{L}_0$.

Also, observe that Example 49 encompasses the so-called *internal consequence relations* of algebraisable substructural sequent calculi with exchange and weakening [2, 3]. In fact, let S be such a calculus and $\mathcal{Q}$ its equivalent algebraic semantics. Upon defining, for finite multisets of $\mathcal{L}_0$-formulas Γ and $\Delta = [\psi_1, \ldots, \psi_m]$,

$$\Gamma \vdash_{\mathcal{S}} \Delta \quad \Longleftrightarrow \quad \vdash_S \Gamma \Rightarrow \psi_1 \cdot \ldots \cdot \psi_m,$$

then it follows from well-known results about substructural logics that $\vdash_{\mathcal{S}} = \vdash_{\mathcal{Q}}$.

If the above examples look a bit contrived, this is due, in part, to the fact that the multiple-conclusion format is unfamiliar to many. As a consequence,

[8] Here and in the sequel, given a multiset $\Gamma = [\varphi_1, \ldots, \varphi_n]$ of $\mathcal{L}_0$-formulas, the notation $\varphi_1 \cdot \ldots \cdot \varphi_n$ will ambiguously refer to any of the $\mathcal{L}_0$-formulas

$$\left(\ldots\left(\varphi_{f(1)} \cdot \varphi_{f(2)}\right) \cdot \ldots \cdot \varphi_{f(n)}\right),$$

where f is a permutation of $\{1, \ldots, n\}$. By way of convention, if Γ is the empty multiset, we formally set $\varphi_1 \cdot \ldots \cdot \varphi_n = 1$.

it would seem expedient to extract from these examples appropriate single-conclusion relations that can be more easily compared, say, to the usual, *external*, consequence relations of substructural sequent calculi. It turns out that single-conclusion relations can be recovered as *fragments* of MDR's [10]:

Definition 50. *Let $\mathcal{L}$ be a propositional language. A* single-conclusion MDR *on $\mathcal{L}$ is a relation $\vdash^u \subseteq Fm^\flat_\mathcal{L} \times Fm_\mathcal{L}$ such that, for some* MDR $\vdash$*,*

$$\Gamma \vdash^u \alpha \text{ iff } \Gamma \vdash [\alpha]\,.$$

It should be observed, though, that we are not claiming that single-conclusion MDR's be themselves instances of MDR's, for they need not be closed w.r.t. all the conditions that define them (see [10] for further discussion). Clearly, for each single-conclusion MDR $\vdash^u$ there exists the least MDR $\vdash$ that has $\vdash^u$ as fragment: namely, the intersection of all such MDR's. In Subsection 5.4 we present an example of two MDR's with the same single-conclusion fragment.

Multiset deductive relations can be taken to subsume TCR's, as the next example shows.

Example 51. *Every finitary substitution-invariant* TCR *can be encoded into a finitary substitution-invariant* MDR*. Indeed, consider such* TCR $\Vdash$ *on language $\mathcal{L}$. Then we define a substitution-invariant* MDR $\vdash$ *on $\mathcal{L}$ by setting, for all $\Gamma = [\varphi_1, \ldots, \varphi_n]\,, \Delta = [\psi_1, \ldots, \psi_m]$ in $Fm^\flat_\mathcal{L}$:*

$$\Gamma \vdash \Delta \iff |\Gamma| \Vdash \psi_k \text{ for all } k \leq m.$$

It is a purely computational matter to check that $\vdash$ is indeed a substitution-invariant MDR*. Moreover, it is clear that $\vdash$ encodes $\Vdash$ in the sense that, whenever $\varphi_j \neq \varphi_k$ for all $j, k \leq n$,*

$$\varphi_1, \ldots, \varphi_n \Vdash \psi \iff [\varphi_1, \ldots, \varphi_n] \vdash [\psi]\,.$$

5.1 Hypermatrices and the first completeness theorem

In this subsection we describe a matrix-based semantics for arbitrary substitution-invariant MDR's. To this end, we work in a fixed (but otherwise arbitrary) language $\mathcal{L}$.

Logical matrices are part and parcel of every algebraic logician's toolbox [12, Ch. 4]. As a consequence, when we are dealing with MDR's over a language $\mathcal{L}$, it seems desirable to be in a position to help ourselves to concepts that inherit at least some of the effectiveness and power of matrix semantics in AAL. Whatever notion of matrix we are bound to adopt, it appears natural that its attendant notion of "Lindenbaum–Tarski matrix" be in keeping with Definition 15: we expect such matrices to have the form $\langle \mathbf{Fm}_\mathcal{L}, F\rangle$ where F is a certain *set of finite multisets* of $\mathcal{L}$-formulas. Therefore, it is all too plausible to focus on "matrices" constituted by an algebra and a certain family of finite submultisets of its universe. The next definitions spell out in detail this basic insight.

Definition 52. *An* $\mathcal{L}$-hypermatrix *is a pair* $\langle \mathbf{A}, F\rangle$*, where* $\mathbf{A}$ *is an* $\mathcal{L}$*-algebra and* F *a* $\leqslant$*-downset in* $A^\flat$*.*

Definition 53. *For a class* H *of* $\mathcal{L}$*-hypermatrices we define a relation* $\models_{\mathsf{H}}$ *on* $Fm^\flat_{\mathcal{L}}$ *as* $\Gamma \models_{\mathsf{H}} \Delta$*, if for every* $\boldsymbol{A} = \langle \mathbf{A}, F\rangle \in \mathsf{H}$*, each context* $\mathfrak{C} \in A^\flat$ *and each homomorphism* $f\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$*:*

$$\mathfrak{C} \uplus f(\Gamma) \in F \text{ implies } \mathfrak{C} \uplus f(\Delta) \in F.$$

Theorem 54. *Let* H *be a class of* $\mathcal{L}$*-hypermatrices. Then* $\models_{\mathsf{H}}$ *is a substitution-invariant* MDR *on* $\mathcal{L}$*.*

Proof. We show the proof for $\mathsf{H} = \{\langle \mathbf{A}, F\rangle\}$; the general statement then follows from the obvious facts that $\models_{\mathsf{H}} = \bigcap \{\models_{\boldsymbol{A}}\colon \boldsymbol{A} \in \mathsf{H}\}$ and that the class of substitution-invariant MDR's is closed under intersections. Transitivity and Substitution-invariance of $\models_{\boldsymbol{A}}$ are obvious.

For Compatibility, assume that $\Gamma \models_{\boldsymbol{A}} \Delta$ and consider a homomorphism $e\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ and context $\mathfrak{C} \in A^\flat$. If $\mathfrak{C} \uplus e(\Gamma) \uplus e(\Pi) = \mathfrak{C} \uplus e(\Gamma \uplus \Pi) \in F$, then by our hypothesis $\mathfrak{C} \uplus e(\Delta) \uplus e(\Pi) \in F$, and thus $\Gamma \uplus \Pi \models_{\boldsymbol{A}} \Delta \uplus \Pi$.

For Generalized Reflexivity, assume that $\Gamma \leqslant \Delta$ and consider a homomorphism $e\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ and context $\mathfrak{C} \in A^\flat$. Clearly, $e(\Gamma) \leqslant e(\Delta)$, and by compatibility of $\leqslant$, $\mathfrak{C} \uplus e(\Gamma) \leqslant \mathfrak{C} \uplus e(\Delta)$. Thus, if $\mathfrak{C} \uplus e(\Delta) \in F$, then $\mathfrak{C} \uplus e(\Gamma)$, because F is a $\leqslant$-downset. □

Note that the fact that F a $\leqslant$-downset in $A^\flat$, and the reference to arbitrary contexts $\mathfrak{C}$, play a crucial role in the previous proof. Lifting both restrictions at once leads to the following definition of a relation $\models'_{\boldsymbol{A}}$ on $Fm^\flat_{\mathcal{L}}$ for an arbitrary pair $\boldsymbol{A} = \langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra and $F \subseteq A^\flat$:

$\Gamma \models'_{\boldsymbol{A}} \Delta$ iff for each homomorphism$f\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$: if $f(\Gamma) \in F$ then $f(\Delta) \in F$.

Lemma 55. *Consider a pair* $\boldsymbol{A} = \langle \mathbf{A}, F\rangle$*, where* $\mathbf{A}$ *is an* $\mathcal{L}$*-algebra and* $F \subseteq A^\flat$*. Then* $\models_{\boldsymbol{A}} \subseteq \models'_{\boldsymbol{A}}$*. Assume further that for some* MDR $\vdash$ *we have* $\vdash \subseteq \models'_{\boldsymbol{A}}$*. Then* $\boldsymbol{A}$ *is an* $\mathcal{L}$*-hypermatrix and* $\vdash \subseteq \models_{\boldsymbol{A}}$*.*

Proof. The first inclusion is trivial. Assume that $\vdash \subseteq \models'_{\boldsymbol{A}}$ and we show that F is $\leqslant$-downset. Note that for any multiset of mutually different atoms $[p_1, \ldots, p_n]$ and $m \leq n$ we have $[p_1, \ldots, p_n] \models'_{\boldsymbol{A}} [p_1, \ldots, p_m]$ and for any multisets $\mathfrak{X} \leqslant \mathfrak{Y}$ there is homomorphism $e\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ such that $\mathfrak{X} = [e(p_1), \ldots, e(p_m)]$ and $\mathfrak{Y} = [e(p_1), \ldots, e(p_n)]$.

To complete the proof we need to show that $\vdash \subseteq \models_{\boldsymbol{A}}$. Assume that $\Gamma \vdash \Delta$, $\mathfrak{C} = [x_1, \ldots, x_n] \in A^\flat$ and $f\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ are such that $\mathfrak{C} \uplus f(\Gamma) \in F$ and we need to prove that $\mathfrak{C} \uplus f(\Delta) \in F$. Let Π be a multiset of mutually different atoms $[p_1, \ldots, p_n]$ not occurring in $\Gamma \uplus \Delta$; we know that $\Gamma \uplus \Pi \vdash \Delta \uplus \Pi$. Next consider the homomorphism e' defined as $e'(p_i) = x_i$ and $e'(p) = e(p)$ for other atoms and note that $e'(\Gamma \uplus \Pi) = \mathfrak{X} \uplus e(\Gamma) \in F$ and so $\mathfrak{X} \uplus e(\Gamma) = e'(\Delta \uplus \Pi) \in F$. □

Next, we provide an example showing that $\models_{\boldsymbol{A}}$ and $\models'_{\boldsymbol{A}}$ are in general different relations.

Example 56. *Consider a two-element set $A = \{0, 1\}$. Then let $F \subseteq A^\flat$ be defined as follows:*

$$F = \{\emptyset, [0], [1], [0, 1]\}.$$

Now, equip A with the structure of an algebra $\mathbf{A} = \langle A, \mathbf{0}, \mathbf{1}\rangle$, whose only operations are constant symbols $\mathbf{0}$ and $\mathbf{1}$ for 0 and 1, respectively. Clearly, F is a $\leqslant$-downset and it is easy to see that for $\boldsymbol{A} = \langle \mathbf{A}, F\rangle$ we have

$$[\mathbf{0}] \models'_{\boldsymbol{A}} [\mathbf{1}] \text{ and } [\mathbf{0}, \mathbf{1}] \not\models'_{\boldsymbol{A}} [\mathbf{1}, \mathbf{1}].$$

Hence the consequence $\models'_{\boldsymbol{A}}$ does not satisfy Compatibility and, therefore, it is not an MDR *and cannot be equal to $\models_{\boldsymbol{A}}$.*

Corollary 57. *Let H be a class of $\mathcal{L}$-hypermatrices. We define a relation $\models'_{\mathsf{H}}$ on $Fm^\flat_{\mathcal{L}}$ as $\models'_{\mathsf{H}} = \bigcap \{\models'_{\boldsymbol{A}} \colon \boldsymbol{A} \in \mathsf{H}\}$. Then $\models'_{\mathsf{H}}$ is an* MDR *iff $\models'_{\mathsf{H}} = \models_{\mathsf{H}}$.*

Now we can define notions of model and filter. Note that the previous lemma renders it immaterial whether we use $\models'_{\boldsymbol{A}}$ or $\models_{\boldsymbol{A}}$ in such definitions.

Definition 58. *Let $\vdash$ be a substitution-invariant* MDR *on $\mathcal{L}$. An $\mathcal{L}$-hypermatrix $\boldsymbol{A} = \langle \mathbf{A}, F\rangle$ is a* model *of $\vdash$ and F is an $\vdash$-*filter *on $\mathbf{A}$ if $\vdash \subseteq \models_{\boldsymbol{A}}$. By $\mathbf{Mod}(\vdash)$ we denote the set of all models of $\vdash$ and by $\mathcal{F}i_\vdash(\mathbf{A})$ the set of all $\vdash$-filters on $\mathbf{A}$.*

It is straightforward to show that $\mathcal{F}i_\vdash(\mathbf{A})$ is a closure system. Given a closure system $\mathcal{C}$ on a set X, let us denote by $\mathcal{C}^p$ the set of its principal members:

$$\mathcal{C}^p = \{\bigcap\{C \in \mathcal{C} : x \in C\} : x \in X\}.$$

Proposition 59. *Let $\vdash$ be a substitution-invariant* MDR *on $\mathcal{L}$.*

1. *For every $\mathcal{L}$-algebra $\mathbf{A}$, the collection $(\mathcal{F}i_\vdash(\mathbf{A}))^p$ is a* DS *on $\mathbf{A}^\flat$ and $\mathcal{F}i_\vdash(\mathbf{A}) = ((\mathcal{F}i_\vdash(\mathbf{A}))^p)^{BJ}$.*
2. *$(\mathcal{F}i_\vdash(\mathbf{Fm}_{\mathcal{L}}))^p = Th^p(\vdash)$.*

Proof. Consider an arbitrary $\mathcal{L}$-algebra $\mathbf{B}$. Recall that $\mathcal{F}i_\vdash(\mathbf{B})$ is a closure system. Then let $\mathrm{Fg}^{\mathbf{B}}_\vdash \colon \wp(B^\flat) \to \wp(B^\flat)$ be its corresponding closure operator. It is easy to see that for every $X \subseteq B^\flat$,

$$\mathrm{Fg}^{\mathbf{B}}_\vdash(X) = \bigcup_{n\in\omega} F_n$$

where $F_0 = X$ and

$$F_{n+1} = F_n \cup \{\mathfrak{X} \in B^\flat : \text{there are } \Gamma, \Delta \in Fm^\flat_{\mathcal{L}} \text{ s.t. } \Gamma \vdash \Delta \text{ and a homomorphism } f \colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{B} \text{ s.t. } f(\Gamma) \in F_n \text{ and } f(\Delta) = \mathfrak{X}\}.$$

From the above remarks it follows that for every $\Gamma \in Fm^\flat_{\mathcal{L}}$,

$$\mathrm{Fg}^{\mathbf{Fm}_{\mathcal{L}}}_\vdash(\Gamma) = \{\Delta \in Fm^\flat_{\mathcal{L}} : \Gamma \vdash \Delta\}.$$

In particular, this means that $(\mathcal{F}i_\vdash(\mathbf{Fm}_{\mathcal{L}}))^p = Th^p(\vdash)$, which proves the second statement.

For the other statement, consider an $\mathcal{L}$-algebra $\mathbf{A}$. We begin by proving that $(\mathcal{F}i_{\vdash}(\mathbf{A}))^p$ is a DS on $\langle A^\flat, \leq, \uplus, \emptyset\rangle$. To this end, we claim that

$$\text{if } \mathfrak{X} \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\mathfrak{Y}), \text{ then } \mathfrak{X} \uplus \mathfrak{C} \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\mathfrak{Y} \uplus \mathfrak{C}) \tag{2}$$

for every $\mathfrak{X}, \mathfrak{Y}, \mathfrak{C} \in A^\flat$.

To prove this claim, fix $\mathfrak{X}, \mathfrak{Y}, \mathfrak{C} = [c_1, \ldots, c_k] \in A^\flat$ and consider decompositions

$$\mathrm{Fg}^{\mathbf{A}}_{\vdash}(\mathfrak{Y}) = \bigcup_{n\in\omega} F_n \text{ and } \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\mathfrak{Y} \uplus \mathfrak{C}) = \bigcup_{n\in\omega} G_n$$

defined at the beginning of this proof. We show, by induction on $n \in \omega$, that

$$\text{if } \mathfrak{X} \in F_n, \text{ then } \mathfrak{X} \uplus \mathfrak{C} \in G_n. \tag{3}$$

The case where $n = 0$ is direct. Then we consider the case $n = s + 1$. Suppose that $\mathfrak{X} \in F_{s+1}$. Then there are Γ, Δ and a homomorphism $f\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ such that

$$\Gamma \vdash \Delta,\ f(\Gamma) \in F_s, \text{ and } f(\Delta) = \mathfrak{X}.$$

Consider the multiset $\Pi = [x_1, \ldots, x_k]$ consisting of fresh pair-wise different variables. By compatibility of $\vdash$, we have that

$$\Gamma \uplus \Pi \vdash \Delta \uplus \Pi. \tag{4}$$

By inductive hypothesis we know that

$$f(\Gamma) \uplus \mathfrak{C} \in G_s. \tag{5}$$

Let $f'\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ be any homomorphism which coincides with f on the variables appearing in the formulas Γ and Δ, and such that $f'(x_i) = c_i$. By (5) we have that

$$f'(\Gamma) \uplus f'(\Pi) \in G_s.$$

Together with (4), this implies that

$$\mathfrak{X} \uplus \mathfrak{C} = f(\Delta) \uplus \mathfrak{C} = f'(\Delta) \uplus f'(\Pi) \in G_{s+1}.$$

This concludes the proof of (3) and, therefore, establishes (2). Now we turn back to the main argument. First observe that

$$(\mathcal{F}i_{\vdash}(\mathbf{A}))^p = \{\mathrm{Fg}^{\mathbf{A}}_{\vdash}(\mathfrak{X}) : \mathfrak{X} \in A^\flat\}.$$

Clearly $(\mathcal{F}i_{\vdash}(\mathbf{A}))^p$ is a family of $\leqslant$-downsets. Then define the map $\delta\colon A^\flat \to (\mathcal{F}i_{\vdash}(\mathbf{A}))^p$ setting

$$\begin{aligned}\delta(\mathfrak{X}) &= \bigcap\{C \in (\mathcal{F}i_{\vdash}(\mathbf{A}))^p : \mathfrak{X} \in C\}\\ &= \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\mathfrak{X})\end{aligned}$$

for every $\mathfrak{X} \in A^\flat$. Clearly $\delta(A^\flat) = (\mathcal{F}i_{\vdash}(\mathbf{A}))^p$. Finally, from (2) we obtain that

$$\text{if } \delta(\mathfrak{X}) \subseteq \delta(\mathfrak{Y}), \text{then } \delta(\mathfrak{X} \uplus \mathfrak{C}) \subseteq \delta(\mathfrak{Y} \uplus \mathfrak{C}).$$

Hence we conclude that $(\mathcal{F}i_{\vdash}(\mathbf{A}))^p$ is a DS on $\mathbf{A}^\flat$ as desired.

Then we turn to prove that $\mathcal{F}i_{\vdash}(\mathbf{A}) = ((\mathcal{F}i_{\vdash}(\mathbf{A}))^p)^{BJ}$. The inclusion from left to right is clear. To prove the other one, consider a family $\{F_i : i \in I\} \subseteq (\mathcal{F}i_{\vdash}(\mathbf{A}))^p$. Then suppose that $\Gamma \vdash \Delta$, and consider a homomorphism $f\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ such that $f(\Gamma) \in \bigcup_{i\in I} F_i$. Clearly there is $j \in I$ such that $f(\Gamma) \in F_j$. Since $F_j \in \mathcal{F}i_{\vdash}(\mathbf{A})$, we obtain that $f(\Delta) \in F_j \subseteq \bigcup_{i\in I} F_i$. Hence we conclude that $\bigcup_{i\in I} F_i \in \mathcal{F}i_{\vdash}(\mathbf{A})$. □

The notions introduced so far are enough to obtain a first completeness theorem for any substitution-invariant MDR.

Theorem 60 (1st completeness theorem)**.** *Let $\vdash$ be a substitution-invariant* MDR *on $\mathcal{L}$. Then*

$$\vdash \;=\; \models_{\mathbf{Mod}(\vdash)} \;=\; \models'_{\mathbf{Mod}(\vdash)}.$$

Proof. From left to right, our claim is obvious. For the reverse direction, assume that $\Gamma \nvdash \Delta$ and define $T = \mathrm{Th}_{\vdash}(\Gamma)$. By Proposition 59, $\langle \mathbf{Fm}_{\mathcal{L}}, T\rangle \in \mathbf{Mod}(\vdash)$ and then the identity mapping is the homomorphism we need to show that $\Gamma \not\models_{\mathbf{Mod}(\vdash)} \Delta$. □

5.2 A bridge to Gentzen systems and the second completeness theorem

We will now establish a connection with the algebraic theory of Gentzen systems, i.e. substitution-invariant ACR's on sequents, a well-trodden research stream in AAL [29, 30, 31, 32, 28, 36], that will serve as a touchstone for our approach based on hypermatrices. Let $\vdash$ be a substitution-invariant MDR on $\mathcal{L}$. We will associate with it a consequence relation $\vdash^g$ between sequents. To this end, consider the set $Seq_{\mathcal{L}}$ of $\mathcal{L}$-sequents of the form

$$\emptyset \rhd \langle \varphi_1, \ldots, \varphi_n\rangle,$$

where $\langle \varphi_1, \ldots, \varphi_n\rangle$ is a finite sequence of formulas. We consider the relation $\vdash^g \subseteq \wp(Seq_{\mathcal{L}}) \times Seq_{\mathcal{L}}$ defined as follows:

$$X \vdash^g \emptyset \rhd \langle \varphi_1, \ldots, \varphi_n\rangle \iff \text{there is } \emptyset \rhd \langle \gamma_1, \ldots, \gamma_m\rangle \in X \text{ s.t. } [\gamma_1, \ldots, \gamma_m] \vdash [\varphi_1, \ldots, \varphi_m].$$

Clearly, $\vdash^g$ is a substitution-invariant ACR on $Seq_{\mathcal{L}}$: for every substitution σ, if $X \vdash^g \emptyset \rhd \langle \varphi_1, \ldots, \varphi_n\rangle$, then

$$\{\emptyset \rhd \langle \sigma(\gamma_1), \ldots, \sigma(\gamma_m)\rangle : \emptyset \rhd \langle \gamma_1, \ldots, \gamma_m\rangle \in X\} \vdash^g \emptyset \rhd \langle \sigma(\varphi_1), \ldots, \sigma(\varphi_n)\rangle.$$

As we remarked above, substitution-invariant ACR's on sequents are the object of study of numerous papers that have appeared under the heading of *algebraisation of Gentzen systems*. Within this theory, a model of a Gentzen system $\Vdash$ on $Seq_{\mathcal{L}}$ is a pair $\langle \mathbf{A}, F\rangle$ where $\mathbf{A}$ is an $\mathcal{L}$-algebra and F is a set of finite sequences of elements of A such that for every set $X \cup \{\emptyset \rhd \langle \varphi_1, \ldots, \varphi_n\rangle\} \subseteq Seq_{\mathcal{L}}$,

if $X \Vdash \emptyset \rhd \langle \varphi_1, \ldots, \varphi_n \rangle$, then for every homomorphism $f \colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$,

$$\begin{aligned} &\text{if } \langle f(\gamma_1), \ldots, f(\gamma_m) \rangle \in F \text{ for every } \emptyset \rhd \langle \gamma_1, \ldots, \gamma_m \rangle \in X, \\ &\qquad\qquad \text{then } \langle f(\varphi_1), \ldots, f(\varphi_n) \rangle \in F. \end{aligned} \tag{6}$$

We denote by $\mathbf{Mod}(\Vdash)$ the class of all models of $\Vdash$. A quick comparison between Definition 58 (see also the comments before the definition) and (6) suggests that the models of $\vdash$ and $\vdash^g$ must be interdefinable. To make this idea precise, we define two maps as follows:

$$(\cdot)^s \colon \mathbf{Mod}(\vdash) \longleftrightarrow \mathbf{Mod}(\vdash^g) \colon (\cdot)^m \tag{7}$$

where $(\cdot)^s$ stands for *sequents* and $(\cdot)^m$ stands for *multisets*. Given $\langle \mathbf{A}, F \rangle \in \mathbf{Mod}(\vdash^g)$ and $\langle \mathbf{B}, G \rangle \in \mathbf{Mod}(\vdash)$, we set:

$$\begin{aligned} \langle \mathbf{A}, F \rangle^m &= \langle \mathbf{A}, \{[a_1, \ldots, a_n] : n \geq 0, \langle a_1, \ldots, a_n \rangle \in F\} \rangle \\ \langle \mathbf{B}, G \rangle^s &= \langle \mathbf{B}, \{f \in B^{\{1,\ldots,n\}} : n \geq 0, [f(1), \ldots, f(n)] = \mathfrak{X} \text{ for some } \mathfrak{X} \in G\} \rangle. \end{aligned}$$

The proof of the following result is straightforward (note that we need to use Lemma 55):

Lemma 61. *The transformations* $(\cdot)^s \colon \mathbf{Mod}(\vdash) \longleftrightarrow \mathbf{Mod}(\vdash^g) \colon (\cdot)^m$ *are well-defined and mutually inverse bijections.*

As a consequence, we can apply the algebraic constructions developed for Gentzen systems in the above-mentioned literature, to the study of substitution-invariant MDR's. We devote the remaining part of this subsection to give a flavour of the resulting theory.

Let $\langle \mathbf{A}, F \rangle$ be a pair consisting of an $\mathcal{L}$-algebra $\mathbf{A}$ and a set F of finite sequences of elements of A. A congruence θ of $\mathbf{A}$ is *compatible* with F if for every $a_1, b_1, \ldots, a_n, b_n \in A$,

$$\text{if } \langle a_1, \ldots, a_n \rangle \in F \text{ and } \langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \in \theta, \text{ then } \langle b_1, \ldots, b_n \rangle \in F.$$

When θ is compatible with F, we set

$$F/\theta = \{\langle a_1/\theta, \ldots, a_n/\theta \rangle : \langle a_1, \ldots, a_n \rangle \in F\}.$$

It turns out that there exists the largest congruence of $\mathbf{A}$ compatible with F. This congruence is called the *Leibniz congruence* of F over $\mathbf{A}$, and is denoted by $\Omega^{\mathbf{A}} F$. The *reduced models* of $\Vdash$ are the following class:

$$\mathbf{Mod}^*(\Vdash) = \mathbb{I}\left\{\langle \mathbf{A}/\Omega^{\mathbf{A}} F, F/\Omega^{\mathbf{A}} F \rangle : \langle \mathbf{A}, F \rangle \in \mathbf{Mod}(\Vdash)\right\}.$$

A general result [12, Proposition 5.111] shows that the Gentzen system $\Vdash$ is *complete* with respect to the semantics $\mathbf{Mod}^*(\Vdash)$: the pairs in $\mathbf{Mod}^*(\Vdash)$ are models of $\Vdash$ and, moreover, if $X \not\Vdash \emptyset \rhd \langle \varphi_1, \ldots, \varphi_n \rangle$, then there is $\langle \mathbf{A}, F \rangle \in \mathbf{Mod}^*(\Vdash)$ and a homomorphism $f \colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ such that

$$\begin{aligned} &\langle f(\gamma_1), \ldots, f(\gamma_m) \rangle \in F \text{ for every } \emptyset \rhd \langle \gamma_1, \ldots, \gamma_m \rangle \in X, \\ &\qquad\qquad \text{and } \langle f(\varphi_1), \ldots, f(\varphi_n) \rangle \notin F. \end{aligned}$$

All the above constructions can be transferred to the study of substitution-invariant MDR's as follows. Let $\langle \mathbf{A}, F\rangle$ be an $\mathcal{L}$-hypermatrix. A congruence θ of $\mathbf{A}$ is *compatible* with F if for every $a_1, b_1, \ldots, a_n, b_n \in A$,

$$\text{if } [a_1, \ldots, a_n] \in F \text{ and } [a_1/\theta, \ldots, a_n/\theta] = [b_1/\theta, \ldots, b_n/\theta]\,, \\ \text{then } [b_1, \ldots, b_n] \in F.$$

When θ is compatible with F, we set:

$$F/\theta = \{[a_1/\theta, \ldots, a_n/\theta] : [a_1, \ldots, a_n] \in F\}\,.$$

Lemma 62. *There exists the largest congruence of* $\mathbf{A}$ *compatible with* F*.*

Proof. We will denote by G the set of finite sequences of elements of A such that $\langle \mathbf{A}, G\rangle = \langle \mathbf{A}, F\rangle^s$. We know that $\Omega^{\mathbf{A}}G$ is the largest congruence of $\mathbf{A}$ compatible with G. In order to conclude the proof, it will be enough to show that $\Omega^{\mathbf{A}}G$ is also the largest congruence of $\mathbf{A}$ compatible with F. However, it is easily proved that a congruence of $\mathbf{A}$ is compatible with F if and only if it is compatible with G, whence our claim follows. □

Given the above result, we denote the largest congruence of $\mathbf{A}$ compatible with F by $\Omega^{\mathbf{A}}F$, and call it the the *Leibniz congruence* of F over $\mathbf{A}$. We define the *reduced models* of an MDR $\vdash$ as follows:

$$\mathbf{Mod}^*(\vdash) = \mathbb{I}\{\langle \mathbf{A}/\Omega^{\mathbf{A}}F, F/\Omega^{\mathbf{A}}F\rangle : \langle \mathbf{A}, F\rangle \in \mathbf{Mod}(\vdash)\}.$$

Corollary 63. *The transformations* $(\cdot)^s \colon \mathbf{Mod}^*(\vdash) \longleftrightarrow \mathbf{Mod}^*(\vdash^g) \colon (\cdot)^m$ *are well-defined and mutually inverse bijections.*

Proof. Pick a model $\langle \mathbf{A}, F\rangle \in \mathbf{Mod}^*(\vdash)$. Let G be the set of finite sequences of elements of A such that $\langle \mathbf{A}, F\rangle^s = \langle \mathbf{A}, G\rangle$. From Lemma 61 we know that $\langle \mathbf{A}, G\rangle \in \mathbf{Mod}(\vdash^g)$. Moreover, since $\langle \mathbf{A}, F\rangle \in \mathbf{Mod}^*(\vdash)$, the congruence $\Omega^{\mathbf{A}}F$ is the identity relation $\mathrm{Id}_{\mathbf{A}}$. Now, in the proof of Lemma 62 we showed that $\Omega^{\mathbf{A}}F = \Omega^{\mathbf{A}}G$. Thus we conclude that $\Omega^{\mathbf{A}}G = \mathrm{Id}_{\mathbf{A}}$. Since $\langle \mathbf{A}, G\rangle \cong \langle \mathbf{A}/\mathrm{Id}_{\mathbf{A}}, G/\mathrm{Id}_{\mathbf{A}}\rangle$ and $\mathbf{Mod}^*(\vdash^g)$ is closed under isomorphisms, we conclude that $\langle \mathbf{A}, F\rangle^s = \langle \mathbf{A}, G\rangle \in \mathbf{Mod}^*(\vdash^g)$. A similar argument shows that if $\langle \mathbf{B}, G\rangle \in \mathbf{Mod}^*(\vdash^g)$, then $\langle \mathbf{B}, G\rangle^m \in \mathbf{Mod}^*(\vdash)$. This means that the maps $(\cdot)^s \colon \mathbf{Mod}^*(\vdash) \longleftrightarrow \mathbf{Mod}^*(\vdash^g) \colon (\cdot)^m$ are well-defined. The fact that they are inverse bijections follows from Lemma 61. □

Theorem 64 (2nd completeness theorem)**.** *Let* $\vdash$ *be a substitution-invariant* MDR *on* $\mathcal{L}$*. Then*

$$\vdash \;=\; \models_{\mathbf{Mod}^*(\vdash)} \;=\; \models'_{\mathbf{Mod}^*(\vdash)}.$$

Proof. From the general theory of the algebraisation of Gentzen systems we know that $\mathbf{Mod}^*(\vdash^g)$ is a class of models of $\vdash^g$. By Lemma 61 this implies that the image of the class $\mathbf{Mod}^*(\vdash^g)$ under the transformation $(\cdot)^m$ is a class of models of $\vdash$. But by the previous corollary we know that this image coincides with $\mathbf{Mod}^*(\vdash)$. Thus $\mathbf{Mod}^*(\vdash)$ is a class of models of $\vdash$. Then suppose that $\mathfrak{X} \nvdash$

$\mathfrak{Y}$. From the very definition of $\vdash^g$ it follows that for any sequence $\langle\varphi_1,\ldots,\varphi_n\rangle$ and $\langle\psi_1,\ldots,\psi_m\rangle$ such that $\mathfrak{X}=[\varphi_1,\ldots,\varphi_n]$ and $\mathfrak{Y}=[\psi_1,\ldots,\psi_m]$ we have:

$$\emptyset \rhd \langle\varphi_1,\ldots,\varphi_n\rangle \not\Vdash \emptyset \rhd \langle\psi_1,\ldots,\psi_m\rangle.$$

From the general completeness result of $\vdash^g$ with respect to $\mathbf{Mod}^*(\vdash^g)$ it follows that there is $\langle\mathbf{A},F\rangle\in\mathbf{Mod}^*(\vdash^g)$ and a homomorphism $f\colon\mathbf{Fm}_{\mathcal{L}}\to\mathbf{A}$ such that

$$\langle f(\varphi_1),\ldots,f(\varphi_n)\rangle\in F \text{ and } \langle f(\psi_1),\ldots,f(\psi_m)\rangle\notin F.$$

Let $G\subseteq A^\flat$ be such that $\langle\mathbf{A},F\rangle^m=\langle\mathbf{A},G\rangle$. Due to the previous corollary we know that $\langle\mathbf{A},G\rangle\in\mathbf{Mod}^*(\vdash)$. Moreover, it is straightforward to check that

$$[f(\varphi_1),\ldots,f(\varphi_n)]\in G \text{ and } [f(\psi_1),\ldots,f(\psi_m)]\notin G. \qquad \square$$

5.3 Monoid matrices and t-norm semantics

It is possible to give hypermatrices a (nearly) equivalent formulation in such a way as to shed further light on the direction in which they generalise ordinary logical matrices. The rough idea is replacing the unstructured set of designated values in a logical matrix by a richer structure. If $\langle\mathbf{A},D\rangle$ is an ordinary logical matrix (i.e., an algebra with a subset), D can be identified with a function in $\{0,1\}^A$; in other words, being designated is an all-or-nothing matter. The set $\{0,1\}$ can also be viewed as the universe of the 2-element join semilattice $\mathbf{2}$. If we replace $\mathbf{2}$ by any dually integral Abelian po-monoid $\mathbf{D}$, however, we can at the same time express an ordering of "degrees of designation", and evaluate the degree of designation of whole submultisets of A, with the monoidal operation in $\mathbf{D}$ ensuring that evaluations behave well with respect to multiset union. This leads to the following:

Definition 65. *Let $\mathcal{L}$ be a language. An* $\mathcal{L}$-monoid matrix *is a quadruple $\boldsymbol{M}=\langle\mathbf{A},\mathbf{D},G,f\rangle$, where:*

1. *$\mathbf{A}$ is an $\mathcal{L}$-algebra.*
2. *$\mathbf{D}=\langle D,\leq,+,0\rangle$ is a dually integral Abelian po-monoid.*
3. *G is a $\leq$-downset in $\mathbf{D}$.*
4. *$f\colon\mathbf{A}^\flat\to\langle D,\leq,+,0\rangle$ is a po-monoid homomorphism.*

The next lemma ensures that monoid matrices are closed w.r.t. a sort of quotient construction. For $E\subseteq D$, $(E]$ will denote the $\leq$-downset generated in $\mathbf{D}$ by E.

Lemma 66. *Let $\boldsymbol{M}=\langle\mathbf{A},\mathbf{D},G,f\rangle$ be an $\mathcal{L}$-monoid matrix, let $\mathbf{D}'$ be a dually integral Abelian po-monoid, and let $g\colon\mathbf{D}\to\mathbf{D}'$ be a po-monoid homomorphism. Then*

$$g_{\mathbf{D}'}(\boldsymbol{M})=\langle\mathbf{A},\mathbf{D}',(g(G)],g\circ f\rangle$$

is an $\mathcal{L}$-monoid matrix.

Proof. $\mathbf{D}'$ is a dually integral Abelian po-monoid by assumption, and likewise $(g(G)]$ is a $\leq$-downset in $\mathbf{D}'$. The map $g \circ f$ is a composition of monoid homomorphisms, and if $\mathfrak{X} \leqslant \mathfrak{Y}$, then $f(\mathfrak{X}) \leq^{\mathbf{D}} f(\mathfrak{Y})$ and so $g(f(\mathfrak{X})) \leq^{\mathbf{D}'} g(f(\mathfrak{Y}))$. □

The relationships between the previously introduced notions are made clear in the next theorem. While every $\mathcal{L}$-hypermatrix arises out of an $\mathcal{L}$-monoid matrix, an $\mathcal{L}$-monoid matrix need not be more than the "homomorphic image" of an $\mathcal{L}$-monoid matrix that arises out of an $\mathcal{L}$-hypermatrix.

Theorem 67. *1. If $\boldsymbol{M} = \langle \mathbf{A}, \mathbf{D}, G, f \rangle$ is an $\mathcal{L}$-monoid matrix, then*

$$\boldsymbol{H}^{\boldsymbol{M}} = \left\langle \mathbf{A}, \left\{ \mathfrak{X} \in A^{\flat} : f(\mathfrak{X}) \in G \right\} \right\rangle$$

is an $\mathcal{L}$-hypermatrix.

2. If $\boldsymbol{H} = \langle \mathbf{A}, F \rangle$ is an $\mathcal{L}$-hypermatrix, then

$$\boldsymbol{M}^{\boldsymbol{H}} = \left\langle \mathbf{A}, \mathbf{A}^{\flat}, F, id \right\rangle$$

is an $\mathcal{L}$-monoid matrix.

3. $\boldsymbol{H}^{\boldsymbol{M}^{\boldsymbol{H}}} = \boldsymbol{H}$.

4. $f_{\mathbf{D}}\left(\boldsymbol{M}^{\boldsymbol{H}^{\boldsymbol{M}}}\right) = \boldsymbol{M}$.

Proof. (1) We have to show that $\left\{ \mathfrak{X} \in A^{\flat} : f(\mathfrak{X}) \in G \right\}$ is a $\leqslant$-downset, i.e. if $f(\mathfrak{X}) \in G$ and $\mathfrak{Y} \leqslant \mathfrak{X}$, then $f(\mathfrak{Y}) \in G$. However, if $\mathfrak{Y} \leqslant \mathfrak{X}$, then $f(\mathfrak{Y}) \leq^{\mathbf{D}} f(\mathfrak{X})$, whence our conclusion follows as G is a $\leq^{\mathbf{D}}$-downset.

(2) Trivial.

(3) $\boldsymbol{H}^{\boldsymbol{M}^{\boldsymbol{H}}} = \left\langle \mathbf{A}, \left\{ \mathfrak{X} \in A^{\flat} : id(\mathfrak{X}) \in F \right\} \right\rangle = \langle \mathbf{A}, F \rangle = \boldsymbol{H}$.

(4) $\boldsymbol{M}^{\boldsymbol{H}^{\boldsymbol{M}}} = \left\langle \mathbf{A}, \mathbf{A}^{\flat}, \{ \mathfrak{X} : f(\mathfrak{X}) \in G \}, id \right\rangle$, whence

$$f_{\mathbf{D}}\left(\boldsymbol{M}^{\boldsymbol{H}^{\boldsymbol{M}}}\right) = \langle \mathbf{A}, \mathbf{D}, (f(\{ \mathfrak{X} : f(\mathfrak{X}) \in G \})], f \rangle .$$

However, $f(\{ \mathfrak{X} : f(\mathfrak{X}) \in G \}) \subseteq G$, so $(f(\{ \mathfrak{X} : f(\mathfrak{X}) \in G \})] = G$, and thus $f_{\mathbf{D}}\left(\boldsymbol{M}^{\boldsymbol{H}^{\boldsymbol{M}}}\right) = \boldsymbol{M}$. □

We now focus on a special class of monoid matrices, namely, those matrices whose underlying po-monoid is just the closed unit real interval $[0, 1]$, endowed with some t-norm $*$ (i.e., a monotone, associative, and commutative operation with unit 1) and with the usual ordering of real numbers. In essence, these monoid matrices can be seen as an algebra together with a fuzzy set of designated values. Although very special in nature, these matrices can be used to yield a semantics for the multiset companions of some fuzzy logics, obtained in the same guise as $\vdash_{\mathcal{MV}}$ (see the remarks immediately following Example 49).

Definition 68. *An* $\mathcal{L}$-fuzzy matrix *is an* $\mathcal{L}$*-monoid matrix* $\boldsymbol{M} = \langle \mathbf{A}, \mathbf{D}, G, f \rangle$*, s.t.* $\mathbf{D} = \langle [0,1], \sqsupseteq, *, 1 \rangle$*, where* $*$ *is some t-norm and* $\sqsubseteq$ *is the usual ordering of* $[0,1]$*.*

Two fuzzy matrices will be called *similar* if their algebra reducts are similar and the t-norm $*$ is the same in both cases.

Definition 69. *If* $\boldsymbol{M} = \langle \mathbf{A}, \mathbf{D}, G, f \rangle$ *is an* $\mathcal{L}$*-fuzzy matrix with t-norm* $*$ *and* $\Gamma, \Delta \in Fm^{\flat}_{\mathcal{L}}$*, we set* $\Gamma \models^{*}_{\boldsymbol{M}} \Delta$ *just in case* $\Gamma \models_{\boldsymbol{H}^{M}} \Delta$*.*

If M *is a class of similar* $\mathcal{L}$*-fuzzy matrices, we write* $\Gamma \models^{*}_{\mathsf{M}} \Delta$ *as a shortcut for:* $\Gamma \models^{*}_{\boldsymbol{M}} \Delta$ *for every* $\boldsymbol{M} \in \mathsf{M}$*.*

Theorem 54 implies that:

Lemma 70. *If* M *is a class of similar* $\mathcal{L}$*-fuzzy matrices,* $\models^{*}_{\mathsf{M}}$ *is a substitution-invariant* MDR *on* $\mathcal{L}$*.*

Observe that, if we fix $\mathbf{A}$ and f while letting M be the class of similar $\mathcal{L}$-fuzzy matrices $\{\langle \mathbf{A}, \mathbf{D}, [a,1], f \rangle : a \in [0,1]\}$, we have that

$$\begin{array}{rcl} \Gamma \models^{*}_{\mathsf{M}} \Delta & \Leftrightarrow & \Gamma \models_{\boldsymbol{H}^{M}} \Delta \text{ for all } \boldsymbol{M} \in \mathsf{M} \\ & \Leftrightarrow & \forall a \forall \mathfrak{C} \forall e \left(f\left(\mathfrak{C} \uplus e\left(\Gamma\right)\right) \in [a,1] \Rightarrow f\left(\mathfrak{C} \uplus e\left(\Delta\right) \in [a,1]\right)\right) \\ & \Leftrightarrow & \forall \mathfrak{C} \forall e \left(f\left(\mathfrak{C} \uplus e\left(\Gamma\right)\right) \sqsubseteq f\left(\mathfrak{C} \uplus e\left(\Delta\right)\right)\right) \\ & \Rightarrow & \forall e \left(f\left(e\left(\Gamma\right)\right) \sqsubseteq f\left(e\left(\Delta\right)\right)\right), \end{array}$$

where the third equivalence uses the downward closure of $\boldsymbol{H}^{\boldsymbol{M}}$. Furthermore, since

$$f\left(e\left[\gamma_1, \ldots, \gamma_n\right]\right) = f\left(e\left(\gamma_1\right)\right) * \cdots * f\left(e\left(\gamma_n\right)\right),$$

the behaviour of f is entirely determined by its behaviour on one-element multisets, whence we lose no generality in taking f to be a function from A to $[0,1]$. In other words, a fuzzy matrix can be viewed — in this special case — as an algebra together with a fuzzy set of designated values. Moreover, if $\mathbf{A}$ itself is some algebra with universe $[0,1]$, the function f becomes a real function.

With this material at hand, we are ready to prove a completeness theorem for $\vdash_{\mathcal{MV}}$ with respect to the class M of all $\mathcal{L}_0$-fuzzy matrices of the form $\langle [\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \mathbf{D}, [a,1], f \rangle$, where:

- $[\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}$ is the standard MV algebra over $[0,1]$, formulated in the language $\mathcal{L}_0$ of commutative residuated lattices.
- $\mathbf{D} = \langle [0,1], \sqsupseteq, \otimes, 1 \rangle$, where $\otimes$ is the Łukasiewicz t-norm.[9]
- $a \in [0,1]$.
- $f\colon [0,1] \to [0,1]$ is strictly monotone and preserves $\otimes$ (this class is nonempty: it contains e.g. the identity function id and the square function $()^2$).

[9] When discussing Łukasiewicz logic and its multiset companion $\vdash_{\mathcal{MV}}$, we write the multiplicative conjunction (residuated lattice product) $\varphi \cdot \psi$ using the more customary notation $\varphi \otimes \psi$.

Theorem 71. *For any* $\Gamma, \Delta \in Fm^{\flat}_{\mathcal{L}_0}$*, the following are equivalent:*

1. $\Gamma \vdash_{\mathcal{MV}} \Delta$.
2. $\Gamma \models^{*}_{\mathsf{M}} \Delta$.
3. $\Gamma \models^{*}_{\mathsf{I}} \Delta$, *where* $\mathsf{I} = \{\boldsymbol{M} \in \mathsf{M} : f = id\}$.

Proof. Assume that $\Gamma = [\varphi_1, \ldots, \varphi_n]$ and $\Delta = [\psi_1, \ldots, \psi_m]$.

(1) $\Rightarrow$ (2). Suppose that $\Gamma \vdash_{\mathcal{MV}} \Delta$. By Chang's completeness theorem, this means that for every homomorphism $e\colon \mathbf{Fm}_{\mathcal{L}_0} \to [\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}$,

$$e(\varphi_1) \otimes \cdots \otimes e(\varphi_n) \sqsubseteq e(\psi_1) \otimes \cdots \otimes e(\psi_m).$$

Let $\boldsymbol{M} = \langle [\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \mathbf{D}, [a, 1], f\rangle \in \mathsf{M}$, $\mathfrak{C} \in [\mathbf{0}, \mathbf{1}]^{\flat}$, and let $e'\colon \mathbf{Fm}_{\mathcal{L}_0} \to [\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}$ be a homomorphism. Suppose further that

$$\begin{aligned} a &\sqsubseteq f(\mathfrak{C} \uplus e'(\Gamma)) \\ &= f(\mathfrak{C}) \otimes f(e'(\Gamma)) \\ &= f(\mathfrak{C}) \otimes f(e'(\varphi_1)) \otimes \cdots \otimes f(e'(\varphi_n)). \end{aligned}$$

However, since f is monotone and $\otimes$-preserving,

$$f(e'(\varphi_1)) \otimes \cdots \otimes f(e'(\varphi_n)) \sqsubseteq f(e'(\psi_1)) \otimes \cdots \otimes f(e'(\psi_m))$$

and, by monotonicity of t-norms, $a \sqsubseteq f(\mathfrak{C}) \otimes f(e'(\varphi_1)) \otimes \cdots \otimes f(e'(\varphi_n)) \sqsubseteq f(\mathfrak{C}) \otimes f(e'(\psi_1)) \otimes \cdots \otimes f(e'(\psi_m)) = f(\mathfrak{C} \uplus e'(\Delta))$, which suffices for our conclusion.

(2) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (1) follows from the observation after Lemma 70. □

5.4 Hilbert systems

We mentioned at the outset that previous attempts at investigating multiset consequence are few and far between. Virtually all authors who undertook this enterprise, however, tried to set up axiomatic calculi of sorts [3, 37, 27]. We now proceed to present our own take on the issue.

Definition 72. *A* consecution[10] *in a propositional language* $\mathcal{L}$ *is a pair* $\langle \Gamma, \Delta\rangle$*, where* Γ *and* Δ *are finite multisets of formulas. A consecution is* single-conclusion *if* $\Delta = [\varphi]$ *for some formula* φ.

Instead of '$\langle \Gamma, \Delta\rangle$', we write '$\Gamma \rhd \Delta$'. With a slight abuse, we also identify the consecution $\emptyset \rhd [\varphi]$ with the formula φ.

Definition 73 (Axiomatic system)**.** *Let* $\mathcal{L}$ *be a propositional language. A* (single-conclusion) axiomatic system *in the language* $\mathcal{L}$ *is a set* AS *of (single-conclusion) consecutions closed under arbitrary substitutions.*[11] *The elements of* AS *of the form* $\Gamma \rhd \Delta$ *are called* axioms *if* $\Gamma = \emptyset$ *and* deduction rules *otherwise.*

[10] The term "consecution" is taken from [1] (the term "sequent" is sometimes used instead).
[11] I.e., if $\Gamma \rhd \Delta \in \mathsf{AS}$, then $\sigma(\Gamma) \rhd \sigma(\Delta) \in \mathsf{AS}$.

Of course, each axiomatic system can also be seen as a collection of *schemata*, i.e. a collection of consecutions and all their substitution instances. Observe that our single-conclusion axiomatic systems are essentially Avron's *multiset Hilbert systems* [3]; however, the upcoming notion of tree-proof is different, as our single-conclusion MDR's (unlike Avron's "simple consequence relations") enjoy the Monotonicity condition.

Definition 74 (Tree-proof)**.** *Let $\mathcal{L}$ be a propositional language and let* AS *be a single-conclusion axiomatic system in $\mathcal{L}$. A* tree-proof *of a formula φ from a multiset of formulas Γ in* AS *is a finite tree t labelled by formulas such that:*

- *The root of t is labelled by φ.*
- *If a leaf of t is labelled by ψ, then either*
  - *ψ is an axiom or*
  - *ψ is an element of Γ and it labels at most $\Gamma(\psi)$ leaves in t.*
- *If a node of t is labelled by ψ and $\Delta \neq \emptyset$ is the multiset of labels of its predecessor nodes, then $\Delta \rhd [\psi] \in$ AS.*

We write $\Gamma \vdash^{t}_{\mathsf{AS}} \varphi$ whenever there is a tree-proof of φ from Γ in AS. Our next goal is to define a notion of derivation for arbitrary axiomatic systems.

Definition 75. *Let $\mathcal{L}$ be a propositional language and let* AS *be an axiomatic system in $\mathcal{L}$. A* derivation *of a finite multiset of formulas Δ from a finite multiset of formulas Γ in* AS *is a finite sequence $\langle \Gamma_1, \ldots, \Gamma_n \rangle$ of finite multisets of formulas such that:*

- $\Gamma_1 = \Gamma$*;*
- *For every Γ_j, $1 < j \leq n$, there is $\Psi \rhd \Psi' \in$ AS, such that $\Psi \leqslant \Gamma_{j-1}$ and $\Gamma_j = (\Gamma_{j-i} \setminus \Psi) \uplus \Psi'$;*
- $\Delta \leqslant \Gamma_n$.

We say that Δ is derivable *from Γ in* AS*, and write $\Gamma \vdash_{\mathsf{AS}} \Delta$, if there is a derivation of Δ from Γ in* AS.

Observe that, if $\Psi = \emptyset$, the second clause above says that in a derivation we are allowed to beef up with finitely many axioms any multiset that has already been derived. The next lemma supports the adequacy of the given definition.

Lemma 76. *Let $\mathcal{L}$ be a propositional language and let* AS *be an axiomatic system in $\mathcal{L}$. Then $\vdash_{\mathsf{AS}}$ is the least substitution-invariant* MDR *containing* AS.

Proof. Generalized Reflexivity being trivial, we prove the remaining conditions one by one.

- (Compatibility). Given a derivation $P = \langle \Gamma_1, \ldots, \Gamma_n \rangle$ of Δ from Γ in AS, it is easy to observe that the sequence $P' = \langle \Gamma_1 \uplus \Pi, \ldots, \Gamma_n \uplus \Pi \rangle$ is a derivation of $\Delta \uplus \Pi$ from $\Gamma \uplus \Pi$ (thanks to the fact that in our notion of proof we can apply rules in an arbitrary context).

- (Transitivity). Suppose we have a derivation $\langle \Gamma_1, \ldots, \Gamma_n \rangle$ of Δ from Γ in AS. Then $\Delta \leqslant \Gamma_n$ and so from $\Delta \vdash_{\mathsf{AS}} \Pi$ we get by monotony $\Gamma_n \vdash_{\mathsf{AS}} \Pi \uplus (\Gamma_n \setminus \Delta)$; let $\langle \Delta_1, \ldots, \Delta_n \rangle$ be the corresponding derivation in AS. Note that $\Delta_1 = \Gamma_n$ and $\Pi \leqslant \Delta_n$. Then clearly the sequence

  $$\langle \Gamma_1, \ldots, \Gamma_n, \Delta_2, \ldots, \Delta_n \rangle$$

  is a derivation of Π from Γ in AS.

- (Substitution-invariance). Given a derivation $P = \langle \Gamma_1, \ldots, \Gamma_n \rangle$ of Δ from Γ in AS, the sequence $P' = \langle \sigma[\Gamma_1], \ldots, \sigma[\Gamma_n] \rangle$ is a derivation of $\sigma[\Delta]$ from $\sigma[\Gamma]$ in AS.

Now for the proof that $\vdash_{\mathsf{AS}}$ is the least substitution-invariant MDR containing AS. Obviously $\mathsf{AS} \subseteq \vdash_{\mathsf{AS}}$. What remains to prove is that for each substitution-invariant MDR $\vdash$, if $\mathsf{AS} \subseteq \vdash$, then $\vdash_{\mathsf{AS}} \subseteq \vdash$. Assume that $\Gamma \vdash_{\mathsf{AS}} \Delta$, i.e. there is a derivation P of Δ from Γ in AS. By induction on the length of P, we can show that for each multiset of formulas Π in P we have $\Gamma \vdash \Pi$, and hence in particular $\Gamma \vdash \Delta$. The base case is settled with an appeal to Reflexivity. As to the induction step: let Π and Π' be labels of successive elements of P and $\Gamma \vdash \Pi$. We know that there is a rule $\Psi \rhd \Psi'$ such that $\Pi' = (\Pi \setminus \Psi) \uplus \Psi'$. Thus $\Psi \vdash \Psi'$, and so by Compatibility $\Psi \uplus (\Pi \setminus \Psi) \vdash \Psi' \uplus (\Pi \setminus \Psi)$, i.e. $\Pi \vdash \Pi'$. An application of Transitivity completes the proof. □

Definition 77. *Let $\mathcal{L}$ be a propositional language, AS an axiomatic system in $\mathcal{L}$, and let $\vdash$ be a substitution-invariant* MDR *on $\mathcal{L}$. We say that AS is an* axiomatic system *for (or a* presentation *of) $\vdash$ if $\vdash = \vdash_{\mathsf{AS}}$.*

Clearly, due to the previous lemma, each MDR can be seen as its own presentation, and so we obtain:

Corollary 78 (Łos-Suszko)**.** *Every substitution-invariant* MDR $\vdash$ *coincides with the derivability relation $\vdash_{\mathsf{AS}}$ of some axiomatic system AS.*

The next lemma spells out the relationship between derivations and tree-proofs.

Lemma 79. *Let AS be a single-conclusion axiomatic system on $\mathcal{L}$, $\Gamma \vdash_{\mathsf{AS}} \Delta$, and $\varphi \in |\Delta|$. Then there are multisets of $\mathcal{L}$-formulas Γ^{φ} and Γ^{r} such that $\Gamma^{\varphi} \uplus \Gamma^{r} = \Gamma$, $\Gamma^{\varphi} \vdash^{t}_{\mathsf{AS}} \varphi$ and $\Gamma^{r} \vdash_{\mathsf{AS}} \Delta \setminus [\varphi]$.*

Proof. Let $P = \langle \Gamma_1, \ldots, \Gamma_n \rangle$ be the assumed derivation of Δ from Γ in AS. For each $i \leq n$, let $\Gamma_i = [\psi^i_1, \ldots, \psi^i_{k_i}]$ and note that without loss of generality we can assume that the rule used in the i-th step of P is $[\psi^i_1, \ldots, \psi^i_{p_i}] \rhd \psi^{i+1}_{c_i}$

for some $p_i \leq k_i$ and $c_i \leq k_{i+1}$. Also note that $k_{i+1} - 1 = k_{i+1} - p_i$ and there is a bijection f between $\Gamma_{i+1} \setminus [\psi^{i+1}_{c_i}]$ and $\Gamma_i \setminus [\psi^i_1, \ldots, \psi^i_{p_i}]$ such that $\psi^{i+1}_j = \psi^i_{f(j)}$ whenever $c_i \neq j \leq k_{i+1}$. We construct the labelled graph G with nodes $N = \{\langle i, j\rangle \mid i \leq n \text{ and } j \leq k_i\}$, where ψ^i_j is the label of $\langle i, j\rangle$, and edges only between the following nodes:

- *rule edges*: $\langle i, k\rangle$ and $\langle i+1, c_i\rangle$ for each $k \leq p_i$;
- *non-rule edges:* $\langle i, f(j)\rangle$ and $\langle i+1, j\rangle$ for $j \neq c_i$.

It is easy to see that G is a forest (a disjoint union of trees). Let t be the subtree of G with root ψ^1_n, and let Γ^φ be the multiset of all labels of leaves in t which are not axioms. Then clearly $\Gamma^\varphi \leqslant \Gamma$ and t is *almost* a tree-proof of φ from Γ^φ; all we have to do is to collapse nodes connected by non-rule edges.

Finally, let Γ^t_i denote the multiset resulting from Γ_i by removing formulas labeling the nodes of t (as many times as it labels some node) and observe that $\Gamma^t_1, \ldots, \Gamma^t_n$ is *almost* a proof of $\Gamma^t_n = \Gamma_n \setminus [\psi^n_1]$ from Γ^t_1: we only need to remove each Γ^t_i which equals its predecessor. Defining $\Gamma^r = \Gamma^t_1$ and observing that $\Gamma^r = \Gamma \setminus \Gamma^\varphi$ and $\Delta \setminus [\psi^n_1] \leqslant \Gamma^t_n$ completes the proof. □

Lemma 80. *Let* AS *be a single-conclusion axiomatic system. Then* $\Gamma \vdash^t_{\mathsf{AS}} \varphi$ *iff* $\Gamma \vdash_{\mathsf{AS}} [\varphi]$.

Proof. One direction follows directly from the previous lemma. To prove the converse one, assume that there is a tree-proof t of φ from Γ in AS. Let n be a node of t, ψ_n its label, P_n the set of its predecessors, Δ_n the multiset of labels of nodes in P_n, and Γ_n the multiset of labels of elements of Γ which are not axioms and occur in leaves of the subtree of t with root n. If we show that $\Gamma_n \vdash_{\mathsf{AS}} [\psi_n]$, the proof is done: indeed for the root r of t we obtain $\Gamma_r \vdash_{\mathsf{AS}} [\varphi]$ and given that $\Gamma_r \leqslant \Gamma$ and $\vdash$ is an MDR, we obtain the claim by Monotonicity. Let us prove the claim: if n is a leaf, the proof is trivial. Otherwise, there is a rule $\Delta_n \rhd [\psi_n]$ and for each $m \in P_n$ we have $\Gamma_m \vdash [\psi_m]$. Thus $\biguplus_{m \in P_n} \Gamma_m \vdash \Delta_n$ and so $\biguplus_{m \in P_n} \Gamma_m \vdash [\psi_n]$. The proof is completed by observing that $\biguplus_{m \in P_n} \Gamma_m = \Gamma_n$. □

Recall that single-conclusion MDR's were introduced in Definition 50 as *fragments* of MDR's. We now observe that the tree-provability relations of single-conclusion axiomatic systems, in a sense, "generate" the corresponding derivability relations.

Corollary 81. *Let* AS *be a single-conclusion axiomatic system. Then* $\vdash^t_{\mathsf{AS}}$ *is a single-conclusion* MDR *and* $\vdash_{\mathsf{AS}}$ *is the least* MDR $\vdash$ *such that*

$$\Gamma \vdash^t_{\mathsf{AS}} \alpha \text{ iff } \Gamma \vdash [\alpha]\,.$$

Proof. The former claim is immediate from the previous lemma. To prove the latter, assume that $\vdash$ is an MDR such that

$$\Gamma \vdash^t_{\mathsf{AS}} \alpha \text{ iff } \Gamma \vdash [\alpha]\,.$$

We need to show that if $\Gamma \vdash_{\mathsf{AS}} \Delta$, then $\Gamma \vdash \Delta$. We prove it by induction on $n = \sum_{\psi \in |\Delta|} \Delta(\psi)$. If $n = 0$ the claim is trivial. Assume now that $\Delta = \Delta_0 \uplus [\varphi]$. By Lemma 79, there are multisets Γ^φ and Γ^r such that $\Gamma^\varphi \uplus \Gamma^r = \Gamma$, $\Gamma^\varphi \vdash^t_{\mathsf{AS}} \varphi$ and $\Gamma^r \vdash_{\mathsf{AS}} \Delta \setminus [\varphi]$. Thus, our assumption on $\vdash$ and the induction hypothesis imply that $\Gamma^\varphi \vdash \varphi$ and $\Gamma^r \vdash \Delta \setminus [\varphi]$. Thus $\Gamma \vdash [\varphi] \uplus \Gamma^r$ and $\Gamma^r \uplus [\varphi] \vdash \Delta$. Transitivity completes the proof. □

We close this subsection by providing an axiomatic system for $\vdash_{\mathcal{MV}}$, the multiset companion of infinite-valued Łukasiewicz logic $\vdash_{\mathrm{L}}$. Although we show that it has no single-conclusion axiomatisation, we also axiomatise its single-conclusion fragment. In this way, we incidentally provide an example of two different MDR's with the same single-conclusion fragment.

Proposition 82. *There is no single-conclusion presentation of* $\vdash_{\mathcal{MV}}$.

Proof. Assume that there is such a system Ax and note that we have $[p \otimes q] \vdash_{\mathsf{Ax}} [p, q]$. Then, due to Lemma 79, either $\vdash_{\mathsf{Ax}} p$ or $\vdash_{\mathsf{Ax}} q$, a contradiction. □

Definition 83. *The axiomatic system* MV^s*, formulated in the language* $\mathcal{L}_0$ *of commutative residuated lattices, contains as axioms all instances of the axioms of Łukasiewicz logic in* $\mathcal{L}_0$*, and as its sole deduction rule the rule (MP):* $[\varphi, \varphi \to \psi] \rhd [\psi]$*. The axiomatic system* MV *is an extension of* MV^s *by the rule (*$\otimes$*-Elim):* $[\varphi \otimes \psi] \rhd [\varphi, \psi]$.

Observe that MV^s is a single-conclusion axiomatic system.

Theorem 84. *Let* $\Gamma, \Delta, [\varphi] \in Fm^\flat_{\mathcal{L}_0}$*. Then*

- $\Gamma \vdash_{\mathsf{MV}^s} [\varphi]$ *iff* $\Gamma \vdash_{\mathcal{MV}} [\varphi]$.
- $\Gamma \vdash_{\mathsf{MV}} \Delta$ *iff* $\Gamma \vdash_{\mathcal{MV}} \Delta$.

Proof. Recall, that given a multiset of $\mathcal{L}_0$-formulas $\Gamma = [\gamma_1, \ldots, \gamma_n]$, we write $\bigotimes \Gamma$ for $\gamma_1 \otimes \cdots \otimes \gamma_n$. Note that due to the standard completeness of $\vdash_{\mathrm{L}}$ we obtain:

$$\Gamma \vdash_{\mathcal{MV}} \Delta \iff \mathcal{MV} \models \bigotimes \Gamma \leq \bigotimes \Delta \iff \vdash_{\mathrm{L}} \bigotimes \Gamma \to \bigotimes \Delta. \tag{8}$$

For the left-to-right direction of the former claim, it suffices to prove the latter. Assume that $\Gamma = \Gamma_1, \ldots, \Gamma_n \geq \Delta$ is a derivation of Δ from Γ. If we show that $\mathcal{MV} \models \bigotimes \Gamma_i \leq \bigotimes \Gamma_{i+1}$ the claim follows as $\mathcal{MV} \models \bigotimes \Gamma_n \leq \bigotimes \Delta$. We distinguish three cases:

- The case when $\Gamma_{i+1} = \Gamma_i \uplus [\varphi]$, where φ is an axiom, is simple, as in this case we have $\mathcal{MV} \models \varphi \approx 1$.
- The case when there is a multiset Δ such that $\Gamma_i = \Delta \uplus [\varphi, \varphi \to \psi]$ and $\Gamma_{i+1} = \Delta \uplus [\psi]$: we know that $\mathcal{MV} \models \varphi \otimes (\varphi \to \psi) \leq \psi$ and so $\mathcal{MV} \models \varphi \otimes (\varphi \to \psi) \otimes \bigotimes \Delta \leq \psi \otimes \bigotimes \Delta$.
- The final case, when there is a multiset Δ such that $\Gamma_i = \Delta \uplus [\varphi \otimes \psi]$ and $\Gamma_{i+1} = \Delta \uplus [\varphi, \psi]$, is simple.

For the converse direction, we first prove the first claim by induction on the length of Γ. Note that thanks to Lemma 79 we can work with tree-proofs. If $\Gamma = \emptyset$, then by the assumption we know that φ is a theorem of Łukasiewicz logic, i.e., there is a proof of φ in the usual Hilbert calculus of Łukasiewicz logic. This proof can be easily transformed into a tree-proof of φ in $\vdash_{\mathsf{MV}^s}$. For the induction step, observe that if $\Gamma \uplus [\psi] \vdash_{\mathcal{MV}} [\varphi]$, then $\Gamma \vdash_{\mathcal{MV}} [\psi \to \varphi]$ and so by induction there is a tree-proof of $\psi \to \varphi$ from Γ from which it is trivial to get a tree-proof of φ from $\Gamma \uplus [\psi]$. Finally, we prove the right-to-left direction of the latter claim. First notice that $\Gamma \vdash_{\mathcal{MV}} \Delta$ entails $\Gamma \vdash_{\mathcal{MV}} \bigotimes \Delta$ and so by the first claim $\Gamma \vdash_{\mathsf{MV}^s} \bigotimes \Delta$ and thus also $\Gamma \vdash_{\mathsf{MV}} \bigotimes \Delta$. Repeated use of the rule ($\otimes$-Elim) completes the proof. □

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