A LOGICAL AND ALGEBRAIC CHARACTERIZATION OF ADJUNCTIONS BETWEEN GENERALIZED QUASI-VARIETIES

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Abstract. We present a logical and algebraic description of right adjoint functors between generalized quasi-varieties, inspired by the work of McKenzie on category equivalence. This result is achieved by developing a correspondence between the concept of adjunction and a new notion of translation between relative equational consequences.

The aim of the paper is to describe a logical and algebraic characterization of adjunctions between generalized quasi-varieties. This characterization is achieved by developing a correspondence between the concept of adjunction and a new notion of translation, called *contextual translation*, between equational consequences relative to classes of algebras. More precisely, given two generalized quasi-varieties K and K', every contextual translation of the equational consequence relative to K into the one relative to K' corresponds to a right adjoint functor from K' to K and vice-versa (Theorems 4.5 and 5.3). In a slogan, contextual translations between relative equational consequences are the *duals* of right adjoint functors. Examples of this correspondence between right adjoint functors and translations abound in the literature, e.g., Gödel's translation of intuitionistic logic into the modal system $\mathcal{S}4$ corresponds to the functor that extracts the Heyting algebra of open elements from an interior algebra (Examples 4.3 and 4.6), and Kolmogorov's translation of classical logic into intuitionistic logic corresponds to the functor that extracts the Boolean algebra of regular elements out of a Heyting algebra (Examples 4.4 and 4.6).

The algebraic aspect of our characterization of adjunctions is inspired by the work of McKenzie on category equivalences [25]. Roughly speaking, McKenzie discovered a combinatorial description of category equivalence between prevarieties of algebras. In particular, he showed that if two prevarieties K and K' are categorically equivalent, then we can transform K into K' by applying two kinds of deformations to K. The first of these deformations is the *matrix power* construction. The matrix power with exponent $n \in \omega$ of an algebra $\boldsymbol{A}$ is a new algebra $\boldsymbol{A}^{[n]}$ with universe A^n and whose basic m-ary operations are all n-sequences of $(m \times n)$-ary term functions of $\boldsymbol{A}$, which are applied component-wise. The other basic deformation is defined as follows. Suppose that σ is a unary term. Then, given an algebra $\boldsymbol{A}$, we let $\boldsymbol{A}(\sigma)$ be the algebra whose universe is the range of the term function $\sigma^{\boldsymbol{A}} \colon A \to A$ and whose m-ary operations are the restrictions of the term functions of $\boldsymbol{A}$ of the form σt, where t is an m-ary term of $\boldsymbol{A}$, to $\sigma[A]$. McKenzie's work shows that the prevarieties categorically equivalent to K are exactly the ones obtained by deforming K by means of a matrix power and the

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σ construction, where σ is a unary *idempotent and invertible* term (here Theorem 3.14). For the relevant definitions of idempotence and invertibility, see Example 3.11. This algebraic approach to the study of category equivalence has been reformulated in categorical terms for example in [28, 29] and has an antecedent in [12].

Building on McKenzie's work and on the theory of *locally presentable* categories [3], we show that every right adjoint functor between generalized quasi-varieties (which are particular kinds of prevarieties) can be decomposed into a combination of two deformations that generalize the ones devised in the special case of category equivalence. These deformations are matrix powers with (possibly) *infinite* exponent and the following generalization of the σ construction. Given an algebra $\boldsymbol{A}$, we say that a set of equations θ in a single variable is *compatible* with a sublanguage $\mathscr{L}$ of the language of $\boldsymbol{A}$ if the set of solutions of θ in $\boldsymbol{A}$ is closed under the restriction of the operations in $\mathscr{L}$. In this case we let $\theta_{\mathscr{L}}(\boldsymbol{A})$ be the algebra obtained by equipping the set of solutions of θ in $\boldsymbol{A}$ with the restriction of the operations in $\mathscr{L}$. The main result of the paper shows that every right adjoint functor between generalized quasi-varieties is, up to a natural isomorphism, a composition of the matrix power construction and the generalized σ construction (Theorem 6.1). Moreover, every functor obtained as a composition of these deformations is indeed a right adjoint. This result can be seen as a purely algebraic formulation of the classical description of adjunctions in categories with a free object, which can be traced back at least to [14] (Remark 6.3).

1. Algebraic preliminaries

For information on standard notions of universal algebra we refer the reader to [6, 9, 26]. We begin with some remarks on notation. Given an algebraic language $\mathscr{L}$ and a set X, we denote the set of terms over $\mathscr{L}$ built up with the variables in X by $Tm(\mathscr{L}, X)$, and the corresponding absolutely free algebra by $\boldsymbol{Tm}(\mathscr{L}, X)$. We also denote the set of equations built up from X by $Eq(\mathscr{L}, X)$. Formally speaking, equations are pairs of terms, i.e., $Eq(\mathscr{L}, X) \coloneqq Tm(\mathscr{L}, X) \times Tm(\mathscr{L}, X)$. When the language $\mathscr{L}$ is clear from the context, we simply write $Tm(X)$, $Eq(X)$ and $\boldsymbol{Tm}(X)$. Since every cardinal κ is a set, sometimes we write $Tm(\mathscr{L}, \kappa)$ to stress the cardinality of the set of variables. The same convention applies to equations and term algebras. Given two cardinals κ and λ, we denote their Cartesian product by $\kappa \times \lambda$, not to be confused with their product as cardinals. We denote the set of natural numbers by ω.

We denote the class operators of isomorphism, homomorphic images, subalgebras, direct products, (isomorphic copies of) subdirect products and ultraproducts respectively by $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\text{SD}}$ and $\mathbb{P}_{\text{U}}$. We assume that product-style class operators admit empty set of indexes and give a trivial algebra as a result. We denote algebras by bold capital letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, etc. (with universes A, B, C, etc.). Given a class of algebras K, we denote its language by $\mathscr{L}_{\mathsf{K}}$.

Given an algebraic language $\mathscr{L}$, a *generalized quasi-equation* is an expression Φ of the form

$$\Phi = (\bigwedge_{i \in I} \alpha_i \approx \beta_i) \to \varphi \approx \psi$$

where I is a possibly infinite *set* and $\alpha_i \approx \beta_i$ and $\varphi \approx \psi$ are equations. A *quasi-equation* is a generalized quasi-equation in which the set I is finite. Given an algebra $\boldsymbol{A}$, we say that a generalized quasi-equation Φ holds in $\boldsymbol{A}$, in symbols $\boldsymbol{A} \vDash \Phi$, if for every assignment $\vec{a} \in A$ we have that

$$\text{if } \alpha_i^{\boldsymbol{A}}(\vec{a}) = \beta_i^{\boldsymbol{A}}(\vec{a}) \text{ for all } i \in I, \text{ then } \varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}).$$

A *prevariety* is a class of algebras axiomatized by (a class of) arbitrary generalized quasi-equations or, equivalently, a class closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. A *generalized quasi-variety* is a class of algebras axiomatized by (a set of) generalized quasi-equations whose number of variables is bounded by some infinite cardinal. These can be equivalently characterized [7] as the classes of algebras closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{U}_\kappa$ (for some infinite cardinal κ), where for every class of algebras K,

$$\mathbb{U}_\kappa(\mathsf{K}) \coloneqq \{\boldsymbol{A} : \boldsymbol{B} \in \mathsf{K} \text{ for every } \kappa\text{-generated subalgebra } \boldsymbol{B} \leqslant \boldsymbol{A}\}.$$

It is well known that a *quasi-variety* is a class of algebras axiomatized by quasi-equations or, equivalently, a class closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_{\text{U}}$. A *variety* is a class of algebras axiomatized by equations or, equivalently, closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$. Given a class of algebras K, we will denote by $\mathbb{GQ}_\kappa(\mathsf{K})$ the models of the generalized quasi-equations in κ-many variables that hold in K and respectively by $\mathbb{Q}(\mathsf{K})$ and $\mathbb{V}(\mathsf{K})$ the quasi-variety and the variety generated by K. It is well known that

$$\mathbb{GQ}_\kappa(\mathsf{K}) = \mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K}) \quad \mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\text{U}}(\mathsf{K}) \quad \mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K}).$$

It is worth remarking that both the existence and the non-existence of a prevariety that is not a generalized quasi-variety are consistent (relative to large cardinals) with von Neumann-Bernays-Gödel class theory (NGB) with the Axiom of Choice. In fact in NBG the assumption that every prevariety is a generalized quasi-variety is equivalent to the *Vopěnka Principle*, which states that every class of pairwise non-embeddable models of a first-order theory is a set [1] (see also [16, Proposition 2.3.18]).

Given a class of algebras K and a set $X \neq \emptyset$, we denote by $\boldsymbol{Tm}_{\mathsf{K}}(X)$ the free algebra in K with free generators X. In general the free algebra $\boldsymbol{Tm}_{\mathsf{K}}(X)$ is constructed as a quotient of the term algebra $\boldsymbol{Tm}(X)$ and its elements are congruence classes of terms equivalent in K. Sometimes we identify the universe of $\boldsymbol{Tm}_{\mathsf{K}}(X)$ with a set of its representatives, i.e., with a set of terms in variables X. It is well known that $\boldsymbol{Tm}_{\mathsf{K}}(X) \in \mathbb{ISP}(\mathsf{K})$. Thus prevarieties contain free algebras with arbitrary large sets of free generators. Prevarieties contain also trivial algebras, which we denote generically by $\mathbf{1}$. Given a class of algebras K, we denote by K_{si} the collection of its subdirectly irreducible members. It is well known that if K is a variety, then $\mathsf{K} = \mathbb{P}_{\text{SD}}(\mathsf{K}_{si})$.

Given a class of algebras K and an algebra $\boldsymbol{A}$, we say that a congruence θ of $\boldsymbol{A}$ is a K-*congruence* if $\boldsymbol{A}/\theta \in \mathsf{K}$, and denote the collection of K-congruences by $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$. In particular, we will denote by $\pi_\theta \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ the canonical map on the quotient and by $0_{\boldsymbol{A}}$ and $1_{\boldsymbol{A}}$ the identity and total congruence of $\boldsymbol{A}$. If K is a prevariety, then $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ forms a closure system when ordered under the inclusion relation. Accordingly, we denote by $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}$ the closure operator of generation of K-congruences. While speaking of congruence generation in the absolute sense, we will simply write $\mathrm{Cg}^{\boldsymbol{A}}$.

Given a class of algebras K and $\Phi \cup \{\varepsilon \approx \delta\} \subseteq Eq(X)$, we define

$$\Phi \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } h\colon \boldsymbol{Tm}(X) \to \boldsymbol{A}$$
$$\text{if } h\varphi = h\psi \text{ for every } \varphi \approx \psi \in \Phi, \text{ then } h\varepsilon = h\delta.$$

The relation $\vDash_{\mathsf{K}}$ is called the *equational consequence relative to* K. The function $\mathsf{C}_{\mathsf{K}}\colon \mathcal{P}(Eq(X)) \to \mathcal{P}(Eq(X))$ defined by the rule

$$\mathsf{C}_{\mathsf{K}}(\Phi) := \{\varepsilon \approx \delta : \Phi \vDash_{\mathsf{K}} \varepsilon \approx \delta\}, \text{ for every } \Phi \subseteq Eq(X)$$

is a closure operator over $Eq(X)$. It is easy to see that the validity of generalized quasi-equations in K corresponds to the validity of deductions in the equational consequence relative to K in the sense that

$$\mathsf{K} \vDash (\bigwedge_{i \in I} \varphi_i \approx \psi_i) \to \varepsilon \approx \delta \iff \{\varphi_i \approx \psi_i : i \in I\} \vDash_{\mathsf{K}} \varepsilon \approx \delta.$$

This is reflected in the fact that if K is a prevariety, then the set of fixed points of $\mathsf{C}_{\mathsf{K}}\colon \mathcal{P}(Eq(X)) \to \mathcal{P}(Eq(X))$ coincides with $\mathrm{Con}_{\mathsf{K}}\boldsymbol{Tm}(X)$. Now let K be a quasi-variety and $\boldsymbol{A}$ an arbitrary algebra. The lattice $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is algebraic and its compact elements $\mathrm{Comp}_{\mathsf{K}}\boldsymbol{A}$ are the finitely generated K-congruences. In particular, the closure operator $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}$ is finitary. An algebra $\boldsymbol{A} \in \mathsf{K}$ is K-*finitely presentable* if there is some $n \in \omega$ and some finitely generated K-congruence θ of $\boldsymbol{Tm}_{\mathsf{K}}(n)$ such that $\boldsymbol{A}$ is isomorphic to $\boldsymbol{Tm}_{\mathsf{K}}(n)/\theta$.

2. Categorical preliminaries

For standard information on category theory we refer the reader to [2, 5, 22], while for categorical universal algebra see [3, 4]. For the sake of simplicity, we chose to organize this section in two three that deal with different but related topics. The reader familiar with basic categorical universal algebra and locally presentable categories may safely chose to skip to the next section.

2.1. **Generalized quasi-varieties.** In this part we explain that prevarieties seen as categories (with algebras as objects and homomorphisms as arrows) are *bicomplete*, i.e., they have small limits and small colimits. To this end, we will conform to the following convention: an algebraic language $\mathscr{L}$ admits *empty* models if and only if $\mathscr{L}$ does not contain constant symbols. In particular, a prevariety K contains the empty algebra if and only if its language does not contain a constant symbol. This convention ensures the existence of the 0-generated free algebra over K.

We are now ready to describe the structure of limits and colimits in prevarieties. Let K be a prevariety. Categorical *products* in K coincide with direct products. Moreover, given a parallel pair of arrows $f, g\colon \boldsymbol{A} \rightrightarrows \boldsymbol{B}$ in K, we have that $h\colon \boldsymbol{C} \to \boldsymbol{A}$ is an *equalizer* of f and g if and only if h is an embedding and $h[C] = \{a \in A : f(a) = g(a)\}$. Observe that if the language of K does not contain constant symbols, then $\boldsymbol{C}$ may be empty. It is well known that all other limits can be obtained as a combination of these two constructions. The description of colimits is slightly more complicated. Consider a family of algebras $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}$ and assume without loss of generality that their universes are pairwise disjoint. For every $i \in I$ we let $\pi_i\colon \boldsymbol{Tm}_{\mathsf{K}}(A_i) \to \boldsymbol{A}_i$ be the unique surjective homomorphism that maps identically A_i onto A_i. Then consider the set $X := \bigcup_{i \in I} A_i$ and define

$$\theta := \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{Tm}_{\mathsf{K}}(X)} \bigcup_{i \in I} \mathrm{Ker}(\pi_i).$$

The algebra $\boldsymbol{Tm}_{\mathsf{K}}(X)/\theta$, together with the maps $p_i \colon \boldsymbol{A}_i \to \boldsymbol{Tm}_{\mathsf{K}}(X)/\theta$ that send $a \in A_i$ to a/θ, is a *coproduct* of $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}$. Observe that the maps p_i need not be injective. Moreover, it is worth remarking that the free κ-generated algebra is a κ-th copower of the free 1-generated algebra. In general, if $\boldsymbol{A}$ is a coproduct in K of the family $\{\boldsymbol{A}_i : i \in I\}$ and $f_i \colon \boldsymbol{A}_i \to \boldsymbol{B}$ with $i \in I$ are arrows in K, then we will denote by $\langle f_i : i \in I\rangle \colon \boldsymbol{A} \to \boldsymbol{B}$ the map induced by the universal property of the coproduct. In case $\boldsymbol{A} = \boldsymbol{Tm}_{\mathsf{K}}(X)/\theta$ as above, the arrow $\langle f_i : i \in I\rangle$ is defined by the rule

$$\varphi(a_1, \dots, a_n)/\theta \longmapsto \varphi^{\boldsymbol{B}}(f_{k_1}(a_1), \dots, f_{k_n}(a_n))$$

for every $\varphi(a_1, \dots, a_n)/\theta \in Tm_{\mathsf{K}}(X)/\theta$ with $a_1 \in A_{k_1}, \dots, a_n \in A_{k_n}$.

Now, we move our attention to the other basic kind of colimit. Given a parallel pair of arrows $f, g \colon \boldsymbol{A} \rightrightarrows \boldsymbol{B}$ in K, we have that $h \colon \boldsymbol{B} \to \boldsymbol{C}$ is a *coequalizer* of f and g if and only if it is surjective and

$$\mathrm{Ker}(h) = \mathrm{Cg}^{\boldsymbol{B}}_{\mathsf{K}}\{\langle f(a), g(a)\rangle : a \in A\}.$$

It is worth remarking that every surjective homomorphism in K arises as the coequalizer of a pair of arrows. In particular, observe that every congruence $\theta \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ of $\boldsymbol{A} \in \mathsf{K}$ can be seen as a subalgebra of the direct product $\boldsymbol{A} \times \boldsymbol{A}$. Keeping this in mind, θ can be associated with two homomorphisms $l, r \colon \theta \rightrightarrows \boldsymbol{A}$ that send a pair $\langle a, b\rangle \in \theta$ respectively to its left and right component. It is easy to prove that π_θ is a coequalizer of l and r. Finally, it is well known that all other colimits can be constructed as a combination of coproducts and coequalizers.

Observe that the *terminal object* of K is the trivial algebra, while its *initial object* is $\boldsymbol{Tm}_{\mathsf{K}}(0)$. Therefore the initial object of K is empty if and only if the language of K contains no constant symbols. Given two prevarieties X and Y, the functors $\mathcal{F} \colon \mathsf{X} \longleftrightarrow \mathsf{Y} \colon \mathcal{G}$, where $\mathcal{F}$ sends everything to the initial object and $\mathcal{G}$ sends every object to the terminal object, always form an adjunction $\mathcal{F} \dashv \mathcal{G}$ (see next subsection, if necessary). We call the adjunctions of this kind *trivial*. In particular, we say that a left (right) adjoint functor between prevarieties is *trivial* if it sends everything to the initial (terminal) object.

It is worth spending some more time on a special kind of colimit constructions. These are *κ-directed colimits*, i.e., colimits of diagrams indexed by posets in which every subset of cardinality $< \kappa$ has an upper bound, for a regular cardinal κ. In varieties they are constructed as usual, by taking the disjoint union of the factors and identifying the elements that become eventually equal. In the case of prevarieties K the only difference is that we have to factor out the resulting algebra by its smallest K-congruence. Remarkably, this last step can be avoided when K is a generalized quasi-variety that can be axiomatized by generalized quasi-equations, whose number of variables is less than κ. Then the κ-directed colimits of families of algebras in K are obtained by just identifying elements that become eventually equal. In particular, in quasi-varieties this is the case for usual $\aleph_0$-directed colimits.

2.2. **Adjunctions.** We will limit our discussion to *locally small categories*, i.e., categories whose hom-sets are ordinary sets. Recall that an *adjunction* between two categories X and Y is a tuple $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta\rangle$ where $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ and $\mathcal{G} \colon \mathsf{Y} \to \mathsf{X}$ are functors and $\eta \colon 1_{\mathsf{X}} \to \mathcal{G}\mathcal{F}$ and $\varepsilon \colon \mathcal{F}\mathcal{G} \to 1_{\mathsf{Y}}$ are natural transformations such that

$$1_{\mathcal{F}(\boldsymbol{A})} = \varepsilon_{\mathcal{F}(\boldsymbol{A})} \circ \mathcal{F}(\eta_{\boldsymbol{A}}) \;\text{ and }\; 1_{\mathcal{G}(\boldsymbol{B})} = \mathcal{G}(\varepsilon_{\boldsymbol{B}}) \circ \eta_{\mathcal{G}(\boldsymbol{B})}$$

for every $\boldsymbol{A} \in \mathsf{X}$ and $\boldsymbol{B} \in \mathsf{Y}$. In this case we say that $\mathcal{F}$ is *left adjoint* to $\mathcal{G}$ and that $\mathcal{G}$ is *right adjoint* to $\mathcal{F}$, in symbols $\mathcal{F} \dashv \mathcal{G}$. Moreover, η and ε are respectively the *unit* and *counit* of the adjunction. We say that a functor is a *right adjoint* (resp a *left adjoint*) if it is right (resp. left) adjoint to some functor. It is worth to remark that if a functor has two right (left) adjoints, these are naturally isomorphic. Right adjoint functors preserve limits and left adjoint functors preserve colimits. A *category equivalence* between two categories X and Y is an adjunction $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ where ε and η are natural isomorphisms. In this case the functors $\mathcal{F}$ and $\mathcal{G}$ preserve all categorical constructions. We say that two categories are *categorically equivalent* when there exists a category equivalence between them.

A *hom-set adjunction* between two categories X and Y is a triple $\langle \mathcal{F}, \mathcal{G}, \mu \rangle$ where $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ and $\mathcal{G} \colon \mathsf{Y} \to \mathsf{X}$ are functors and μ is a natural isomorphism between the functors:

$$\hom_{\mathsf{Y}}(\mathcal{F}(\cdot), \cdot) \colon \mathsf{X}^{op} \times \mathsf{Y} \to \mathsf{Set} \;\text{ and }\; \hom_{\mathsf{X}}(\cdot, \mathcal{G}(\cdot)) \colon \mathsf{X}^{op} \times \mathsf{Y} \to \mathsf{Set}.$$

When $\langle \mathcal{F}, \mathcal{G}, \mu \rangle$ is a hom-set adjunction as above, we say that $\mathcal{F}$ is *left adjoint* to $\mathcal{G}$ and that $\mathcal{G}$ is *right adjoint* to $\mathcal{F}$, in symbols $\mathcal{F} \dashv \mathcal{G}$.

Adjunctions and hom-set adjunctions are two sides of the same coin. To explain why, consider an adjunction $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ between X and Y with $\mathcal{F} \dashv \mathcal{G}$. Then for every $\langle \boldsymbol{A}, \boldsymbol{B} \rangle \in \mathsf{X}^{op} \times \mathsf{Y}$ we let

$$\gamma_{\langle \boldsymbol{A}, \boldsymbol{B} \rangle} \colon \hom_{\mathsf{Y}}(\mathcal{F}(\boldsymbol{A}), \boldsymbol{B}) \to \hom_{\mathsf{X}}(\boldsymbol{A}, \mathcal{G}(\boldsymbol{B}))$$

be the map that sends an arrow f to $\mathcal{G}(f) \circ \eta_{\boldsymbol{A}}$. It turns out that the global map

$$\gamma \colon \hom_{\mathsf{Y}}(\mathcal{F}(\cdot), \cdot) \to \hom_{\mathsf{X}}(\cdot, \mathcal{G}(\cdot))$$

is a natural isomorphism. Thus the triple $\langle \mathcal{F}, \mathcal{G}, \gamma \rangle$ is a hom-set adjunction between X and Y with $\mathcal{F} \dashv \mathcal{G}$. Vice-versa consider a hom-set adjunction $\langle \mathcal{F}, \mathcal{G}, \gamma \rangle$ between X and Y with $\mathcal{F} \dashv \mathcal{G}$. For every $\boldsymbol{A} \in \mathsf{X}$ and $\boldsymbol{B} \in \mathsf{Y}$ we define $\eta_{\boldsymbol{A}} \coloneqq \gamma_{\langle \boldsymbol{A}, \mathcal{F}(\boldsymbol{A}) \rangle}(1_{\mathcal{F}(\boldsymbol{A})})$ and $\varepsilon_{\boldsymbol{B}} \coloneqq \gamma^{-1}_{\langle \mathcal{G}(\boldsymbol{B}), \boldsymbol{B} \rangle}(1_{\mathcal{G}(\boldsymbol{B})})$. It turns out that the global maps $\eta \colon 1_{\mathsf{X}} \to \mathcal{G}\mathcal{F}$ and $\varepsilon \colon \mathcal{F}\mathcal{G} \to 1_{\mathsf{Y}}$ are natural transformations and that $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ is an adjunction with $\mathcal{F} \dashv \mathcal{G}$. Keeping this in mind we can speak of the hom-set adjunction associated with an adjunction and vice-versa. This justifies the usage of the same symbol $\dashv$.

2.3. **Locally presentable categories.** For standard information on locally presentable categories we refer the reader to [3]. Let κ be a regular cardinal and let K be a locally small category. An object $\boldsymbol{A}$ in K is *κ-presentable* if the functor $\hom(\boldsymbol{A}, \cdot)$ preserves κ-directed colimits. More explicitly, this means that for every κ-directed diagram $\{\boldsymbol{B}_i : i \in I\}$ with colimit $g_i \colon \boldsymbol{B}_i \to \boldsymbol{B}$ and for every arrow $h \colon \boldsymbol{A} \to \boldsymbol{B}$ the following conditions hold:

1. There is $i \in I$ and an arrow $p \colon \boldsymbol{A} \to \boldsymbol{B}_i$ such that $g_i \circ p = h$.
2. The map p is essentially unique, in the sense that for every other arrow $q \colon \boldsymbol{A} \to \boldsymbol{B}_m$ such that $g_m \circ q = h$ there is $j \geqslant i, m$ such that $f_{ij} \circ p = f_{mj} \circ q$, where $f_{ij} \colon \boldsymbol{B}_i \to \boldsymbol{B}_j$ and $f_{mj} \colon \boldsymbol{B}_m \to \boldsymbol{B}_j$ are arrows of the κ-directed diagram.

In generalized quasi-varieties the κ-presentable objects can be described as follows:

Lemma 2.1. *Let κ be a regular cardinal and K be a generalized quasi-variety axiomatized by generalized quasi-equations in less than κ variables. An algebra $\boldsymbol{A} \in \mathsf{K}$ is κ-presentable in the categorical sense if and only if it is (isomorphic to) a quotient of $\boldsymbol{Tm}_{\mathsf{K}}(\lambda)$ under a μ-generated K-congruence for some $\lambda, \mu < \kappa$.*

Let κ be a regular cardinal and K be a locally small category. K is *locally κ-presentable* if it is cocomplete, and has a *set* J of κ-presentable objects such that every object in K is a κ-directed colimit of objects in J. Moreover, K is *locally presentable* if it is locally κ-presentable for some regular cardinal κ.

Lemma 2.2. *Generalized quasi-varieties are locally presentable categories.*

Adámek and Rosický proved in [3, Theorem 1.66] the following characterization of right adjoint functors between locally presentable categories. By Lemma 2.2 it applies to generalized quasi-varieties as well.

Theorem 2.3 (Adámek and Rosický)**.** *A functor between locally presentable categories is right adjoint if and only if it preserves limits and κ-directed colimits for some regular cardinal κ.*

3. The two basic deformations

In this section we describe two general methods to deform a given generalized quasi-variety, obtaining a new generalized quasi-variety that is related to the first one by an adjunction. In particular, it turns out that every right adjoint between generalized quasi-varieties arises (up to natural isomorphism) as a combination of these deformations (Theorem 6.1). Remarkably, in the particular case of category equivalence, these deformations coincide with the ones identified by McKenzie in [25] (see Examples 3.6 and 3.11).

The first deformation that we consider is just an infinite version of the usual *finite matrix power* construction. Let X be a class of similar algebras and κ be a cardinal. Then observe that every term $\varphi \in Tm(\kappa)$ induces a map $\varphi\colon A^\kappa \to A$ for every $\boldsymbol{A} \in \mathsf{X}$.

Definition 3.1. Let $\kappa > 0$ be a cardinal and X a class of similar algebras. Then $\mathscr{L}^\kappa_{\mathsf{X}}$ is the algebraic language whose n-ary operations (for every $n \in \omega$) are all κ-sequences $\langle t_i : i < \kappa\rangle$ of terms t_i of the language of X built up with variables

$$\{x^j_m : 1 \leqslant m \leqslant n \text{ and } j < \kappa\}.$$

Observe that each t_i has a finite number of variables, possibly none, of each sequence $\vec{x}_m \coloneqq \langle x^j_m : j < \kappa\rangle$ with $1 \leqslant m \leqslant n$. We will write $t_i = t_i(\vec{x}_1, \dots, \vec{x}_n)$ to denote this fact.

Example 3.2. Consider the variety of bounded distributive lattices DL_{01}. Examples of basic operations of $\mathscr{L}^2_{\mathsf{DL}_{01}}$ are:

$$\begin{aligned}
\langle x^1, x^2\rangle \sqcap \langle y^1, y^2\rangle &\coloneqq \langle x^1 \wedge y^1, x^2 \vee y^2\rangle\\
\langle x^1, x^2\rangle \sqcup \langle y^1, y^2\rangle &\coloneqq \langle x^1 \vee y^1, x^2 \wedge y^2\rangle\\
\neg\langle x^1, x^2\rangle &\coloneqq \langle x^2, x^1\rangle\\
1 &\coloneqq \langle 1, 0\rangle\\
0 &\coloneqq \langle 0, 1\rangle.
\end{aligned}$$

Observe that the first two operations are binary, the third is unary and the last two are constants. ⊠

Definition 3.3. Consider an algebra $\boldsymbol{A} \in \mathsf{X}$ and a cardinal $\kappa > 0$. We let $\boldsymbol{A}^{[\kappa]}$ be the algebra of type $\mathscr{L}_{\mathsf{X}}^{\kappa}$ with universe A^{κ} where a n-ary operation $\langle t_i : i < \kappa \rangle$ is interpreted as

$$\langle t_i : i < \kappa \rangle(a_1, \dots, a_n) = \langle t_i^{\boldsymbol{A}}(a_1/\vec{x}_1, \dots, a_n/\vec{x}_n) : i < \kappa \rangle$$

for every $a_1, \dots, a_n \in A^{\kappa}$ (the notation $a_m/\vec{x}_m$ means that we are assigning the tuple a_m of elements of A to the tuple of variables $\vec{x}_m$). In other words $\langle t_i : i < \kappa \rangle(a_1, \dots, a_n)$ is the κ-sequence of elements of A defined as follows. Consider $i < \kappa$. Observe that only a finite number of variables occurs in t_i, say

$$t_i = t_i(x_1^{\alpha_1^1}, \dots, x_1^{\alpha_{m_1}^1}, \dots, x_n^{\alpha_1^n}, \dots, x_n^{\alpha_{m_n}^n}),$$

where $\alpha_1^1, \dots, \alpha_{m_1}^1, \dots, \alpha_1^n, \dots, \alpha_{m_n}^n < \kappa$. Then the i-th component of the sequence $\langle t_i : i < \kappa \rangle(a_1, \dots, a_n)$ is

$$t_i^{\boldsymbol{A}}(a_1(\alpha_1^1), \dots, a_1(\alpha_{m_1}^1), \dots, a_n(\alpha_1^n), \dots, a_1(\alpha_{m_n}^n)).$$

If X is a class of similar algebras, we set

$$\mathsf{X}^{[\kappa]} := \mathbb{I}\{\boldsymbol{A}^{[\kappa]} : \boldsymbol{A} \in \mathsf{X}\}$$

and call it the κ-th *matrix power* of X.

Now, let $[\kappa]$ be the map defined as follows:

$$\boldsymbol{A} \longmapsto \boldsymbol{A}^{[\kappa]}$$
$$f \colon \boldsymbol{A} \to \boldsymbol{B} \longmapsto f^{[\kappa]} \colon \boldsymbol{A}^{[\kappa]} \to \boldsymbol{B}^{[\kappa]}$$

where $f^{[\kappa]}\langle a_i : i < \kappa \rangle := \langle f(a_i) : i < \kappa \rangle$, for every $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{X}$ and every homomorphism f. It is easy to check that the map $f^{[\kappa]} \colon \boldsymbol{A}^{[\kappa]} \to \boldsymbol{B}^{[\kappa]}$ is indeed a homomorphism.

Example 3.4. In Example 3.2 we highlighted some operations of $\mathscr{L}_{\mathsf{DL}_{01}}^{2}$. Let us explain how are they interpreted in the matrix power construction. Consider $\boldsymbol{A} \in \mathsf{DL}_{01}$. The universe of $\boldsymbol{A}^{[2]}$ is just the Cartesian product $A \times A$. We have that:

$$\begin{aligned}
\langle a, b \rangle \sqcap \langle c, d \rangle &= \langle a \wedge c, b \vee d \rangle \\
\langle a, b \rangle \sqcup \langle c, d \rangle &= \langle a \vee c, b \wedge d \rangle \\
\neg \langle a, b \rangle &= \langle b, a \rangle \\
1 &= \langle 1^{\boldsymbol{A}}, 0^{\boldsymbol{A}} \rangle \\
0 &= \langle 0^{\boldsymbol{A}}, 1^{\boldsymbol{A}} \rangle
\end{aligned}$$

for every $\langle a, b \rangle, \langle c, d \rangle \in A \times A$. Examples of matrix powers with infinite exponent are technically, but not conceptually, more involved. We review one of them in Example 6.6. ⊠

Theorem 3.5. *Let X be a generalized quasi-variety and $\kappa > 0$ a cardinal. If Y is a generalized quasi-variety such that $\mathsf{X}^{[\kappa]} \subseteq \mathsf{Y}$, then $[\kappa] \colon \mathsf{X} \to \mathsf{Y}$ is a right adjoint functor.*

Proof. It is not difficult to see that the map $[\kappa]$ is a functor that preserves direct products and equalizers. Since all limits can be constructed as combination of products and equalizers, we conclude that $[\kappa]$ preserves limits. In view of Theorem 2.3 it only remains to show that it preserves λ-directed colimits for some regular cardinal λ. To this end, let λ be a regular cardinal larger than the number of variables occurring in the generalized quasi-equations axiomatizing X and Y. This makes sense, since X and Y are generalized quasi-varieties. Moreover, take λ to be larger than κ. Then consider a λ-directed diagram $\{\boldsymbol{A}_i : i \in I\}$ in X with arrows $f_{ij}\colon \boldsymbol{A}_i \to \boldsymbol{A}_j$ when $i \leqslant j$. By the first requirement on λ, the directed colimit of this diagram is the algebra $\boldsymbol{A}$ obtained as follows. First we consider the disjoint union $\{\langle a, i\rangle : a \in A_i \text{ and } i \in I\}$. Then we factor out by the quotient with respect to the following equivalence relation

$$\theta \coloneqq \{\langle\langle a, i\rangle, \langle b, j\rangle\rangle : \text{ there is } k \geqslant i, j \text{ such that } f_{ik}(a) = f_{jk}(b)\}$$

and define operations in the natural way. Analogously, the colimit in Y of the λ-directed diagram $\{\boldsymbol{A}_i^{[\kappa]} : i \in I\}$ with arrows $f_{ij}^{[\kappa]}\colon \boldsymbol{A}_i^{[\kappa]} \to \boldsymbol{A}_j^{[\kappa]}$ when $i \leqslant j$ is the algebra $\boldsymbol{B}$ obtained as follows. We first consider the disjoint union $\{\langle \vec{a}, i\rangle : \vec{a} \in A_i^{\kappa} \text{ and } i \in I\}$, then we factor it out by the equivalence relation

$$\phi \coloneqq \{\langle\langle \vec{a}, i\rangle, \langle \vec{b}, j\rangle\rangle : \text{ there is } k \geqslant i, j \text{ such that } f_{ik}^{[\kappa]}(\vec{a}) = f_{jk}^{[\kappa]}(\vec{b})\}$$

and finally we define operations in the natural way.

We claim that the map $g\colon \boldsymbol{B} \to \boldsymbol{A}^{[\kappa]}$ defined as

$$g(\langle \vec{a}, i\rangle/\phi) \coloneqq \langle\langle \vec{a}(r), i\rangle/\theta : r < \kappa\rangle$$

for every $\langle \vec{a}, i\rangle/\phi \in B$ is an isomorphism. It is very easy to see that g is well defined. To see that it is injective, we reason as follows. Suppose that $g(\langle \vec{a}, i\rangle/\phi) = g(\langle \vec{b}, j\rangle/\phi)$. This means that for every $r < \kappa$ there is $k_r \geqslant i, j$ such that $f_{ik_r}(\vec{a}(r)) = f_{jk_r}(\vec{b}(r))$. But since our diagram is λ-directed and $\kappa < \lambda$, there is $k \in I$ such that $k_r \leqslant k$ for every $r < \kappa$. In particular, this implies that $f_{ik}(\vec{a}(r)) = f_{jk}(\vec{b}(r))$ for every $r < \kappa$ and, therefore, that $f_{ik}^{[\kappa]}(\vec{a}) = f_{jk}^{[\kappa]}(\vec{b})$. This means that $\langle \vec{a}, i\rangle/\phi = \langle \vec{b}, j\rangle/\phi$, as desired.

Then we turn to show that g is surjective. Consider an element $\langle\langle a_r, i_r\rangle/\theta : r < \kappa\rangle \in \boldsymbol{A}^{[\kappa]}$. Again, since our diagram is λ-directed and $\kappa < \lambda$, there is $k \in I$ such that $k \geqslant i_r$ for every $r < \kappa$. In particular, this implies that $\langle\langle a_r, i_r\rangle/\theta : r < \kappa\rangle = \langle\langle f_{i_r k}(a_r), k\rangle/\theta : r < \kappa\rangle$. Now observe that the element $\vec{b} \coloneqq \langle f_{i_r k}(a_r) : r < \kappa\rangle$ belongs to $A_k^{[\kappa]}$. Moreover, we have that $g(\langle \vec{b}, k\rangle/\phi) = \langle\langle f_{i_r k}(a_r), k\rangle/\theta : r < \kappa\rangle$, as desired.

To complete the proof of the claim, it remains to show that g is a homomorphism. To this end, let φ be a basic n-ary operation of Y and consider $\langle \vec{a}_1, i_1\rangle/\phi, \dots, \langle \vec{a}_n, i_n\rangle/\phi \in B$. Consider an index $j \geqslant i_1, \dots, i_n$. Then for every $s < \kappa$, we have the following (where φ_s is the s-th component of φ):

$$\begin{aligned}
&\varphi^{\boldsymbol{A}^{[\kappa]}}(g\langle \vec{a}_1, i_1\rangle/\phi, \dots, g\langle \vec{a}_n, i_n\rangle/\phi)(s)\\
=&\varphi^{\boldsymbol{A}^{[\kappa]}}(\langle\langle \vec{a}_1(r), i_1\rangle/\theta : r < \kappa\rangle, \dots, \langle\langle \vec{a}_n(r), i_n\rangle/\theta : r < \kappa\rangle)(s)\\
=&\varphi^{\boldsymbol{A}^{[\kappa]}}(\langle\langle f_{i_1 j}(\vec{a}_1(r)), j\rangle/\theta : r < \kappa\rangle, \dots, \langle\langle f_{i_n j}(\vec{a}_n(r)), j\rangle/\theta : r < \kappa\rangle)(s)\\
=&\varphi_s^{\boldsymbol{A}}(\vec{x}_1/\langle\langle f_{i_1 j}(\vec{a}_1(r)), j\rangle/\theta : r < \kappa\rangle, \dots, \vec{x}_n/\langle\langle f_{i_n j}(\vec{a}_n(r)), j\rangle/\theta : r < \kappa\rangle)\\
=&\langle \varphi_s^{\boldsymbol{A}_j}(\vec{x}_1/f_{i_1 j}(\vec{a}_1), \dots, \vec{x}_n/f_{i_n j}(\vec{a}_n)), j\rangle/\theta
\end{aligned}$$

$$
\begin{aligned}
&= \langle \varphi^{\boldsymbol{A}_j^{[\kappa]}}(f_{i_1 j}^{[\kappa]}(\vec{a}_1), \dots, f_{i_n j}^{[\kappa]}(\vec{a}_n))(s), j\rangle/\theta\\
&= g(\langle \varphi^{\boldsymbol{A}_j^{[\kappa]}}(f_{i_1 j}^{[\kappa]}(\vec{a}_1), \dots, f_{i_n j}^{[\kappa]}(\vec{a}_n)), j\rangle/\phi)(s)\\
&= g(\varphi^{\boldsymbol{B}}(\langle f_{i_1 j}^{[\kappa]}(\vec{a}_1), j\rangle/\phi, \dots, \langle f_{i_n j}^{[\kappa]}(\vec{a}_n), j\rangle/\phi))(s)\\
&= g(\varphi^{\boldsymbol{B}}(\langle \vec{a}_1, i_1\rangle/\phi, \dots, \langle \vec{a}_n, i_n\rangle/\phi))(s).
\end{aligned}
$$

This concludes the proof of our claim.

To prove that $[\kappa]$ preserves λ-directed colimits, it only remains to show that $g \circ q_i = p_i^{[\kappa]}$ for every $i \in I$, where $q_i \colon \boldsymbol{A}_i^{[\kappa]} \to \boldsymbol{B}$ and $p_i \colon \boldsymbol{A}_i \to \boldsymbol{A}$ are the maps associated with the colimits $\boldsymbol{B}$ and $\boldsymbol{A}$ respectively. But this is an easy consequence of the fact that

$$q_i(\vec{a}) := \langle \vec{a}, i\rangle/\phi \text{ and } p_i(a) := \langle a, i\rangle/\theta$$

for every $\vec{a} \in A_i^{\kappa}$ and $a \in A_i$. Therefore we can apply Theorem 2.3, concluding that $[\kappa]$ is a right adjoint functor. ⊠

Example 3.6 (Finite Exponent)**.** It is not difficult to see that if X is a class of similar algebras and $\kappa > 0$, then the functor $[\kappa] \colon \mathsf{X} \to \mathsf{X}^{[\kappa]}$ is a category equivalence (see for example [25, Theorem 2.3.(i)] where this is stated under the assumption that κ is finite). Moreover, when κ is finite, it happens that if X is a prevariety (or a generalized quasi-variety, a quasi-variety, a variety), then so is $\mathsf{X}^{[\kappa]}$. However this is not the case in general: when κ is infinite it may happen that X is a prevariety and that $[\kappa] \colon \mathsf{X} \to \mathsf{Y}$ fails to be a category equivalence for every prevariety Y containing $\mathsf{X}^{[\kappa]}$. In particular, this implies that $\mathsf{X}^{[\kappa]}$ can fail to be a prevariety, even if X is one.

To construct the necessary counterexample, we reason as follows. First observe that if K is a prevariety, then an *infinite* algebra $\boldsymbol{A} \in \mathsf{K}$ has cardinality λ if and only if the following conditions hold:

1. The set $\mathrm{hom}(\boldsymbol{B}, \boldsymbol{A})$ has cardinality $\leqslant \lambda$ for every finitely generated algebra $\boldsymbol{B} \in \mathsf{K}$.
2. There is a finitely generated algebra $\boldsymbol{B} \in \mathsf{K}$ such that $\mathrm{hom}(\boldsymbol{B}, \boldsymbol{A})$ has exactly cardinality λ.

To see this, observe if $\boldsymbol{A}$ has infinite cardinality λ and $\boldsymbol{B}$ is n-generated, then the cardinality of $\mathrm{hom}(\boldsymbol{B}, \boldsymbol{A})$ is less or equal than $\lambda^n = \lambda$. Moreover there is a finitely generated algebra $\boldsymbol{B}$, e.g., the one-generated free algebra, such that $\mathrm{hom}(\boldsymbol{B}, \boldsymbol{A})$ has cardinality λ. This justifies the equivalence between having cardinality λ and satisfying conditions 1 and 2.

Together with the fact that the notion of a finitely generated algebra is categorical in prevarieties [25, Theorem 3.1.(5)] and that category equivalences preserve the cardinality of hom-sets, this implies that category equivalences preserve also infinite cardinalities.[1] We will use this fact to construct the desired counterexample. Consider a generalized quasi-variety X of finite type and an infinite cardinal κ. We know that the free algebra $\boldsymbol{Tm}_{\mathsf{X}}(\kappa)$ has cardinality κ and that its matrix power $\boldsymbol{Tm}_{\mathsf{X}}(\kappa)^{[\kappa]}$ has cardinality κ^{κ}. Since $\kappa < \kappa^{\kappa}$, we conclude that the functor $[\kappa]$ does

[1] This contrasts with the fact that category equivalences between prevarieties do not preserve the cardinality of *finite* algebras. Nevertheless, they preserve the fact of being *finite* [25, Theorem 3.1.(7)].

not preserve infinite cardinalities. Thus $[\kappa]\colon \mathsf{X} \to \mathsf{Y}$ is not a category equivalence, for every prevariety Y containing $\mathsf{X}^{[\kappa]}$. ⊠

In order to describe the second kind of deformation, we need to introduce a new concept:

Definition 3.7. Let X be a class of similar algebras and $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. A set of equations $\theta \subseteq Eq(\mathscr{L}_{\mathsf{X}}, 1)$ is *compatible* with $\mathscr{L}$ in X if for every n-ary operation $\varphi \in \mathscr{L}$ we have that

$$\theta(x_1) \cup \dots \cup \theta(x_n) \vDash_{\mathsf{X}} \theta(\varphi(x_1, \dots, x_n)).$$

In other words θ is compatible with $\mathscr{L}$ in X when the solution sets of θ in X are closed under the interpretation of the operations and constants in $\mathscr{L}$.

Now we will explain how is it possible to build a functor out of a set of equations θ compatible with $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. For every $\boldsymbol{A} \in \mathsf{X}$, we let $\theta_{\mathscr{L}}(\boldsymbol{A})$ be the algebra of type $\mathscr{L}$ whose universe is

$$\theta_{\mathscr{L}}(A) := \{a \in A : \boldsymbol{A} \vDash \theta(a)\}$$

equipped with the restriction of the operations in $\mathscr{L}$. We know that $\theta_{\mathscr{L}}(\boldsymbol{A})$ is well-defined, since its universe is closed under the interpretation of the operations in $\mathscr{L}$ and contains the interpretation of the constants in $\mathscr{L}$. Observe that by definition of compatibility $\theta_{\mathscr{L}}(\boldsymbol{A})$ can be empty if and only if $\mathscr{L}$ contains no constant symbol.

Given a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ in X, we denote its restriction to $\theta_{\mathscr{L}}(A)$ by

$$\theta_{\mathscr{L}}(f)\colon \theta_{\mathscr{L}}(\boldsymbol{A}) \to \theta_{\mathscr{L}}(\boldsymbol{B}).$$

It is easy to see that $\theta_{\mathscr{L}}(f)$ is a well-defined homomorphism. Now, consider the following class of algebras:

$$\theta_{\mathscr{L}}(\mathsf{X}) := \mathbb{I}\{\theta_{\mathscr{L}}(\boldsymbol{A}) : \boldsymbol{A} \in \mathsf{X}\}.$$

Let $\theta_{\mathscr{L}}\colon \mathsf{X} \to \theta_{\mathscr{L}}(\mathsf{X})$ be the map defined by the following rule:

$$\boldsymbol{A} \longmapsto \theta_{\mathscr{L}}(\boldsymbol{A})$$
$$f\colon \boldsymbol{A} \to \boldsymbol{B} \longmapsto \theta_{\mathscr{L}}(f)\colon \theta_{\mathscr{L}}(\boldsymbol{A}) \to \theta_{\mathscr{L}}(\boldsymbol{B}).$$

It is easy to check that $\theta_{\mathscr{L}}$ is a functor.

Theorem 3.8. *Let X be a generalized quasi-variety and $\theta \subseteq Eq(\mathscr{L}_{\mathsf{X}}, 1)$ a set of equations compatible with $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. If Y is a generalized quasi-variety such that $\theta_{\mathscr{L}}(\mathsf{X}) \subseteq \mathsf{Y}$, then $\theta_{\mathscr{L}}\colon \mathsf{X} \to \mathsf{Y}$ is a right adjoint functor.*

Proof. By Theorem 2.3 we know that the functor $\theta_{\mathscr{L}}$ is a right adjoint if and only if it preserves limits and κ-directed colimits for some regular cardinal κ. We begin by proving that $\theta_{\mathscr{L}}$ preserves limits. It will be enough to show that it preserves direct products and equalizers. To do this, consider a family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{X}$. It is easy to see that

$$\theta_{\mathscr{L}}(\prod_{i \in I} \boldsymbol{A}_i) = \prod_{i \in I} \theta_{\mathscr{L}}(\boldsymbol{A}_i)$$

and that projections are sent to projections. As to equalizers, the situation is analogous. Consider two homomorphisms $f, g\colon \boldsymbol{A} \rightrightarrows \boldsymbol{B}$ in X. Their equalizer is the inclusion map $e\colon \boldsymbol{C} \to \boldsymbol{A}$, where $\boldsymbol{C}$ is the subalgebra of $\boldsymbol{A}$ with universe $\{a \in A : f(a) = g(a)\}$. Keeping this in mind, it is clear that $\theta_{\mathscr{L}}(e)\colon \theta_{\mathscr{L}}(\boldsymbol{C}) \to \theta_{\mathscr{L}}(\boldsymbol{A})$ is an inclusion map, whose range consists of objects on which $\theta_{\mathscr{L}}(f)$ and $\theta_{\mathscr{L}}(g)$ are identical. Therefore it only remains to prove that $\theta_{\mathscr{L}}(e)$ covers all the elements on which $\theta_{\mathscr{L}}(f)$ and $\theta_{\mathscr{L}}(g)$ coincide. Pick $a \in \theta_{\mathscr{L}}(A)$ such that $\theta_{\mathscr{L}}(f)(a) = \theta_{\mathscr{L}}(g)(a)$. This means that $f(a) = g(a)$ and, therefore, that $a \in C$. Moreover a is a solution to all the equations in θ; therefore we obtain that $a \in \theta_{\mathscr{L}}(C)$. This concludes the proof that $\theta_{\mathscr{L}}$ preserves limits.

It only remains to prove that $\theta_{\mathscr{L}}$ preserves κ-directed colimits for some regular cardinal κ. Let κ be a regular cardinal larger than the number of variables occurring in the generalized quasi-equations axiomatizing X. Consider a κ-directed diagram $\{\boldsymbol{A}_i : i \in I\}$ with arrows $f_{ij} \colon \boldsymbol{A}_i \to \boldsymbol{A}_j$ when $i \leqslant j$ in X. Its directed colimit is the algebra $\boldsymbol{A}$ obtained as follows. We first consider the disjoint union $\{\langle a, i\rangle : a \in A_i \text{ and } i \in I\}$. Then we factor out by the following equivalence relation

$$\phi := \{\langle\langle a, i\rangle, \langle b, j\rangle\rangle : \text{ there is } k \geqslant i, j \text{ such that } f_{ik}(a) = f_{jk}(b)\}$$

and define operations in the natural way. It is now clear that the algebra $\theta_{\mathscr{L}}(\boldsymbol{A})$ is obtained analogously out of the κ-directed diagram $\{\theta_{\mathscr{L}}(\boldsymbol{A}_i) : i \in I\}$ and $\theta_{\mathscr{L}}(f_{ij})$ for $i \leqslant j$. Therefore the directed colimit of this diagram is the quotient of $\theta_{\mathscr{L}}(\boldsymbol{A})$ with respect to its smallest Y-congruence. But this congruence is the identity, because $\theta_{\mathscr{L}}(\boldsymbol{A}) \in \mathsf{Y}$. Therefore we conclude that $\theta_{\mathscr{L}}(\boldsymbol{A})$ is the directed colimit of the diagram as desired. ⊠

A familiar instance of the above construction is the following:

Example 3.9 (Subreducts)**.** Let X be a (generalized) quasi-variety and $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. An *$\mathscr{L}$-subreduct* of an algebra $\boldsymbol{A} \in \mathsf{X}$ is a subalgebra of the $\mathscr{L}$-reduct of $\boldsymbol{A}$. From [16, Proposition 2.3.19] it is easy to infer that the class Y of $\mathscr{L}$-subreducts of algebras in X is a (generalized) quasi-variety. For quasi-varieties this fact was proved by Maltsev [24]. Consider the forgetful functor $\mathcal{U} \colon \mathsf{X} \to \mathsf{Y}$. It is easy to see that $\mathcal{U} = \theta_{\mathscr{L}}$ where $\theta = \emptyset$. From Theorem 3.8 it follows that $\mathcal{U}$ has a left adjoint. ⊠

In the next examples we illustrate how the two deformations introduced so far can be combined to describe right adjoint functors.

Example 3.10 (Kleene Algebras)**.** A *Kleene algebra* $\boldsymbol{A} = \langle A, \sqcap, \sqcup, \neg, 0, 1\rangle$ is a De Morgan algebra in which the equation $x \sqcap \neg x \leqslant y \sqcup \neg y$ holds. We denote by KA the variety of Kleene algebras and by DL_{01} the variety of bounded distributive lattices. In [10] (but see also [18]) a way of constructing Kleene algebras out of bounded distributive lattices is described. More precisely, given $\boldsymbol{A} \in \mathsf{DL}_{01}$, the Kleene algebra $\mathcal{G}(\boldsymbol{A})$ has universe

$$G(A) := \{\langle a, b\rangle \in A^2 : a \wedge b = 0\}$$

and operations defined as in Example 3.4. Moreover, given a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ in DL_{01}, the map $\mathcal{G}(f) \colon \mathcal{G}(\boldsymbol{A}) \to \mathcal{G}(\boldsymbol{B})$ is defined by replicating f component-wise. It turns out that $\mathcal{G} \colon \mathsf{DL}_{01} \to \mathsf{KA}$ is a right adjoint functor [10, Theorem 1.7].

It is worth remarking that DL_{01} and KA are *not* categorically equivalent and, therefore, $\mathcal{G}$ is not a category equivalence. This follows from the following observations:

1. Category equivalences between prevarieties preserve the fact of being a nontrivial subdirectly irreducible algebra.

2. DL_{01} has, up to isomorphism, only one non-trivial subdirectly irreducible member (the two-element chain), while KA has two (the two- and the three-element chains).

In order to decompose $\mathcal{G}$ into a combination of our two deformations, we reason as follows. First consider the matrix power functor $[2]\colon \mathsf{DL}_{01} \to \mathsf{DL}_{01}^{[2]}$. Recall from Example 3.6 that it is a category equivalence and that $\mathsf{DL}_{01}^{[2]}$ is a variety. Now consider the following sublanguage $\mathscr{L}$ of the language of $\mathsf{DL}_{01}^{[2]}$ defined in Example 3.2. Consider also the set of equations

$$\theta := \{\langle x^1 \wedge x^2, x^1 \wedge x^2\rangle \approx \langle 0,0\rangle\} \subseteq Eq(\mathscr{L}_{\mathsf{DL}_{01}^{[2]}}, 1).$$

It is easy to see that θ is compatible with $\mathscr{L}$. For example the compatibility of θ w.r.t. $\sqcap$ amounts to the following condition: For every $\boldsymbol{A} \in \mathsf{DL}_{01}$ and $\langle a,b\rangle, \langle c,d\rangle \in A \times A$, if

$$\langle x^1 \wedge x^2, x^1 \wedge x^2\rangle^{\boldsymbol{A}^{[2]}}(\langle a,b\rangle) = \langle 0,0\rangle \text{ and } \langle x^1 \wedge x^2, x^1 \wedge x^2\rangle^{\boldsymbol{A}^{[2]}}(\langle c,d\rangle) = \langle 0,0\rangle,$$

then

$$\langle x^1 \wedge x^2, x^1 \wedge x^2\rangle^{\boldsymbol{A}^{[2]}}(\langle a,b\rangle \sqcap^{\boldsymbol{A}^{[2]}} \langle c,d\rangle) = \langle 0,0\rangle.$$

The condition above is equivalent to the following elementary fact: For every $\boldsymbol{A} \in \mathsf{DL}_{01}$ and $\langle a,b\rangle, \langle c,d\rangle \in A \times A$,

$$\text{if } a \wedge b = 0 \text{ and } c \wedge d = 0, \text{ then } (a \wedge c) \wedge (b \vee d) = 0.$$

This shows that θ is compatible with $\sqcap$. A similar argument shows that θ is compatible with the whole $\mathscr{L}$.

Moreover, for every $\boldsymbol{A} \in \mathsf{DL}_{01}$ and $a, b \in A$ we have that

$$\begin{aligned}
\langle a,b\rangle \in \mathcal{G}(\boldsymbol{A}) &\iff a \wedge b = 0 \\
&\iff \langle a \wedge b, a \wedge b\rangle = \langle 0,0\rangle \\
&\iff \boldsymbol{A}^{[2]} \vDash \langle x^1 \wedge x^2, x^1 \wedge x^2\rangle \approx \langle 0,0\rangle [\![a,b]\!] \\
&\iff \langle a,b\rangle \in \theta_{\mathscr{L}}(A).
\end{aligned}$$

Hence we conclude that $\theta_{\mathscr{L}}(\boldsymbol{A}^{[2]}) = \mathcal{G}(\boldsymbol{A}) \in \mathsf{KA}$ for every $\boldsymbol{A} \in \mathsf{DL}_{01}^{[2]}$. But this implies that $\theta_{\mathscr{L}}\colon \mathsf{DL}_{01}^{[2]} \to \mathsf{KA}$ is a right adjoint functor by Theorem 3.8. Finally, the functor $\mathcal{G}$ coincides with the composition $\theta_{\mathscr{L}} \circ [2]$ as desired. Observe that we showed that $\mathcal{G}$ is the composition of two right adjoint functors. Thus we obtained a new and purely combinatorial proof of the fact that $\mathcal{G}$ is a right adjoint functor. ⊠

Before concluding this section, we show that the deformations described until now can be applied to decompose equivalence functors between prevarieties. This will make the connection with McKenzie's work [25] explicit. To this end, let us recall the definition of a special version of the $\theta_{\mathscr{L}}$ construction.

Example 3.11 (Idempotent and Invertible Terms)**.** Suppose that X is a prevariety and $\sigma(x)$ a unary term. We say that $\sigma(x)$ is *idempotent* if $\mathsf{X} \vDash \sigma\sigma(x) \approx \sigma(x)$ and that $\sigma(x)$ is *invertible* if there are an n-ary term t and unary terms $t_1, \dots, t_n$ such that

$$\mathsf{X} \vDash t(\sigma t_1(x), \dots, \sigma t_n(x)) \approx x.$$

Given a unary and idempotent term $\sigma(x)$ of X, we define

$$\mathscr{L} := \{\sigma t : t \text{ is a basic symbol of } \mathsf{X}^{[1]}\}$$

and $\theta := \{x \approx \sigma(x)\}$. Moreover, we define

$$\mathsf{X}(\sigma) := \theta_{\mathscr{L}}(\mathsf{X}^{[1]}).$$

McKenzie proved that the functor $\sigma \colon \mathsf{X} \to \mathsf{X}(\sigma)$ defined as the composition $\theta_{\mathscr{L}} \circ [1]$ is a category equivalence [25, Theorem 2.2.(ii)]. Moreover, if X is a prevariety (or a generalized quasi-variety, a quasi-variety, a variety), then so is $\mathsf{X}(\sigma)$. Following the literature, we will write $\boldsymbol{A}(\sigma)$ instead of $\sigma(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{X}$. ⊠

To introduce McKenzie's characterization of category equivalence, we restrict to prevarieties without constant symbols. It should be kept in mind that this restriction is somehow immaterial, since, given a prevariety K, we can always replace the constant symbols of K by constant unary operations obtaining a new prevariety K' whose only difference with K is the presence of the empty algebra.

We need to recall some basic concepts [6, Definitions 4.76 and 4.77]:

Definition 3.12. Let X and Y be prevarieties without constant symbols. An *interpretation* of X in Y is a map $\boldsymbol{\tau} \colon \mathscr{L}_{\mathsf{X}} \to Tm(\mathscr{L}_{\mathsf{Y}}, \omega)$ such that:

1. $\boldsymbol{\tau}$ sends n-ary basic symbols to at most n-ary terms for every $n \geqslant 1$.
2. $\boldsymbol{A}^{\boldsymbol{\tau}} := \langle A, \{\boldsymbol{\tau}(\lambda) : \lambda \in \mathscr{L}_{\mathsf{X}}\}\rangle \in \mathsf{X}$ for every $\boldsymbol{A} \in \mathsf{Y}$.

Definition 3.13. Two prevarieties X and Y without constant symbols are *term-equivalent* if there are interpretations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ of X in Y and of Y in X respectively such that for every $\boldsymbol{A} \in \mathsf{X}$ and $\boldsymbol{B} \in \mathsf{Y}$,

$$(\boldsymbol{A}^{\boldsymbol{\rho}})^{\boldsymbol{\tau}} = \boldsymbol{A} \text{ and } (\boldsymbol{B}^{\boldsymbol{\tau}})^{\boldsymbol{\rho}} = \boldsymbol{B}.$$

When two prevarieties X and Y without constant symbols are term-equivalent, the map that sends $\boldsymbol{A} \in \mathsf{X}$ to $\boldsymbol{A}^{\boldsymbol{\rho}} \in \mathsf{Y}$ and that is the identity on arrows is a category equivalence $\mathcal{F}^{\boldsymbol{\rho}} \colon \mathsf{X} \to \mathsf{Y}$. Then we have the following [25, Theorem 6.1]:

Theorem 3.14 (McKenzie)**.** *If $\mathcal{G} \colon \mathsf{X} \to \mathsf{Y}$ is a category equivalence between prevarieties without constant symbols, then there are a natural number $n > 0$ and a unary idempotent and invertible term $\sigma(x)$ of $\mathsf{X}^{[n]}$ such that*

1. *Y is term-equivalent to $\mathsf{X}^{[n]}(\sigma)$ under some interpretation $\boldsymbol{\rho}$ of Y in $\mathsf{X}^{[n]}(\sigma)$.*
2. *The functors $\mathcal{G}$ and $\mathcal{F}^{\boldsymbol{\rho}} \circ (\sigma \circ [n])$ are naturally isomorphic.*

4. From translations to right adjoints

As we mentioned, our aim is to develop a correspondence between the *adjunctions* between two generalized quasi-varieties X and Y and the *translations* between the equational consequences relative to X and Y. The first step we make in this direction is to introduce a precise notion of translation between relative equational consequences. Subsequently, we use these translations to construct right adjoint functors (Theorem 4.5). To simplify the notation, we will assume throughout this section that X and Y are two fixed generalized quasi-varieties (possibly in different languages).

Definition 4.1. Consider a cardinal $\kappa > 0$. A *κ-translation* $\boldsymbol{\tau}$ of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$ is a map from $\mathscr{L}_{\mathsf{X}}$ to $\mathscr{L}_{\mathsf{Y}}^{\kappa}$ that preserves the arities of function symbols.

In other words, if a basic symbol $\varphi \in \mathscr{L}_{\mathsf{X}}$ is n-ary, we have that $\boldsymbol{\tau}(\varphi) = \langle t_i : i < \kappa\rangle$ for some terms $t_i = t_i(\vec{x}_1, \dots, \vec{x}_n)$ of language of Y, where $\vec{x}_m = \langle x_m^j : j < \kappa\rangle$. It is worth remarking that $\boldsymbol{\tau}$ sends constant symbols to sequences of constant symbols.

Thus if $\mathscr{L}_{\mathsf{X}}$ contains a constant symbol, then also $\mathscr{L}_{\mathsf{Y}}$ must contain one for a translation to exist.

A κ-translation $\boldsymbol{\tau}$ extends naturally to arbitrary terms. Let us explain briefly how. Given a cardinal λ, let $Tm(\mathscr{L}_{\mathsf{X}}, \lambda)$ be the set of terms of X written with variables in $\{x_j : j < \lambda\}$ and let $Tm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)$ be the set of terms of Y written with variables in $\{x_j^i : j < \lambda, i < \kappa\}$. We define recursively a map

$$\boldsymbol{\tau}_* \colon Tm(\mathscr{L}_{\mathsf{X}}, \lambda) \to Tm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)^\kappa.$$

For variables and constants we set

$$\begin{aligned} \boldsymbol{\tau}_*(x_j) &\coloneqq \langle x_j^i : i < \kappa \rangle, \text{ for every } j < \lambda \\ \boldsymbol{\tau}_*(c) &\coloneqq \boldsymbol{\tau}(c). \end{aligned}$$

For complex terms, let $\psi \in \mathscr{L}_{\mathsf{X}}$ be n-ary and $\varphi_1, \dots, \varphi_n \in Tm(\mathscr{L}_{\mathsf{X}}, \lambda)$. We have that $\boldsymbol{\tau}(\psi) = \langle t_i : i < \kappa \rangle$ where $t_i = t_i(\vec{x}_1, \dots, \vec{x}_n)$. Keeping this in mind, we set

$$\boldsymbol{\tau}_*(\psi(\varphi_1, \dots, \varphi_n))(i) \coloneqq t_i(\boldsymbol{\tau}_*(\varphi_1)/\vec{x}_1, \dots, \boldsymbol{\tau}_*(\varphi_n)/\vec{x}_n) \text{ for every } i < \kappa.$$

The map $\boldsymbol{\tau}_*$ can be lifted to sets of equations yielding a new function

$$\boldsymbol{\tau}^* \colon \mathcal{P}(Eq(\mathscr{L}_{\mathsf{X}}, \lambda)) \to \mathcal{P}(Eq(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda))$$

defined by the following rule:

$$\Phi \longmapsto \{\boldsymbol{\tau}_*(\varepsilon)(i) \approx \boldsymbol{\tau}_*(\delta)(i) : i < \kappa \text{ and } \varepsilon \approx \delta \in \Phi\}.$$

Definition 4.2. Consider a cardinal $\kappa > 0$. A *contextual κ-translation* of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$ is a pair $\langle \boldsymbol{\tau}, \Theta \rangle$ where $\boldsymbol{\tau}$ is a κ-translation of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$ and $\Theta(\vec{x}) \subseteq Eq(\mathscr{L}_{\mathsf{Y}}, \kappa)$ is a set of equations written with variables among $\{x^i : i < \kappa\}$ that satisfies the following conditions:

1. For every cardinal λ and equations $\Phi \cup \{\varepsilon \approx \delta\} \subseteq Eq(\mathscr{L}_{\mathsf{X}}, \lambda)$ written with variables among $\{x_j : j < \lambda\}$,
$$\text{if } \Phi \vDash_{\mathsf{X}} \varepsilon \approx \delta, \text{ then } \boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j) \vDash_{\mathsf{Y}} \boldsymbol{\tau}^*(\varepsilon \approx \delta).$$
2. For every n-ary operation $\psi \in \mathscr{L}_{\mathsf{X}}$,
$$\Theta(\vec{x}_1) \cup \dots \cup \Theta(\vec{x}_n) \vDash_{\mathsf{Y}} \Theta(\boldsymbol{\tau}_*\psi(x_1, \dots, x_n)).$$

In 1 and 2 it is intended that $\vec{x}_j = \langle x_j^i : i < \kappa \rangle$. The set Θ is the *context* of the contextual translation $\langle \boldsymbol{\tau}, \Theta \rangle$.

A contextual κ-translation $\langle \boldsymbol{\tau}, \Theta \rangle$ of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$ is *non-trivial*[2] provided that if there is a (non-empty) sequence $\vec{\varphi} \in Tm(\mathscr{L}_{\mathsf{Y}}, 0)^\kappa$ of constant symbols such that $\mathsf{Y} \vDash \Theta(\vec{\varphi})$, then there is $i_0 < \kappa$ and sequences of variables

$$\vec{x} = \langle x^i : i < \kappa \rangle \text{ and } \vec{y} = \langle y^i : i < \kappa \rangle$$

such that

$$\Theta(\vec{x}) \cup \Theta(\vec{y}) \nvDash_{\mathsf{Y}} x^{i_0} \approx y^{i_0}.$$

[2]This condition of *non-triviality* is designed in order to identify contextual translations that correspond to non-trivial adjunctions. This will become clear in the proof of Theorem 4.5.

Several translations between logics classically considered in the literature provide examples of this general notion of contextual translation between relative equational consequences. However some of these translations between logics send variables x to complex terms φ, while we do not allow this in our definition of contextual translation. The behaviour of variables is then described by adding the equation $x \approx \varphi$ to the context of the contextual translation.

Example 4.3 (Heyting and Interior Algebras)**.** As shown by Gödel in [15] (see also [13, 23, 27]), it is possible to interpret the intuitionistic propositional calculus $\mathcal{IPC}$ into the consequence relation associated with the global modal system $\mathcal{S}4$ [20, 21]. Since these two logics are algebraizable [8] with equivalent algebraic semantics the variety of Heyting algebras HA and of interior algebras IA respectively, this interpretation can be lifted from terms to equations. More precisely, let $\boldsymbol{\tau}$ be the 1-translation of $\mathscr{L}_{\mathsf{HA}}$ into $\mathscr{L}_{\mathsf{IA}}$ defined as follows:

$$x \star y \longmapsto x \star y \quad \neg x \longmapsto \Box\neg x \quad x \to y \longmapsto \Box(x \to y)$$

for $\star \in \{\wedge, \vee\}$. The interpretation of $\mathcal{IPC}$ into $\mathcal{S}4$ can now be presented as follows:

$$\Gamma \vdash_{\mathcal{IPC}} \varphi \Longleftrightarrow \sigma\boldsymbol{\tau}_*(\Gamma) \vdash_{\mathcal{S}4} \sigma\boldsymbol{\tau}_*(\varphi) \tag{1}$$

for every $\Gamma \cup \{\varphi\} \subseteq Tm(\mathscr{L}_{\mathsf{HA}}, \lambda)$, where σ is the substitution sending every variable x to its necessitation $\Box x$. In order to present this translation in our framework, we have to deal with the fact that we allow only translations that send variables to variables. As we mentioned, this problem is overcome by introducing a *context* in the premises. To explain how, we recall that the terms of $Tm(\mathscr{L}_{\mathsf{HA}}, \lambda)$ are written with variables among $\{x_j : j < \lambda\}$. Then we have that:

$$\sigma\boldsymbol{\tau}_*(\Gamma) \vdash_{\mathcal{S}4} \sigma\boldsymbol{\tau}_*(\varphi) \Longleftrightarrow \boldsymbol{\tau}_*(\Gamma) \cup \{x_j \leftrightarrow \Box x_j : j < \lambda\} \vdash_{\mathcal{S}4} \boldsymbol{\tau}_*(\varphi). \tag{2}$$

The left-to-right direction of (2) follows from the fact that the algebraic meaning of $x_j \leftrightarrow \Box x_j$ is $x_j \approx \Box x_j$. To prove the other direction, suppose that the right-hand deduction holds. Then by structurality we can apply the substitution σ to it. This fact, together with $\emptyset \vdash_{\mathcal{S}4} \Box x \leftrightarrow \Box\Box x$, yields the desired conclusion. Now, using the completeness of $\mathcal{IPC}$ and $\mathcal{S}4$ with respect to the corresponding equivalent algebraic semantics, we obtain that

$$\Phi \vDash_{\mathsf{HA}} \varepsilon \approx \delta \Longleftrightarrow \boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(x_j) \vDash_{\mathsf{IA}} \boldsymbol{\tau}^*(\varepsilon \approx \delta) \tag{3}$$

for every $\Phi \cup \{\varepsilon \approx \delta\} \subseteq Eq(\mathscr{L}_{\mathsf{HA}}, \lambda)$, where $\Theta(x) = \{x \approx \Box x\}$. Observe that (3) implies condition 1 of Definition 4.2. Moreover, observe that in this case condition 2 of the same definition amounts to the following deductions, which are all easy to check (the first and the last are trivial, since $\mathsf{IA} \vDash \Box x \approx \Box\Box x$):

$$x \approx \Box x \vDash_{\mathsf{IA}} \Box\neg x \approx \Box\Box\neg x$$
$$x \approx \Box x, y \approx \Box y \vDash_{\mathsf{IA}} x \star y \approx \Box(x \star y)$$
$$x \approx \Box x, y \approx \Box y \vDash_{\mathsf{IA}} \Box(x \to y) \approx \Box\Box(x \to y)$$

for each $\star \in \{\wedge, \vee\}$. Therefore we conclude that $\langle \boldsymbol{\tau}, \Theta \rangle$ is a contextual translation of $\vDash_{\mathsf{HA}}$ into $\vDash_{\mathsf{IA}}$. ⊠

Example 4.4 (Heyting and Boolean Algebras)**.** The same trick can be applied to subsume Kolmogorov's interpretation of classical propositional calculus $\mathcal{CPC}$ into $\mathcal{IPC}$ [19] in our framework. Let $\boldsymbol{\tau}$ be the 1-translation defined as follows:

$$0 \longmapsto 0 \quad 1 \longmapsto 1 \quad \neg x \longmapsto \neg x \quad x \star y \longmapsto \neg\neg(x \star y)$$

for every $\star \in \{\wedge, \vee, \to\}$. The original translation of Kolmogorov states that

$$\Gamma \vdash_{\mathcal{CPC}} \varphi \Longleftrightarrow \sigma\boldsymbol{\tau}_*(\Gamma) \vdash_{\mathcal{IPC}} \sigma\boldsymbol{\tau}_*(\varphi)$$

for every $\Gamma \cup \{\varphi\} \subseteq Tm(\mathscr{L}, \lambda)$, where σ is the substitution sending every variable x to its double negation $\neg\neg x$. Combining it with the observation that $\emptyset \vdash_{\mathcal{IPC}} \neg x \leftrightarrow \neg\neg\neg x$, it is easy to see that $\langle \boldsymbol{\tau}, \Theta \rangle$ with $\Theta = \{x \approx \neg\neg x\}$ is a contextual translation of $\vDash_{\mathsf{BA}}$ into $\vDash_{\mathsf{HA}}$, where BA is the variety of Boolean algebras. ⊠

The importance of non-trivial contextual κ-translations of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$ is that they correspond to non-trivial right adjoint functors from Y to X. Notice that right adjoints reverse the direction of contextual translations and vice-versa. We now proceed to establish one half of this correspondence by showing how to construct a right adjoint functor out of a contextual translation. Consider a non-trivial contextual κ-translation $\langle \boldsymbol{\tau}, \Theta \rangle$ of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$. Then consider the set:

$$\mathscr{L} := \{\boldsymbol{\tau}(\psi) : \psi \in \mathscr{L}_{\mathsf{X}}\} \subseteq \mathscr{L}_{\mathsf{Y}}^{\kappa}. \tag{4}$$

Observe that $\mathscr{L}$ is a sublanguage of the language of the matrix power $\mathsf{Y}^{[\kappa]}$. Then consider the set

$$\theta := \{\vec{\varepsilon} \approx \vec{\delta} : \varepsilon \approx \delta \in \Theta\}$$

where $\vec{\varepsilon}$ and $\vec{\delta}$ are the κ-sequences constantly equal to ε and δ respectively. Observe that θ is a set of identities between κ-sequences of terms of Y in κ variables. Now, κ-sequences of terms of Y in κ-many variables can be viewed as *unary* terms of the matrix power $\mathsf{Y}^{[\kappa]}$. Thus θ can be viewed as a set of equations in one variable in the language of $\mathsf{Y}^{[\kappa]}$. Hence we have the three basic ingredients of our construction: a matrix power $\mathsf{Y}^{[\kappa]}$, a sublanguage $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{Y}}^{\kappa}$, and a set of equations $\theta \subseteq Eq(\mathscr{L}_{\mathsf{Y}}^{\kappa}, 1)$.

There is still a technicality we must take into account: when κ is infinite the matrix power $\mathsf{Y}^{[\kappa]}$ may fail to be a generalized quasi-variety. Let K be the class of algebras defined as follows:

$$\mathsf{K} := \begin{cases} \mathbb{Q}(\mathsf{Y}^{[\kappa]}) & \text{if } \mathsf{X} \text{ and } \mathsf{Y} \text{ are quasi-varieties} \\ & \text{and } \mathrm{Cg}_{\mathsf{Y}}^{\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)}(\Theta) \text{ is finitely generated} \\ \mathbb{GQ}_{\lambda}(\mathsf{Y}^{[\kappa]}) & \text{otherwise, where } \lambda \text{ is infinite and } \mathbb{U}_{\lambda}(\mathsf{X}) = \mathsf{X} \end{cases}$$

where the expressions $\mathbb{Q}$ and $\mathbb{GQ}_{\lambda}$ have been introduced at pag. 3. Observe that in the above definition λ is not uniquely determined, but any choice will be equivalent for our purposes.

Theorem 4.5. *Let X and Y be generalized quasi-varieties, let $\langle \boldsymbol{\tau}, \Theta \rangle$ be a non-trivial contextual κ-translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$, and let K be the class just introduced. The maps $[\kappa] \colon \mathsf{Y} \to \mathsf{K}$ and $\theta_{\mathscr{L}} \colon \mathsf{K} \to \mathsf{X}$ defined above are right adjoint functors. In particular, the composition $\theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{X}$ is a non-trivial right adjoint.*

Proof. Observe that K is a generalized quasi-variety. Therefore we can apply Theorem 3.5, yielding that $[\kappa] \colon \mathsf{Y} \to \mathsf{K}$ is a right adjoint functor. Now we turn to prove the same for $\theta_{\mathscr{L}}$. We will detail the case where X and Y are quasi-varieties and $\mathrm{Cg}_{\mathsf{Y}}^{\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)}(\Theta)$ finitely generated, since the other case is analogous. Since Y is a quasi-variety and $\mathrm{Cg}_{\mathsf{Y}}^{\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)}(\Theta)$ is finitely generated, there is a finite set $\{\langle \alpha_i, \beta_i \rangle : i < n\} \subseteq \Theta$ such that $\{\langle \alpha_i, \beta_i \rangle : i < n\} \dashv\vDash_{\mathsf{Y}} \Theta$. It is easy to see that

$$\{\vec{\alpha}_i \approx \vec{\beta}_i : i < n\} \dashv\vDash_{\mathsf{Y}^{[\kappa]}} \theta \tag{5}$$

where $\vec{\alpha}_i$ and $\vec{\beta}_i$ are the κ-sequences constantly equal to α_i and β_i respectively.

Now from condition 2 of Definition 4.2 it follows that θ is compatible with $\mathscr{L}$ in $\mathsf{Y}^{[\kappa]}$, where $\mathscr{L}$ is the language defined in (4). From (5) we know that this compatibility condition can be expressed by a set of deductions, whose antecedent is finite, of the equational consequence relative to $\mathsf{Y}^{[\kappa]}$, i.e.,

$$\bigcup_{j \leqslant m} \{\vec{\alpha}_i \approx \vec{\beta}_i : i < n\}(\vec{x}_j) \vDash_{\mathsf{Y}^{[\kappa]}} \theta(\boldsymbol{\tau}(\psi)(\vec{x}_1, \dots, \vec{x}_n))$$

for every m-ary $\psi \in \mathscr{L}$. In particular, this implies that θ is still compatible with $\mathscr{L}$ in K (recall that K is the *quasi-variety* generated by $\mathsf{Y}^{[\kappa]}$).

We claim that $\theta_{\mathscr{L}}(\boldsymbol{A}) \in \mathsf{X}$ for every $\boldsymbol{A} \in \mathsf{K}$. To prove this, consider any finite deduction

$$\varphi_1 \approx \psi_1, \dots, \varphi_m \approx \psi_m \vDash_{\mathsf{X}} \varepsilon \approx \delta.$$

Let $x_1, \dots, x_p$ be the variables that occur in it. From condition 1 of Definition 4.2 it follows that

$$\{\boldsymbol{\tau}_*(\varphi_t) \approx \boldsymbol{\tau}_*(\psi_t) : t \leqslant m\} \cup \bigcup_{j \leqslant p} \theta(\vec{x}_j) \vDash_{\mathsf{Y}^{[\kappa]}} \boldsymbol{\tau}_*(\varepsilon) \approx \boldsymbol{\tau}_*(\delta)$$

where $\vec{x}_j = \langle x_j^i : i < \kappa \rangle$. Thanks to (5) the above deduction can be expressed by a collection of deductions, whose antecedent is finite, of the equational consequence relative to $\mathsf{Y}^{[\kappa]}$, i.e.,

$$\{\boldsymbol{\tau}_*(\varphi_t) \approx \boldsymbol{\tau}_*(\psi_t) : t \leqslant m\} \cup \bigcup_{j \leqslant p} \{\vec{\alpha}_i \approx \vec{\beta}_i : i < n\}(\vec{x}_j) \vDash_{\mathsf{Y}^{[\kappa]}} \boldsymbol{\tau}_*(\varepsilon) \approx \boldsymbol{\tau}_*(\delta).$$

Since K is the *quasi-variety* generated by $\mathsf{Y}^{[\kappa]}$, we know that the above deduction persists in K. Together with the fact that $\{\vec{\alpha}_i \approx \vec{\beta}_i : i < n\} \subseteq \theta$, this implies that for every $\boldsymbol{A} \in \mathsf{K}$ and every $a_1, \dots, a_p \in \theta_{\mathscr{L}}(A)$, we have that:

$$\begin{aligned} &\text{if } \boldsymbol{A} \vDash \boldsymbol{\tau}_*(\varphi_1) \approx \boldsymbol{\tau}_*(\psi_1), \dots, \boldsymbol{\tau}_*(\varphi_m) \approx \boldsymbol{\tau}_*(\psi_m)[\![a_1, \dots, a_p]\!],\\ &\text{then } \boldsymbol{A} \vDash \boldsymbol{\tau}_*(\varepsilon) \approx \boldsymbol{\tau}_*(\delta)[\![a_1, \dots, a_p]\!]. \end{aligned}$$

But this means exactly that

$$\begin{aligned} &\text{if } \theta_{\mathscr{L}}(\boldsymbol{A}) \vDash \varphi_1 \approx \psi_1, \dots, \varphi_m \approx \psi_m[\![a_1, \dots, a_p]\!],\\ &\text{then } \theta_{\mathscr{L}}(\boldsymbol{A}) \vDash \varepsilon \approx \delta[\![a_1, \dots, a_p]\!]. \end{aligned}$$

Thus we showed that $\theta_{\mathscr{L}}(\boldsymbol{A})$ satisfies every quasi-equation that holds in X. Since X is a quasi-variety, we conclude that $\theta_{\mathscr{L}}(\boldsymbol{A}) \in \mathsf{X}$. This establishes our claim. Hence we can apply Theorem 3.8, yielding that $\theta_{\mathscr{L}} \colon \mathsf{K} \to \mathsf{X}$ is a right adjoint functor. We conclude that $\theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{X}$ is a right adjoint functor.

It only remains to prove that $\theta_{\mathscr{L}} \circ [\kappa]$ is non-trivial, i.e., that it does not send every algebra to the trivial one. First consider the case where there is no sequence $\vec{\varphi} \in Tm(\mathscr{L}_{\mathsf{Y}}, 0)^\kappa$ of constant symbols such that $\mathsf{Y} \vDash \Theta(\vec{\varphi})$. Then consider the free algebra $\boldsymbol{Tm}_{\mathsf{Y}}(0)$. We have that $\theta_{\mathscr{L}}(\boldsymbol{Tm}_{\mathsf{Y}}(0)^{[\kappa]}) = \emptyset$, otherwise the equations

Θ would have a constant solution (which is not the case). Thus in this case the functor $\theta_{\mathscr{L}} \circ [\kappa]$ is non-trivial. Then consider the case where there is a non-empty sequence $\vec{\varphi} \in Tm(\mathscr{L}_{\mathsf{Y}}, 0)^{\kappa}$ such that $\mathsf{Y} \vDash \Theta(\vec{\varphi})$. Since $\langle \boldsymbol{\tau}, \Theta \rangle$ is non-trivial, we have that

$$\Theta(\vec{x}) \cup \Theta(\vec{y}) \nvDash_{\mathsf{Y}} x^i \approx y^i.$$

This means that there is an algebra $\boldsymbol{A} \in \mathsf{Y}$ and sequences $\vec{a}, \vec{c} \in A^{\kappa}$ such that that $\vec{a}, \vec{c} \in \theta_{\mathscr{L}}(A^{[\kappa]})$ and $\vec{a} \neq \vec{c}$. Thus the algebra $\theta_{\mathscr{L}}(\boldsymbol{A}^{[\kappa]})$ has at least two elements and, therefore, is non-trivial as desired. ⊠

If we apply the above construction to Gödel and Kolmogorov's translations, we obtain some well-known transformations:

Example 4.6 (Open and Regular Elements)**.** Given $\boldsymbol{A} \in \mathsf{IA}$, an element $a \in A$ is *open* if $\Box a = a$. The set of open elements $\mathrm{Op}(\boldsymbol{A})$ of $\boldsymbol{A}$ is closed under the lattice operations and contains the bounds. Moreover we can equip it with an implication $\multimap$ and with a negation $\sim$ defined for every $a, b \in \mathrm{Op}(\boldsymbol{A})$ as follows:

$$a \multimap b := \Box^{\boldsymbol{A}}(a \to^{\boldsymbol{A}} b) \text{ and } \sim a := \Box^{\boldsymbol{A}} \neg^{\boldsymbol{A}} a.$$

It is well known that

$$\mathrm{Op}(\boldsymbol{A}) := \langle \mathrm{Op}(\boldsymbol{A}), \wedge, \vee, \multimap, \sim, 0, 1 \rangle$$

is a Heyting algebra. Now, every homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ between interior algebras restricts to a homomorphism $f \colon \mathrm{Op}(\boldsymbol{A}) \to \mathrm{Op}(\boldsymbol{B})$. Therefore the map $\mathrm{Op} \colon \mathsf{IA} \to \mathsf{HA}$ can be regarded as a functor. As the reader may have guessed, it is in fact the right adjoint functor induced by Gödel's translation of $\mathcal{IPC}$ into $\mathcal{S}4$ (Example 4.3).

A similar correspondence arises from Kolmogorov's translation of $\mathcal{CPC}$ into $\mathcal{IPC}$. More precisely, given $\boldsymbol{A} \in \mathsf{HA}$, an element $a \in A$ is *regular* if $\neg\neg a = a$. It is well known that the set of regular elements $\mathrm{Reg}(\boldsymbol{A})$ of $\boldsymbol{A}$ is closed under $\wedge, \neg$ and $\to$ and contains the bounds. Moreover we can equip it with a new join $\sqcup$ defined for every $a, b \in \mathrm{Reg}(\boldsymbol{A})$ as follows:

$$a \sqcup b := \neg^{\boldsymbol{A}} \neg^{\boldsymbol{A}}(a \vee b).$$

It is well known that

$$\mathrm{Reg}(\boldsymbol{A}) := \langle \mathrm{Reg}(\boldsymbol{A}), \wedge, \sqcup, \to, \neg, 0, 1 \rangle$$

is a Boolean algebra. Now, every homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ between Heyting algebras restricts to a homomorphism $f \colon \mathrm{Reg}(\boldsymbol{A}) \to \mathrm{Reg}(\boldsymbol{B})$. Therefore the map $\mathrm{Reg} \colon \mathsf{HA} \to \mathsf{BA}$ can be regarded as a functor, which is exactly the right adjoint functor induced by Kolmogorov's translation (Example 4.4).

The reader may wonder how the left adjoints to Op and Reg look like. We detail only the construction of the left adjoint to Op, since a similar construction works for Reg as well. To this end, let $\langle \boldsymbol{\tau}, \Theta \rangle$ be the contextual translation arising from Gödel's translation of $\mathcal{IPC}$ into $\mathcal{S}4$ (Example 4.3). Then consider $\boldsymbol{A} \in \mathsf{HA}$. Let θ be the kernel of the natural surjective homomorphism $p \colon \boldsymbol{Tm}(\mathscr{L}_{\mathsf{HA}}, A) \to \boldsymbol{A}$. We denote by $\theta_{\boldsymbol{A}}$ the least IA-congruence of $\boldsymbol{Tm}(\mathscr{L}_{\mathsf{IA}}, A)$ containing the set

$$\boldsymbol{\tau}^*(\theta) \cup \{\langle a, \Box a \rangle : a \in A\}.$$

Finally, we define $\mathcal{F}(\boldsymbol{A}) := \boldsymbol{Tm}(\mathscr{L}_{\mathsf{IA}}, A)/\theta_{\boldsymbol{A}}$. It is clear that $\mathcal{F}(\boldsymbol{A}) \in \mathsf{IA}$. Now, consider a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ in HA. We let $\mathcal{F}(f) \colon \mathcal{F}(\boldsymbol{A}) \to \mathcal{F}(\boldsymbol{B})$ be the map defined for every $\varphi(\vec{a}) \in Tm(\mathscr{L}_{\mathsf{IA}}, A)$ as

$$\mathcal{F}(f)(\varphi(\vec{a})/\theta_{\boldsymbol{A}}) = \varphi(f(\vec{a}))/\theta_{\boldsymbol{B}}.$$

It turns out that $\mathcal{F}(f) \colon \mathcal{F}(\boldsymbol{A}) \to \mathcal{F}(\boldsymbol{B})$ is a well-defined homomorphism. Indeed $\mathcal{F}$ is a functor, which is left adjoint to Op. This construction shows that the left adjoint to Op can be fully described in terms of the contextual translation $\langle \boldsymbol{\tau}, \Theta \rangle$. A general formulation of this observation will be given in Corollary 6.2. ⊠

5. From right adjoints to translations

In the previous section we described one half of the correspondence between contextual translations and adjunctions, namely how to build an adjunction out of a contextual translation. Now we provide the other half, showing how to construct a contextual translation (between relative equational consequences) out of an adjunction (between generalized quasi-varieties). To this end, in this section we will work with a fixed (but arbitrary) non-trivial left adjoint functor $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ between generalized quasi-varieties. Our goal is to construct a contextual translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$. We will rely on the following observation:

Lemma 5.1. *Let $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ be a non-trivial left adjoint functor between generalized quasi-varieties. The universe of $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1))$ is non-empty.*

Proof. Suppose towards a contradiction that $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)) = \emptyset$. Then for every $\boldsymbol{A} \in \mathsf{Y}$, there is a unique homomorphism $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)) \to \boldsymbol{A}$. Since $\mathcal{F} \dashv \mathcal{G}$, this means that there is a unique homomorphism $\boldsymbol{Tm}_{\mathsf{X}}(1) \to \mathcal{G}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{Y}$. Hence $\mathcal{G}(\boldsymbol{A})$ is the trivial algebra, for every $\boldsymbol{A} \in \mathsf{Y}$. But this contradicts the assumption that $\mathcal{F}$ is non-trivial. ⊠

Now we construct the announced contextual translation $\langle \boldsymbol{\tau}, \Theta \rangle$ out of $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$. By Lemma 5.1 we know that $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)) \neq \emptyset$. Then we can choose a cardinal $\kappa > 0$ and a surjective homomorphism $\pi_1 \colon \boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \to \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1))$. Let Θ be the kernel of π and observe that it can be viewed as a set of equations in $Eq(\mathscr{L}_{\mathsf{Y}}, \kappa)$. This is context of our contextual translation.

In order to construct the κ-translation $\boldsymbol{\tau}$ of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$, we do the following. Consider a cardinal $\lambda > 0$. Since $\mathcal{F}$ preserves copowers and the algebra $\boldsymbol{Tm}_{\mathsf{X}}(\lambda)$ is the λ-th copower of $\boldsymbol{Tm}_{\mathsf{X}}(1)$, we know that $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda))$ is the λ-th copower of $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1))$. Keeping in mind how coproducts look like in prevarieties (see Subsection 2.1), we can identify $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda))$ with the quotient of the free algebra $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda)$ with free generators $\{x^i_j : i < \kappa, j < \lambda\}$ under the Y-congruence generated by $\bigcup_{j<\lambda} \Theta(\vec{x}_j)$ where $\vec{x}_j = \langle x^i_j : i < \kappa \rangle$.

The above construction can be carried out also for $\lambda = 0$ as follows. Recall that $\mathcal{F}$ preserves initial objects, since these are special colimits. Thus we can assume that $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(0)) = \boldsymbol{Tm}_{\mathsf{Y}}(0)$. Now we have that $\boldsymbol{Tm}_{\mathsf{Y}}(0)$ is exactly the quotient of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times 0)$ under the Y-congruence generated by the union of zero-many copies of Θ, i.e., under the identity relation. Thus we identify $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda))$ with a quotient of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda)$ for every cardinal λ. Accordingly, we denote by $\pi_\lambda \colon \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda) \to \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda))$ the corresponding canonical map.

Definition 5.2. Let λ be a cardinal and $\varphi \in Tm(\mathscr{L}_{\mathsf{X}}, \lambda)$. We denote also by $\varphi \colon \boldsymbol{Tm}_{\mathsf{X}}(1) \to \boldsymbol{Tm}_{\mathsf{X}}(\lambda)$ the unique homomorphism that sends x to φ, where x is the free generator of $\boldsymbol{Tm}_{\mathsf{X}}(1)$.

We are finally ready to construct the κ-translation $\boldsymbol{\tau}$ of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$. Consider an n-ary basic operation $\psi \in \mathscr{L}_{\mathsf{X}}$. By the above definition it can be viewed as an arrow $\psi \colon \boldsymbol{Tm}_{\mathsf{X}}(1) \to \boldsymbol{Tm}_{\mathsf{X}}(n)$. Since π_n is surjective and $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)$ is onto-projective in Y, there is a homomorphism

$$\boldsymbol{\tau}(\psi) \colon \boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times n)$$

that makes the following diagram commute:

$$\begin{tikzcd}
 & \boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \arrow[d, "\pi_1"] \arrow[ddl, dotted, bend right, "\boldsymbol{\tau}(\psi)"'] \\
 & \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)) \arrow[d, "\mathcal{F}(\psi)"] \\
\boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times n) \arrow[r, "\pi_n"'] & \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(n))
\end{tikzcd} \tag{6}$$

The map $\boldsymbol{\tau}(\psi)$ can be identified with its values on the generators $\{x^i : i < \kappa\}$ of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)$. In this way it becomes a κ-sequence

$$\langle \boldsymbol{\tau}(\psi)(x^i) : i < \kappa \rangle$$

of terms in variables $\{x^i_j : i < \kappa, 1 \leqslant j \leqslant n\}$.

Let $\boldsymbol{\tau}$ be the κ-translation of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$ obtained by applying this construction to every $\psi \in \mathscr{L}_{\mathsf{X}}$. Hence we constructed a pair $\langle \boldsymbol{\tau}, \Theta \rangle$, where $\boldsymbol{\tau}$ is a κ-translation of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$ and $\Theta \subseteq Eq(\mathscr{L}_{\mathsf{Y}}, \kappa)$.

Theorem 5.3. *Let $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ be a non-trivial left adjoint functor between generalized quasi-varieties. The pair $\langle \boldsymbol{\tau}, \Theta \rangle$ defined above is a non-trivial contextual translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$.*

Proof. Consider a cardinal λ. We know that $\boldsymbol{\tau}$ can be extended to a function $\boldsymbol{\tau}_* \colon Tm(\mathscr{L}_{\mathsf{X}}, \lambda) \to Tm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)^\kappa$, where the terms $Tm(\mathscr{L}_{\mathsf{X}}, \lambda)$ and $Tm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)$ are built respectively with variables among $\{x_j : j < \lambda\}$ and $\{x^i_j : i < \kappa, j < \lambda\}$. Then consider $\varphi \in Tm(\mathscr{L}_{\mathsf{X}}, \lambda)$. Observe that $\boldsymbol{\tau}_*(\varphi)$ is a κ-sequence of terms of Y in variables $\{x^i_j : i < \kappa, j < \lambda\}$. Thus $\boldsymbol{\tau}_*(\varphi)$ can be regarded as a map from the free generators of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)$ to $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda)$. Since $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)$ is a free algebra, this assignment extends uniquely to a homomorphism

$$\boldsymbol{\tau}_*(\varphi) \colon \boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda).$$

Claim 5.3.1. *For every cardinal λ and every $\varphi \in Tm(\mathscr{L}_{\mathsf{X}}, \lambda)$, the following diagram commutes:*

$$\begin{tikzcd}
\boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \arrow[r, "\boldsymbol{\tau}_*(\varphi)"] \arrow[d, "\pi_1"'] & \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda) \arrow[d, "\pi_\lambda"] \\
\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)) \arrow[r, "\mathcal{F}(\varphi)"'] & \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda))
\end{tikzcd}$$

The proof works by induction on φ. We begin by the base case: φ is either a variable or a constant. We can assume without loss of generality that the identification of $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda))$ with a quotient of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda)$ described above is done in such a way that the claim holds for variables. Then we consider the case where φ is a constant c. Then consider the following diagram.

$$\begin{tikzcd}
 & \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)) \arrow[d, "\mathcal{F}(c)"] \arrow[dr, bend left, "\mathcal{F}(c)"] & \\
\boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \arrow[ur, bend left, "\pi_1"] \arrow[r, "\boldsymbol{\tau}(c)"'] \arrow[dr, bend right, "\boldsymbol{\tau}_*(c)"'] & \boldsymbol{Tm}_{\mathsf{Y}}(0) \arrow[r, "f"'] & \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda)) \\
 & \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda) \arrow[ur, bend right, "\pi_\lambda"'] &
\end{tikzcd} \tag{7}$$

Recall that we identified $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(0))$ with $\boldsymbol{Tm}_{\mathsf{Y}}(0)$ and that, under this identification, the map π_0 becomes the identity map $1\colon \boldsymbol{Tm}_{\mathsf{Y}}(0) \to \boldsymbol{Tm}_{\mathsf{Y}}(0)$. Keeping this in mind, we look at the left upper quadrant of diagram (7). It is an instance of diagram (6), where we deleted the identity map π_0 since it plays no significant role. Therefore this quadrant commutes by construction of $\boldsymbol{\tau}$. Then we consider the right upper quadrant of diagram (7), where f is the unique homomorphism given by the universal property of the initial object, i.e., the map that sends each constant term to its interpretation in $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda))$. Then let $g\colon \boldsymbol{Tm}_{\mathsf{X}}(0) \to \boldsymbol{Tm}_{\mathsf{X}}(\lambda)$ be the inclusion map. It is clear that the following diagram commutes.

$$\begin{tikzcd}
\boldsymbol{Tm}_{\mathsf{X}}(1) \arrow[d, "c"'] \arrow[dr, bend left, "c"] & \\
\boldsymbol{Tm}_{\mathsf{X}}(0) \arrow[r, "g"'] & \boldsymbol{Tm}_{\mathsf{X}}(\lambda)
\end{tikzcd}$$

In particular, this implies that the image under $\mathcal{F}$ of the above diagram commutes too. But observe that $\mathcal{F}(g) = f$, since $\boldsymbol{Tm}_{\mathsf{Y}}(0)$ is the initial object of Y. This shows that the right upper quadrant of diagram (7) commutes. We are now ready to prove the claim for $\varphi = c$. Let $\{x^i : i < \kappa\}$ be the free generators of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)$. Then consider $i < \kappa$ and let $c_i \in \mathscr{L}_{\mathsf{Y}}$ be the constant symbol that is the i-th component of the κ-sequence $\boldsymbol{\tau}(c)$. Since the upper part of diagram (7) commutes, we have that:

$$\mathcal{F}(c) \circ \pi_1(x^i) = f \circ \boldsymbol{\tau}(c)(x^i) = c_i^{\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda))}.$$

Moreover, observe that $\boldsymbol{\tau}_*(c) = \boldsymbol{\tau}(c)$ by definition of $\boldsymbol{\tau}_*$. Together with the fact that c_i is a constant, this implies that

$$\pi_\lambda \circ \boldsymbol{\tau}_*(c)(x^i) = \pi_\lambda(c_i^{\boldsymbol{Tm}_{\mathsf{Y}}(\kappa\times\lambda)}) = c_i^{\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda))}.$$

We conclude that $\pi_\lambda \circ \boldsymbol{\tau}_*(c) = \mathcal{F}(c) \circ \pi_1$. This establishes the base case.

Then we turn to prove the inductive case. Consider a basic n-ary operation $\psi \in \mathscr{L}_{\mathsf{X}}$ and $\varphi_1, \dots, \varphi_n \in Tm(\mathscr{L}_{\mathsf{Y}}, \lambda)$. Recall that the angle-bracket notation was introduced in Subsection 2.1 to denote arrows induced by the universal property of the coproduct. Applying in succession the inductive hypothesis and the fact that $\mathcal{F}$ preserves coproducts, we obtain that

$$\begin{aligned}\pi_\lambda \circ \langle \boldsymbol{\tau}_*(\varphi_1), \dots, \boldsymbol{\tau}_*(\varphi_n)\rangle =& \langle \mathcal{F}(\varphi_1), \dots, \mathcal{F}(\varphi_n)\rangle \circ \pi_n \\ =& \mathcal{F}\langle \varphi_1, \dots, \varphi_n\rangle \circ \pi_n\end{aligned}$$

where $\varphi_j \colon \boldsymbol{Tm}_{\mathsf{X}}(1) \to \boldsymbol{Tm}_{\mathsf{X}}(\lambda)$ and $\boldsymbol{\tau}_*(\varphi_j) \colon \boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda)$ for every $j \leqslant n$. Recall from the definition of $\boldsymbol{\tau}$ that $\pi_n \circ \boldsymbol{\tau}(\psi) = \mathcal{F}(\psi) \circ \pi_1$, where $\psi \colon \boldsymbol{Tm}_{\mathsf{X}}(1) \to \boldsymbol{Tm}_{\mathsf{X}}(n)$. Hence we conclude that

$$\begin{aligned}\mathcal{F}(\psi(\varphi_1, \dots, \varphi_n)) \circ \pi_1 =& \mathcal{F}(\langle \varphi_1, \dots, \varphi_n\rangle \circ \psi) \circ \pi_1 \\ =& \mathcal{F}\langle \varphi_1, \dots, \varphi_n\rangle \circ \mathcal{F}(\psi) \circ \pi_1 \\ =& \mathcal{F}\langle \varphi_1, \dots, \varphi_n\rangle \circ \pi_n \circ \boldsymbol{\tau}(\psi) \\ =& \pi_\lambda \circ \langle \boldsymbol{\tau}_*(\varphi_1), \dots, \boldsymbol{\tau}_*(\varphi_n)\rangle \circ \boldsymbol{\tau}(\psi) \\ =& \pi_\lambda \circ \boldsymbol{\tau}_*(\psi(\varphi_1, \dots, \varphi_n)).\end{aligned}$$

This establishes the claim.

Claim 5.3.2. *$\langle \boldsymbol{\tau}, \Theta\rangle$ satisfies condition 1 of Definition 4.2.*

Consider a cardinal λ and equations $\Phi \cup \{\varepsilon \approx \delta\} \subseteq Eq(\mathscr{L}_{\mathsf{X}}, \lambda)$ such that $\Phi \vDash_{\mathsf{X}} \varepsilon \approx \delta$. Define $\mu := |\Phi|$. For the sake of simplicity we identify μ with the set Φ. Then consider the map $\boldsymbol{\tau}_* \colon Tm(\mathscr{L}_{\mathsf{X}}, \lambda) \to Tm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)^\kappa$. Consider also the free algebras $\boldsymbol{Tm}_{\mathsf{X}}(\mu)$ and $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \mu)$ with free generators $\{x_{\alpha\approx\beta} : \alpha \approx \beta \in \Phi\}$ and $\{x^i_{\alpha\approx\beta} : i < \kappa, \alpha \approx \beta \in \Phi\}$ respectively. Then let

$$p_l \colon \boldsymbol{Tm}_{\mathsf{X}}(\mu) \to \boldsymbol{Tm}_{\mathsf{X}}(\lambda) \quad \text{and} \quad q_l \colon \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \mu) \to \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda)$$

be the homomorphisms defined respectively by the following rules:

$$x_{\alpha\approx\beta} \longmapsto \alpha \text{ and } x^i_{\alpha\approx\beta} \longmapsto \boldsymbol{\tau}_*(\alpha)(i).$$

Observe that the following diagram commutes.

$$\begin{tikzcd} \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \mu) \arrow[r, "q_l"] \arrow[d, "\pi_\mu"'] & \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda) \arrow[d, "\pi_\lambda"] \\ \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\mu)) \arrow[r, "\mathcal{F}(p_l)"'] & \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda)) \end{tikzcd} \tag{8}$$

To prove this, it will be enough to show that $\pi_\lambda \circ q_l(x^i_{\alpha\approx\beta}) = \mathcal{F}(p_l) \circ \pi_\mu(x^i_{\alpha\approx\beta})$ for every $i < \kappa$ and $\alpha \approx \beta \in \Phi$. Consider the maps

$$\begin{aligned}\boldsymbol{\tau}_*(x_{\alpha\approx\beta}) \colon& \boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \mu) \\ \boldsymbol{\tau}_*(\alpha) \colon& \boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda) \\ x_{\alpha\approx\beta} \colon& \boldsymbol{Tm}_{\mathsf{X}}(1) \to \boldsymbol{Tm}_{\mathsf{X}}(\mu) \\ \alpha \colon& \boldsymbol{Tm}_{\mathsf{X}}(1) \to \boldsymbol{Tm}_{\mathsf{X}}(\lambda).\end{aligned}$$

Applying Claim 5.3.1 to the 2nd and 5th equalities below, we obtain that

$$\begin{aligned}\mathcal{F}(p_l) \circ \pi_\mu(x^i_{\alpha\approx\beta}) &= \mathcal{F}(p_l) \circ (\pi_\mu \circ \boldsymbol{\tau}_*(x_{\alpha\approx\beta}))(x^i)\\ &= \mathcal{F}(p_l) \circ (\mathcal{F}(x_{\alpha\approx\beta}) \circ \pi_1)(x^i)\\ &= \mathcal{F}(p_l \circ x_{\alpha\approx\beta}) \circ \pi_1(x^i)\\ &= \mathcal{F}(\alpha) \circ \pi_1(x^i)\\ &= \pi_\lambda \circ \boldsymbol{\tau}_*(\alpha)(x^i)\\ &= \pi_\lambda \circ q_l(x^i_{\alpha\approx\beta}).\end{aligned}$$

Thus we conclude that diagram (8) commutes.

Now, observe that we can define two maps p_r and q_r (dual to p_l and q_r) respectively by the rules:

$$x_{\alpha\approx\beta} \longmapsto \beta \;\text{ and }\; x^i_{\alpha\approx\beta} \longmapsto \boldsymbol{\tau}_*(\beta)(i).$$

A reasoning analogous to the one described above yields that $\pi_\lambda \circ q_r = \mathcal{F}(p_r) \circ \pi_\mu$. Hence we showed that

$$\pi_\lambda \circ q_l = \mathcal{F}(p_l) \circ \pi_\mu \;\text{ and }\; \pi_\lambda \circ q_r = \mathcal{F}(p_r) \circ \pi_\mu. \tag{9}$$

Now, let ϕ be the X-congruence of $\boldsymbol{Tm}_{\mathsf{X}}(\lambda)$ generated by Φ. It is clear that π_ϕ is a coequalizer of p_l and p_r. Since $\mathcal{F}$ preserves colimits, this implies that $\mathcal{F}(\pi_\phi)$ is a coequalizer of $\mathcal{F}(p_l)$ and $\mathcal{F}(p_r)$. Keeping in mind that π_μ is surjective, this means that $\mathcal{F}(\pi_\phi)$ is also a coequalizer of $\mathcal{F}(p_l) \circ \pi_\mu$ and $\mathcal{F}(p_r) \circ \pi_\mu$. Finally, with an application of (9), we conclude that $\mathcal{F}(\pi_\phi)$ is a coequalizer of $\pi_\lambda \circ q_l$ and $\pi_\lambda \circ q_r$. In particular, this implies that the kernel of $\mathcal{F}(\pi_\phi) \circ \pi_\lambda$ is the Y-congruence of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda)$ generated by

$$\boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j) \tag{10}$$

where $\vec{x}_j = \langle x^i_j : i < \kappa\rangle$. Now observe that $\pi_\phi \circ \varepsilon = \pi_\phi \circ \delta$, where $\varepsilon, \delta \colon \boldsymbol{Tm}_{\mathsf{X}}(1) \rightrightarrows \boldsymbol{Tm}_{\mathsf{X}}(\lambda)$ since $\langle \varepsilon, \delta\rangle \in \phi$. By Claim 5.3.1 this implies that

$$\begin{aligned}\mathcal{F}(\pi_\phi) \circ \pi_\lambda \circ \boldsymbol{\tau}_*(\varepsilon) &= \mathcal{F}(\pi_\phi) \circ \mathcal{F}(\varepsilon) \circ \pi_1\\ &= \mathcal{F}(\pi_\phi) \circ \mathcal{F}(\delta) \circ \pi_1\\ &= \mathcal{F}(\pi_\phi) \circ \pi_\lambda \circ \boldsymbol{\tau}_*(\delta).\end{aligned}$$

Together with the description of the kernel of $\mathcal{F}(\pi_\phi) \circ \pi_\lambda$ given in (10), this implies that

$$\boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j) \vDash_{\mathsf{Y}} \boldsymbol{\tau}^*(\varepsilon \approx \delta).$$

This establishes Claim 5.3.2.

Claim 5.3.3. *$\langle \boldsymbol{\tau}, \Theta\rangle$ satisfies condition 2 of Definition 4.2.*

Consider an n-ary operation symbol $\psi \in \mathscr{L}_{\mathsf{X}}$ and $\varepsilon \approx \delta \in \Theta$. Claim 5.3.1 and the fact that the kernel of π_1 is the Y-congruence of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)$ generated by Θ imply that

$$\begin{aligned}\pi_n(\varepsilon(\boldsymbol{\tau}_*(\psi)/\vec{x})) &= \pi_n \circ \boldsymbol{\tau}_*(\psi)(\varepsilon) = \mathcal{F}(\psi) \circ \pi_1(\varepsilon) = \mathcal{F}(\psi) \circ \pi_1(\delta)\\ &= \pi_n \circ \boldsymbol{\tau}_*(\psi)(\delta) = \pi_n(\delta(\boldsymbol{\tau}_*(\psi)/\vec{x})).\end{aligned}$$

Since π_n is the kernel of the Y-congruence of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times n)$ generated by $\Theta(\vec{x}_1) \cup \cdots \cup \Theta(\vec{x}_n)$, we conclude that

$$\Theta(\vec{x}_1) \cup \cdots \cup \Theta(\vec{x}_n) \vDash_{\mathsf{Y}} \varepsilon(\boldsymbol{\tau}_*(\psi)/\vec{x}) \approx \delta(\boldsymbol{\tau}_*(\psi)/\vec{x}).$$

This establishes Claim 5.3.3.

Claim 5.3.4. *$\langle \boldsymbol{\tau}, \Theta \rangle$ is a non-trivial contextual translation.*

From Claims 5.3.2 and 5.3.3 it follows that $\langle \boldsymbol{\tau}, \Theta \rangle$ is a contextual translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$. It only remains to prove that $\langle \boldsymbol{\tau}, \Theta \rangle$ is non-trivial. Suppose that there is a tuple $\vec{\varphi} \in Tm(\mathscr{L}_{\mathsf{Y}}, 0)^{\kappa}$ such that $\mathsf{Y} \vDash \Theta(\vec{\varphi})$. Then let $\mathcal{G}$ be the functor right adjoint to $\mathcal{F}$. Since $\mathcal{F}$ is non-trivial, there is $\boldsymbol{A} \in \mathsf{Y}$ such that $\mathcal{G}(\boldsymbol{A})$ is non-trivial. Now observe that the solution set of Θ in $\boldsymbol{A}$ is in bijection with $\mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \boldsymbol{A})$, since $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1))$ is the quotient of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)$ under the Y-congruence generated by Θ. It is easy to see that $\vec{\varphi}^{\boldsymbol{A}}$ a solution of Θ in $\boldsymbol{A}$. Thus $\mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \boldsymbol{A}) \neq \emptyset$. By the hom-set adjunction associated with $\mathcal{F} \dashv \mathcal{G}$ and the universal property of the free 1-generated algebra we have that

$$0 \neq |\,\mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \boldsymbol{A})| = |\,\mathrm{hom}(\boldsymbol{Tm}_{\mathsf{X}}(1), \mathcal{G}(\boldsymbol{A}))| = |\mathcal{G}(\boldsymbol{A})|.$$

Since $\mathcal{G}(\boldsymbol{A})$ is non-trivial, it has at least two elements. Again, this implies that there are two different solutions $\vec{a}, \vec{c} \in A^{\kappa}$ to the equations Θ. In particular, this shows that there is $i < \kappa$ such that

$$\Theta(\vec{x}) \cup \Theta(\vec{y}) \nvDash_{\mathsf{Y}} x^i \approx y^i.$$

This establishes Claim 5.3.4. ⊠

As an exemplification of the construction above, we will describe the contextual translation associated with the adjunction between Kleene algebras and bounded distributive lattices.

Example 5.4 (Kleene Algebras)**.** Let $\mathcal{G}\colon \mathsf{DL}_{01} \to \mathsf{KA}$ be the functor described in Example 3.10. In [10] a functor $\mathcal{F}$ left adjoint to $\mathcal{G}$ is described. Let us briefly recall its behaviour. Given $\boldsymbol{A} \in \mathsf{KA}$, we let $\mathrm{Pr}(\boldsymbol{A})$ be the Priestley space dual to the bounded lattice reduct of $\boldsymbol{A}$ [11]. Moreover, we equip it with a map $g\colon \mathrm{Pr}(\boldsymbol{A}) \to \mathrm{Pr}(\boldsymbol{A})$ defined by the rule

$$g(F) \longmapsto A \smallsetminus \{\neg a : a \in F\}, \text{ with } F \in \mathrm{Pr}(\boldsymbol{A}).$$

Now observe that

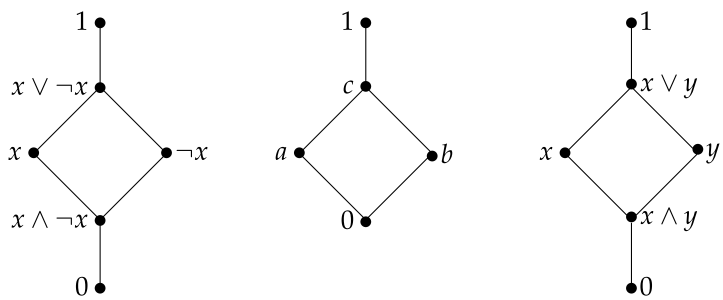
$$\mathrm{Pr}(\boldsymbol{A})^+ := \{F \in \mathrm{Pr}(\boldsymbol{A}) : F \subseteq g(F)\}$$

is the universe of a Priestley subspace of $\mathrm{Pr}(\boldsymbol{A})$. Keeping this in mind, we let $\mathcal{F}(\boldsymbol{A})$ be the bounded distributive lattice dual to $\mathrm{Pr}(\boldsymbol{A})^+$. Moreover, given a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ in KA, we let $\mathcal{F}(f)\colon \mathcal{F}(\boldsymbol{A}) \to \mathcal{F}(\boldsymbol{B})$ be the map defined by the rule

$$U \longmapsto \{F \in \mathrm{Pr}(\boldsymbol{B})^+ : f^{-1}(F) \in U\}, \text{ for each } U \in \mathcal{F}(\boldsymbol{A}).$$

The map $\mathcal{F}\colon \mathsf{KA} \to \mathsf{DL}_{01}$ is the functor left adjoint to $\mathcal{G}$.

Now we turn to describe the contextual translation associated with the adjunction $\mathcal{F} \dashv \mathcal{G}$. To this end, observe that the free Kleene algebra $\boldsymbol{Tm}_{\mathsf{KA}}(1)$, its image $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{KA}}(1))$ in DL_{01} and the free bounded distributive lattice $\boldsymbol{Tm}_{\mathsf{DL}_{01}}(2)$ are respectively the algebras depicted below.



Then let $\pi\colon \boldsymbol{Tm}_{\mathsf{DL}_{01}}(2) \to \mathcal{F}(\boldsymbol{Tm}_{\mathsf{KA}}(1))$ be the unique (surjective) homomorphism determined by the assignment $\pi(x) = a$ and $\pi(y) = b$. Following the general construction described above, we should identify Θ with the kernel of π viewed as a set of equations in 2 variables. But the only equation of this kind that is not vacuously satisfied is $x \wedge y \approx 0$. Hence we can set without loss of generality $\Theta := \{x \wedge y \approx 0\}$.

The description of $\boldsymbol{\tau}$ is more complicated and we will detail it only for the case of negation. First observe that $\neg\colon \boldsymbol{Tm}_{\mathsf{KA}}(1) \to \boldsymbol{Tm}_{\mathsf{KA}}(1)$ is the unique endomorphism that sends x to $\neg x$. Then, applying the definition of $\mathcal{F}$, it is easy to see that $\mathcal{F}(\neg)$ is the endomorphism of $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{KA}}(1))$ that behaves as the identity except that it interchanges a and b. Now we have to choose an endomorphism $\boldsymbol{\tau}(\neg)$ of $\boldsymbol{Tm}_{\mathsf{DL}_{01}}(2)$ such that $\pi \circ \boldsymbol{\tau}(\neg) = \mathcal{F}(\neg) \circ \pi$. It is easy to see that the unique homomorphism $\boldsymbol{\tau}(\neg)$ determined by the assignment $\boldsymbol{\tau}(\neg)(x) = y$ and $\boldsymbol{\tau}(\neg)(y) = x$ fulfils this condition. Hence the translation of $\neg$ consists of the pair $\langle y, x\rangle$. The same idea allows us to extend $\boldsymbol{\tau}$ to the other constant and binary basic symbols of KA as follows:[3]

$$x \sqcap y \longmapsto \langle x^1, x^2\rangle \sqcap \langle y^1, y^2\rangle := \langle x^1 \wedge y^1, x^2 \vee y^2\rangle$$
$$x \sqcup y \longmapsto \langle x^1, x^2\rangle \sqcup \langle y^1, y^2\rangle := \langle x^1 \vee y^1, x^2 \wedge y^2\rangle$$

and

$$\neg x \mapsto \neg\langle x^1, x^2\rangle := \langle x^2, x^1\rangle \qquad 1 \mapsto 1 := \langle 1, 0\rangle \qquad 0 \mapsto 0 := \langle 0, 1\rangle.$$

By Theorem 5.3 the pair $\langle \boldsymbol{\tau}, \Theta\rangle$ is a contextual translation of $\vDash_{\mathsf{KA}}$ into $\vDash_{\mathsf{DL}_{01}}$.

For the reader familiar with the theory of algebraizable logics [8] it may be interesting to observe that this contextual translation is not induced by a translation between two propositional logics (as was the case in Examples 4.3 and 4.4). This is due to the fact that DL_{01} and KA are not the equivalent algebraic semantics of any algebraizable logics. $\boxtimes$

6. Decomposition of right adjoints

In the preceding sections we drew a correspondence between adjunctions and contextual translations, by showing how can we convert one into the other and vice-versa. Now we are ready to present the main outcome of this correspondence, namely the discovery that every every right adjoint functor between generalized

[3] At this stage the reader may find it useful to compare the translation displayed here with the sublanguage $\mathscr{L}$ of the matrix power DL_{01} that we considered in Example 3.10.

quasi-varieties can be decomposed into a combination of two canonical deformations, namely the matrix power with (possibly) infinite exponents and the $\theta_{\mathscr{L}}$ construction. More precisely, we have the following:

Theorem 6.1. *Let X and Y be generalized quasi-varieties.*

1. *For every non-trivial right adjoint $\mathcal{G}\colon \mathsf{Y} \to \mathsf{X}$ there are a generalized quasi-variety K and functors $[\kappa]\colon \mathsf{Y} \to \mathsf{K}$ and $\theta_{\mathscr{L}}\colon \mathsf{K} \to \mathsf{X}$ (where θ is compatible with $\mathscr{L}$ in K) such that $\mathcal{G}$ is naturally isomorphic to $\theta_{\mathscr{L}} \circ [\kappa]$.*
2. *Every functor of the form $\theta_{\mathscr{L}} \circ [\kappa]\colon \mathsf{Y} \to \mathsf{X}$ (where θ is compatible with $\mathscr{L}$ in $\mathsf{Y}^{[\kappa]}$) is a right adjoint.*

Proof. 1. Let $\mathcal{F}$ be the functor left adjoint to $\mathcal{G}$ and let η, ε be the unit and counit of the adjunction respectively. In Theorem 5.3 we showed that $\mathcal{F}$ gives rise to a contextual translation $\langle \boldsymbol{\tau}, \Theta \rangle$ of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$. Then consider the generalized quasi-variety K and the right adjoint functors $[\kappa]\colon \mathsf{Y} \to \mathsf{K}$ and $\theta_{\mathscr{L}}\colon \mathsf{K} \to \mathsf{X}$ associated with $\langle \boldsymbol{\tau}, \Theta \rangle$ as in Theorem 4.5. We will prove that $\mathcal{G}$ and the composition $\theta_{\mathscr{L}} \circ [\kappa]$ are naturally isomorphic.

To this end, it will be convenient to work with some substitutes of $\mathcal{G}$ and $\theta_{\mathscr{L}} \circ [\kappa]$. Let $\mathsf{ALG}_{\mathsf{X}}$ be the category of all algebras of the type of X. Then let $\mathcal{G}^*\colon \mathsf{Y} \to \mathsf{ALG}_{\mathsf{X}}$ be the functor defined by the rule

$$\begin{aligned} \boldsymbol{A} &\longmapsto \mathrm{hom}(\boldsymbol{Tm}_{\mathsf{X}}(1), \mathcal{G}(\boldsymbol{A})) \\ f &\longmapsto \mathcal{G}(f) \circ (\cdot) \end{aligned}$$

for every algebra $\boldsymbol{A}$ and homomorphism f in Y. The operations of the algebra $\mathcal{G}^*(\boldsymbol{A})$ are defined as follows. Given an n-ary operation $\psi \in \mathscr{L}_{\mathsf{X}}$ with corresponding arrow $\psi\colon \boldsymbol{Tm}_{\mathsf{X}}(1) \to \boldsymbol{Tm}_{\mathsf{X}}(n)$, we set

$$\psi^{\mathcal{G}^*(\boldsymbol{A})}(f_1, \dots, f_n) \coloneqq \langle f_1, \dots, f_n \rangle \circ \psi$$

for every $f_1, \dots, f_n \in \mathcal{G}^*(\boldsymbol{A})$. Now observe that the map $\zeta_{\boldsymbol{A}}\colon \mathcal{G}(\boldsymbol{A}) \to \mathcal{G}^*(\boldsymbol{A})$ that takes an element $a \in \mathcal{G}(\boldsymbol{A})$ to the unique arrow $f \in \mathcal{G}^*(\boldsymbol{A})$ such that $f(x) = a$ is an isomorphism for every $\boldsymbol{A} \in \mathsf{Y}$. It is easy to see that the global map $\zeta\colon \mathcal{G} \to \mathcal{G}^*$ is a natural isomorphism between $\mathcal{G}, \mathcal{G}^*\colon \mathsf{Y} \to \mathsf{ALG}_{\mathsf{X}}$. As a consequence, we obtain the following:

Fact 6.1.1. *The map $\mathcal{G}^*$ can be viewed as a functor from Y to X naturally isomorphic to $\mathcal{G}$.*

Then we construct our substitute for $\theta_{\mathscr{L}} \circ [\kappa]$. Consider the functor

$$\mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \cdot)\colon \mathsf{Y} \to \mathsf{ALG}_{\mathsf{X}}.$$

In particular, given $\boldsymbol{A} \in \mathsf{Y}$, the operations on $\mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \boldsymbol{A})$, for short $\mathrm{hom}(\boldsymbol{A})$, are defined as follows:

$$\psi^{\mathrm{hom}(\boldsymbol{A})}(f_1, \dots, f_n) \coloneqq \langle f_1, \dots, f_n \rangle \circ \mathcal{F}(\psi),$$

for every $f_1, \dots, f_n \in \mathrm{hom}(\boldsymbol{A})$. Now, given $\boldsymbol{A} \in \mathsf{Y}$, we consider the map $\sigma_{\boldsymbol{A}}\colon \mathrm{hom}(\boldsymbol{A}) \to \theta_{\mathscr{L}}(\boldsymbol{A}^{[\kappa]})$ defined by the rule

$$f \longmapsto \langle f \circ \pi_1(x^i) : i < \kappa \rangle$$

where $\pi_1\colon \boldsymbol{Tm}_{\mathsf{Y}}(\kappa) \to \mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1))$ is the map defined right before Definition 5.2. Keeping in mind that the kernel of π_1 is the Y-congruence of $\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)$ generated by Θ, it is easy to see that $\sigma_{\boldsymbol{A}}$ is a well-defined bijection. It is an isomorphism too: for $f_1, \ldots, f_n \in \mathrm{hom}(\boldsymbol{A})$, we have that

$$\begin{aligned}
\sigma_{\boldsymbol{A}} \psi^{\mathrm{hom}(\boldsymbol{A})}(f_1, \ldots, f_n) &= \sigma_{\boldsymbol{A}}(\langle f_1, \ldots, f_n \rangle \circ \mathcal{F}(\psi)) \\
&= \langle \langle f_1, \ldots, f_n \rangle \circ \mathcal{F}(\psi) \circ \pi_1(x^i) : i < \kappa \rangle \\
&= \langle \langle f_1, \ldots, f_n \rangle \circ \pi_n \circ \boldsymbol{\tau}(\psi)(x^i) : i < \kappa \rangle \\
&= \psi^{\theta_{\mathscr{L}}(\boldsymbol{A}^{[\kappa]})}(\langle \langle \langle f_1, \ldots, f_n \rangle \circ \pi_n(x^i_j) : i < \kappa \rangle : j < n) \\
&= \psi^{\theta_{\mathscr{L}}(\boldsymbol{A}^{[\kappa]})}(\langle \langle f_j \circ \pi_1(x^i) : i < \kappa \rangle : j < n) \\
&= \psi^{\theta_{\mathscr{L}}(\boldsymbol{A}^{[\kappa]})}(\sigma_{\boldsymbol{A}}(f_1), \ldots, \sigma_{\boldsymbol{A}}(f_n)).
\end{aligned}$$

The third equality above follows from the commutativity of diagram (6). This shows that the global map $\sigma\colon \mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \cdot) \to \theta_{\mathscr{L}} \circ [\kappa]$ is a natural isomorphism between $\mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \cdot), \theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{ALG}_{\mathsf{X}}$. As a consequence we obtain the following:

Fact 6.1.2. *The map* $\mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \cdot)$ *can be viewed as a functor from* Y *to* X *naturally isomorphic to* $\theta_{\mathscr{L}} \circ [\kappa]$.

Thanks to Facts 6.1.1 and 6.1.2, in order to complete the proof it will be enough to construct a natural isomorphism

$$\mu\colon \mathcal{G}^* \to \mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \cdot).$$

This is what we do now. For every $\boldsymbol{A} \in \mathsf{Y}$, the component $\mu_{\boldsymbol{A}}$ of the natural transformation μ is the following map:

$$\varepsilon_{\boldsymbol{A}} \circ \mathcal{F}(\cdot)\colon \mathrm{hom}(\boldsymbol{Tm}_{\mathsf{X}}(1), \mathcal{G}(\boldsymbol{A})) \to \mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \boldsymbol{A}).$$

From the hom-set adjunction associated with $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ it follows that $\mu_{\boldsymbol{A}}$ is a bijection. Then consider $f_1, \ldots, f_n \in \mathrm{hom}(\boldsymbol{Tm}_{\mathsf{X}}(1), \mathcal{G}(\boldsymbol{A}))$. Since $\mathcal{F}$ preserves coproducts, we have that

$$\begin{aligned}
\mu_{\boldsymbol{A}} \psi^{\mathcal{G}^*(\boldsymbol{A})}(f_1, \ldots, f_n) &= \mu_{\boldsymbol{A}}(\langle f_1, \ldots, f_n \rangle \circ \psi) \\
&= \varepsilon_{\boldsymbol{A}} \circ \mathcal{F}(\langle f_1, \ldots, f_n \rangle \circ \psi) \\
&= \varepsilon_{\boldsymbol{A}} \circ \langle \mathcal{F}(f_1), \ldots, \mathcal{F}(f_n) \rangle \circ \mathcal{F}(\psi) \\
&= \langle \varepsilon_{\boldsymbol{A}} \circ \mathcal{F}(f_1), \ldots, \varepsilon_{\boldsymbol{A}} \circ \mathcal{F}(f_n) \rangle \circ \mathcal{F}(\psi) \\
&= \langle \mu_{\boldsymbol{A}}(f_1), \ldots, \mu_{\boldsymbol{A}}(f_n) \rangle \circ \mathcal{F}(\psi) \\
&= \psi^{\mathrm{hom}(\boldsymbol{A})}(\mu_{\boldsymbol{A}}(f_1), \ldots, \mu_{\boldsymbol{A}}(f_n)).
\end{aligned}$$

Therefore we conclude that $\mu_{\boldsymbol{A}}$ is an isomorphism.

It only remains to prove that the global map μ satisfies the commutative condition required of natural transformations. In order to do this, consider any homomorphism $g\colon \boldsymbol{A} \to \boldsymbol{B}$ in Y and an element $f \in \mathcal{G}^*(\boldsymbol{A})$. From the hom-set adjunction associated with $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ it follows that

$$\begin{aligned}
\mathrm{hom}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), g) \circ \mu_{\boldsymbol{A}}(f) &= g \circ \mu_{\boldsymbol{A}}(f) \\
&= \mu_{\boldsymbol{B}}(\mathcal{G}(g) \circ f) \\
&= (\mu_{\boldsymbol{B}} \circ \mathcal{G}^*(g))(f).
\end{aligned}$$

Hence μ is a natural isomorphism as desired.

2. Consider an infinite cardinal λ such that $\mathbb{U}_\lambda(\mathsf{X}) = \mathsf{X}$ and define $\mathsf{K} := \mathbb{GQ}_\lambda(\mathsf{Y}^{[\kappa]})$. Since θ is compatible with $\mathscr{L}$ in $\mathsf{Y}^{[\kappa]}$ and the compatibility condition is expressible by a set of generalized quasi-equations each of which is written with finitely many variables, we conclude that θ is compatible with $\mathscr{L}$ in K too. Moreover, from the fact that $\theta_{\mathscr{L}}(\mathsf{Y}^{[\kappa]}) \subseteq \mathsf{X}$ and $\mathbb{U}_\lambda(\mathsf{X}) = \mathsf{X}$ it follows that the functor $\theta_{\mathscr{L}} \colon \mathsf{K} \to \mathsf{X}$ is well defined. By Theorems 3.5 and 3.8 we know that the maps $[\kappa] \colon \mathsf{Y} \to \mathsf{K}$ and $\theta_{\mathscr{L}} \colon \mathsf{K} \to \mathsf{X}$ are right adjoint functors. As a consequence their composition $\theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{X}$ is also a right adjoint. ⊠

Corollary 6.2. *Let $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ be a non-trivial left adjoint functor between generalized quasi-varieties and $\phi \in \mathrm{Con}_{\mathsf{X}}\boldsymbol{Tm}_{\mathsf{X}}(\lambda)$. Assume that the right adjoint of $\mathcal{F}$ decomposes as $\theta_{\mathscr{L}} \circ [\kappa]$. Then*

$$\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(\lambda)/\phi) \cong \boldsymbol{Tm}_{\mathsf{Y}}(\kappa \times \lambda)/\mathrm{Cg}_{\mathsf{Y}}(\boldsymbol{\tau}^*(\phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j)).$$

Remark 6.3. The description of right adjoints given in Theorem 6.1 can be seen as a purely algebraic formulation of the classical description of adjunctions in categories with a free object, which can be traced back at least to [14].

To see why, suppose that $\mathcal{F} \colon \mathsf{X} \longleftrightarrow \mathsf{Y} \colon \mathcal{G}$ is an adjunction $\mathcal{F} \dashv \mathcal{G}$, and that X and Y are prevarieties. We proceed to sketch the general description of $\mathcal{G}(\boldsymbol{A})$ in [14]. Since X contains free algebras, the universe of the algebra $\mathcal{G}(\boldsymbol{A})$ can be identified with $\mathrm{hom}_{\mathsf{X}}(\boldsymbol{Tm}_{\mathsf{X}}(1), \mathcal{G}(\boldsymbol{A}))$. By the hom-set adjunction induced by $\mathcal{F} \dashv \mathcal{G}$, we know that

$$\mathrm{hom}_{\mathsf{X}}(\boldsymbol{Tm}_{\mathsf{X}}(1), \mathcal{G}(\boldsymbol{A})) \cong \mathrm{hom}_{\mathsf{Y}}(\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)), \boldsymbol{A}).$$

Since Y contains arbitrarily large free algebras, the algebra $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1))$ can be expressed as a suitable quotient of a free algebra, i.e. $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)) \cong \boldsymbol{Tm}_{\mathsf{Y}}(\kappa)/\theta$ for some cardinal κ and some congruence θ. Thus the universe of $\mathcal{G}(\boldsymbol{A})$ can be identified with $\mathrm{hom}_{\mathsf{Y}}(\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)/\theta, \boldsymbol{A})$. More in general $\mathcal{G}(\boldsymbol{A})$ can be identified with the set $\mathrm{hom}_{\mathsf{Y}}(\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)/\theta, \boldsymbol{A})$, equipped with a suitable algebraic structure. This provides a full arrow-theoretic description of the algebra $\mathcal{G}(\boldsymbol{A})$ as

$$\mathcal{G}(\boldsymbol{A}) \cong \mathrm{hom}_{\mathsf{Y}}(\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)/\theta, \boldsymbol{A}).$$

The main contribution of the present work is to recognize that the algebra $\mathrm{hom}_{\mathsf{Y}}(\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)/\theta, \boldsymbol{A})$ in the above display can be given a very transparent description in terms of matrix powers and compatible equations. This is a consequence of the fact that the set $\mathrm{hom}_{\mathsf{Y}}(\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)/\theta, \boldsymbol{A})$ can be identified with the set of solutions of the equations $\theta(\vec{x})$ in $\boldsymbol{A}$, which is exactly the universe of the algebra $\theta_{\mathscr{L}}(\boldsymbol{A}^{[\kappa]})$. Moreover, this identification respect the algebraic structures, yielding an isomorphism

$$\mathrm{hom}_{\mathsf{Y}}(\boldsymbol{Tm}_{\mathsf{Y}}(\kappa)/\theta, \boldsymbol{A}) \cong \theta_{\mathscr{L}}(\boldsymbol{A}^{[\kappa]}).$$

As a consequence the structure of the algebra $\mathcal{G}(\boldsymbol{A})$ can be expressed in purely algebraic and combinatorial terms as $\theta_{\mathscr{L}}(\boldsymbol{A}^{[\kappa]})$. In particular, this description of $\mathcal{G}(\boldsymbol{A})$ was exploited to establish the correspondence between between adjunctions and contextual translations. ⊠

Until now we showed that every right adjoint functor $\mathcal{G} \colon \mathsf{Y} \to \mathsf{X}$ between generalized quasi-varieties induces a contextual translation $\langle \boldsymbol{\tau}, \theta \rangle$ of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$, and vice-versa. In general, contextual translations $\langle \boldsymbol{\tau}, \theta \rangle$ are infinite objects, in the sense that $\boldsymbol{\tau}$ is a map that translates terms into possibly infinite sequences of terms and θ is a possibly infinite set of equations. It is therefore natural to ask under which conditions these contextual translations can be finitized. In other words, we can ask for sufficient and necessary conditions under which a right adjoint functor $\mathcal{G}$ can be canonically decomposed as $\theta_{\mathscr{L}} \circ [\kappa]$ in such a way that $\kappa \in \omega$ and θ is a finite set of equations. The next lemma provides an answer in the case where X and Y are quasi-varieties.

Lemma 6.4. *Let $\mathcal{F} \colon \mathsf{X} \longleftrightarrow \mathsf{Y} \colon \mathcal{G}$ be an adjunction $\mathcal{F} \dashv \mathcal{G}$ between quasi-varieties. The following conditions are equivalent:*

(i) *$\mathcal{F}$ preserves finitely presentable algebras.*
(ii) *$\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1))$ is finitely presentable.*
(iii) *$\mathcal{G}$ preserves directed colimits.*
(iv) *$\mathcal{G}$ can be decomposed as $\theta_{\mathscr{L}} \circ [\kappa]$ with both κ and θ finite.*

Proof. The equivalence between (i) and (iii) is well known, and is a consequence of the fact that the finitely X-presentable algebras are exactly the algebras $\boldsymbol{A} \in \mathsf{X}$ for which the functor $\mathrm{hom}(\boldsymbol{A}, \cdot) \colon \boldsymbol{A} \to \mathsf{Set}$ preserves directed colimits (see Lemma 2.1). Part (i)⇒(ii) is trivial and part (iv)⇒(i) is a consequence of Corollary 6.2.

(ii)⇒(iv): Assume that $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1))$ is finitely presentable. Then there are $n \in \omega$ and a compact Y-congruence Θ such that $\mathcal{F}(\boldsymbol{Tm}_{\mathsf{X}}(1)) = \boldsymbol{Tm}_{\mathsf{Y}}(n)/\Theta$. Now, Θ is generated by a finite set $\Phi \subseteq \Theta$. This means that $\mathcal{G}$ can be decomposed as $\theta_{\mathscr{L}} \circ [n]$, where $\theta \coloneqq \{\vec{\varepsilon} \approx \vec{\delta} : \langle \varepsilon, \delta \rangle \in \Phi\}$ and $\vec{\varepsilon}, \vec{\delta}$ are sequences of length n. ⊠

Remark 6.5. Even if we do not pursue the details here, it is interesting to observe that if $\mathcal{G} \colon \mathsf{Y} \to \mathsf{X}$ is a right adjoint between quasi-varieties satisfying any of the equivalent conditions in the above lemma, then $\mathcal{G}$ is indeed induced by a *model theoretic* interpretation of the language of X into the language of Y, see [17, Chapter 5] for the relevant definitions. ⊠

The next example shows that there are adjunctions between quasi-varieties that do not meet the equivalent conditions of Lemma 6.4. In other words, it shows that there are contextual translations between finitary relative equational consequences that cannot be finitized.

Example 6.6 (Ring Hom-Functor)**.** Consider a generalized quasi-variety X and an algebra $\boldsymbol{A} \in \mathsf{X}$. Then let $\mathrm{hom}(\boldsymbol{A}, \cdot) \colon \mathsf{X} \to \mathsf{Set}$ be the functor defined by the following rule:

$$\boldsymbol{B} \longmapsto \mathrm{hom}(\boldsymbol{A}, \boldsymbol{B})$$
$$f \colon \boldsymbol{B} \to \boldsymbol{C} \longmapsto f \circ (\cdot) \colon \mathrm{hom}(\boldsymbol{A}, \boldsymbol{B}) \to \mathrm{hom}(\boldsymbol{A}, \boldsymbol{C}).$$

The functor $\mathrm{hom}(\boldsymbol{A}, \cdot)$ has a left adjoint $\mathcal{F} \colon \mathsf{Set} \to \mathsf{X}$ defined as follows. Given a set I, the algebra $\mathcal{F}(I)$ is the copower of $\boldsymbol{A}$ indexed by I. Moreover, given a function $f \colon I \to J$ between sets, we let $\mathcal{F}(f) \colon \mathcal{F}(I) \to \mathcal{F}(J)$ be the map $\langle p_{f(i)} : i \in I \rangle$ induced by the universal propery of the coproduct $\mathcal{F}(I)$, where $\{p_j \colon \boldsymbol{A} \to \mathcal{F}(J) : j \in J\}$ are the maps associated with the copower $\mathcal{F}(J)$.

Now consider the special case where X is the variety R of commutative rings with unit. Then consider the functor $\mathcal{F}$ that is left adjoint to $\mathrm{hom}(\boldsymbol{Q}, \cdot) \colon \mathsf{R} \to \mathsf{Set}$, where $\boldsymbol{Q}$ is the ring of rational numbers. First observe that $\mathcal{F}$ does not preserve finitely generated algebras. Observe that finitely generated algebras are exactly the quotients of the finitely presentable ones. Since $\mathcal{F}$ preserves surjective homomorphisms, we conclude that it does not preserve finitely presentable algebras. $\boxtimes$

In this paper contextual translations have been presented as translations between *relative equational consequences*. Nevertheless, the contextual translations coming from some of the motivating examples (such as Gödel and Kolmogorov's ones) originated as translations between *propositional logics*. More precisely, it is a general fact that if two generalized quasi-varieties X and Y are the equivalent algebraic semantics of two algebraizable logics $\mathcal{L}$ and $\mathcal{L}'$ in the sense of [8], then every contextual translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$ can be viewed as translation of $\mathcal{L}$ into $\mathcal{L}'$. In this sense, contextual translation may provide a useful notion of translation between algebraizable logics (possibly in different languages).

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