Assertional logics, truth-equational logics, and the hierarchies of abstract algebraic logic

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Dedicated to Professor Don Pigozzi
on the occasion of his 80th birthday

Abstract We establish some relations between the class of truth-equational logics, the class of assertional logics, other classes in the Leibniz hierarchy, and the classes in the Frege hierarchy. We argue that the class of assertional logics belongs properly in the Leibniz hierarchy. We give two new characterizations of truth-equational logics in terms of their full generalized models, and use them to obtain further results on the internal structure of the Frege hierarchy and on the relations between the two hierarchies. Some of these results and several counterexamples contribute to answer a few open problems in abstract algebraic logic, and open a new one.

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1 Introduction

The contribution of Don Pigozzi to the recent evolution of algebraic logic is enormous. He has shaped the field in such a way that the community has adopted a special name, *abstract algebraic logic*, promoted by him,[1] to refer to the area of algebraic logic that studies in abstract mathematical terms the very process of algebraization of logics, associates with every logic an algebraic counterpart, relates properties of a logic with properties of its algebraic counterpart, and classifies logics according to the type of relation they enjoy with it; all this, with the purpose that once one knows where a logic fits in the classification, the application of the theory built around the classification criteria and their consequences can immediately reveal many of its properties.

Don's fundamental work was developed mostly, but not exclusively, in his long standing collaborations with Willem J. Blok and with Janusz Czelakowski. At its center we find the construction of an impressive edifice, the so-called *Leibniz hierarchy* (Blok and Pigozzi, 1992), based on the notions of *algebraizable logic* (Blok and Pigozzi, 1989) and of *protoalgebraic logic* (Blok and Pigozzi, 1986). Other scholars (such as Janusz Czelakowski, Burghard Herrmann, Ramon Jansana, James Raftery) have also contributed to the enlargement and further study of this hierarchy. The latest addition to the Leibniz hierarchy is the class of *truth-equational logics*, characterized in Raftery (2006); up to now, it is the only class in this hierarchy not contained in the class of protoalgebraic logics (but see below). Don also laid the foundations (Pigozzi, 1991) of the study with algebraic logic tools of the distinction between Fregean and non-Fregean logics (due to Roman Suszko); this gave rise to the technical notion of *Fregean logic* and later on to the construction of a simpler hierarchy, the *Frege hierarchy*,[2] where logics are classified according to replacement properties they (or their models) satisfy. We adress the reader to Czelakowski (2001) for more information on the Leibniz hierarchy, and to Font (2015, 2016) for both hierarchies.

One of the goals of the present paper is to contend that the (already well-known) class of *assertional logics* (also called "1-assertional" in the literature) should be counted among those in the Leibniz hierarchy (notice that it is not included in the class of protoalgebraic logics). We also study the relations between this class and that of truth-equational logics, and between

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[1] In this Don followed a suggestion of Hajnal Andréka and István Németi, who first used the term, in Section 5.3 of Henkin, Monk, and Tarski (1985), for their abstract model theoretic approach to the algebraization of first-order logic. It was first applied in the present sense (i.e., to the study of sentential logics) in the *Workshop on abstract algebraic logic* organized in Barcelona, under Don's chairmanship, in 1997. In the 2010 version of the *Mathematics Subject Classification*, the term appears with code 03G27. Notice that these initial paragraphs of our Introduction just intend to put the paper in context; a complete exposition of Don's work is found elsewhere in this volume.

[2] See the detailed references given after Definition 12.

each of these and the classes in the Frege hierarchy, in order to further clarify the internal structure of this hierarchy and some relations between the two hierarchies. In particular, we show that the Frege hierarchy becomes considerably simplified inside large portions of the Leibniz hierarchy. Our results, and the construction of some *ad hoc* counterexamples, allow us to answer several open problems on these issues; one of these answers is only partial and opens another new problem.

The structure of the paper is as follows. After summarizing the indispensable preliminaries in Section 2, we introduce assertional logics and truth-equational logics in Section 3. The analysis of several characterizations of the former among the latter supports our claim that the class of assertional logics should be considered as belonging to the Leibniz hierarchy, as it can be characterized by conditions formulated purely in terms of the Leibniz congruence, in the same way as, say, regularly algebraizable logics are characterized among the algebraizable ones. We see that the class of truth-equational logics occupies an intermediate position between the class of assertional logics and the class of logics having an algebraic semantics (the latter not belonging to the Leibniz hierarchy). Then, in Section 4 we introduce the fundamental notion of full generalized model of a logic, present the Frege hierarchy, and establish that Fregean logics with theorems are all assertional, and hence truth-equational, and that for a fully selfextensional logic, to be assertional is the same as to be truth-equational. In Section 5 we give two characterizations, of independent interest, of truth-equational logics in terms of their full generalized models. In Section 6, using these characterizations and the appropriate counterexamples, we prove that for truth-equational logics the Frege hierarchy reduces to exactly three classes, and that for finitary weakly algebraizable logics it reduces to two. Finally, we combine our results in order to answer several open problems on the structure of the Frege hierarchy posed in Font and Jansana (1996) and Font (2003, 2006): we prove that the class of selfextensional logics is not the union of the classes of Fregean and fully selfextensional logics, that there are finitely regularly algebraizable logics that are selfextensional but not fully selfextensional, and that for logics with theorems the class of fully Fregean logics is the intersection of the classes of Fregean logics and of fully selfextensional logics. This last result opens a new problem, that of whether the assumption that the logic has theorems can be deleted from it.

2 Preliminaries

We assume the reader is acquainted with the standard notions, terminology and notations of abstract algebraic logic, as given for instance in Blok and Pigozzi (1989); Czelakowski (2001); Font (2015, 2016); Font and Jansana

(1996); Font, Jansana, and Pigozzi (2003); Raftery (2006); Wójcicki (1988). We recall here just the most central to the paper.

All logics and all algebras we deal with are assumed to share an arbitrary but fixed algebraic language. A (sentential) logic $\mathcal{L}$ is identified with its consequence relation $\vdash_{\mathcal{L}}$.

Three kinds of algebra-based structures play a rôle in this area as models of logics: just plain ***algebras*** (denoted by $\mathbf{A}, \mathbf{B}$, etc., with universes A, B, resp.), ***matrices*** in the usual sense (i.e., pairs $\langle \mathbf{A}, F \rangle$ where $F \subseteq A$), and ***generalized matrices*** (***g-matrices*** for short), which are pairs $\langle \mathbf{A}, \mathcal{C} \rangle$ where $\mathcal{C}$ is a closure system of subsets of A. If $\mathbf{A}$ is an algebra, the set of all the ***$\mathcal{L}$-filters*** of $\mathbf{A}$ is a closure system and is denoted by $\mathcal{F}i_{\mathcal{L}}\mathbf{A}$. A matrix $\langle \mathbf{A}, F \rangle$ is a ***model*** of a logic $\mathcal{L}$ when $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, and a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is a ***generalized model*** (***g-model*** for short) of $\mathcal{L}$ when $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{L}}\mathbf{A}$. Thus, the largest g-model of $\mathcal{L}$ on $\mathbf{A}$ is the g-matrix $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{L}}\mathbf{A} \rangle$.

Given an algebra $\mathbf{A}$ and a subset F of its universe A, the ***Leibniz congruence*** of F, denoted by $\varOmega^{\mathbf{A}} F$, is the largest congruence of $\mathbf{A}$ that is compatible with F in the sense that it does not identify elements in F with elements of A not in F. Note that this congruence is a purely algebraic object and does not depend on any logic. However, when studying a sentential logic $\mathcal{L}$, the term ***Leibniz operator*** on $\mathbf{A}$ refers to the map $F \mapsto \varOmega^{\mathbf{A}} F$ restricted to $\mathcal{F}i_{\mathcal{L}}\mathbf{A}$. Several classes of logics with particularly well-behaved matrix semantics can be characterized in terms of properties of this operator, constituting the so-called ***Leibniz hierarchy*** (the part of this hierarchy relevant for the paper is depicted in Figure 1 on page 10). A matrix is ***reduced*** when its Leibniz congruence is the identity relation. The class of reduced models of $\mathcal{L}$ is denoted by $\mathsf{Mod}^*\mathcal{L}$, and the class of its algebraic reducts by $\mathsf{Alg}^*\mathcal{L}$. This class of algebras was classically taken to be the most natural algebraic counterpart of the logic $\mathcal{L}$, but in Font and Jansana (1996) it was shown that this may not be the case for some non-protoalgebraic logics, and that a more general algebra-based semantics where generalized matrices replace ordinary matrices seems to yield better results. To introduce it we need a few more definitions.

If $\mathcal{C}$ is a closure system over the universe A of an algebra $\mathbf{A}$, its ***Tarski congruence*** is $\widetilde{\varOmega}^{\mathbf{A}} \mathcal{C} := \bigcap \{ \varOmega^{\mathbf{A}} F : F \in \mathcal{C} \}$; it is the largest congruence of $\mathbf{A}$ compatible with all $F \in \mathcal{C}$. The map $\mathcal{C} \longmapsto \widetilde{\varOmega}^{\mathbf{A}} \mathcal{C}$ is called the ***Tarski operator*** on $\mathbf{A}$. The ***reduction*** of a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is the result of factoring it out by its Tarski congruence, that is, the quotient g-matrix $\langle \mathbf{A}/\widetilde{\varOmega}^{\mathbf{A}} \mathcal{C}, \mathcal{C}/\widetilde{\varOmega}^{\mathbf{A}} \mathcal{C} \rangle$. A g-matrix is a g-model of a logic if and only if its reduction is. A g-matrix is ***reduced*** when its Tarski congruence is the identity relation (obviously, the reduction of a g-matrix is always reduced).

The class $\mathsf{Alg}\mathcal{L}$ is defined as the class of the algebraic reducts of the reduced g-models of a logic $\mathcal{L}$. This class of algebras provides another algebraic counterpart of a logic that is useful for all logics, even if they are not protoalgebraic. Moreover, when $\mathcal{L}$ is protoalgebraic, $\mathsf{Alg}\mathcal{L} = \mathsf{Alg}^*\mathcal{L}$. In particular, if $\mathcal{L}$ is algebraizable, then $\mathsf{Alg}\mathcal{L}$ coincides with its largest equivalent algebraic semantics introduced by Blok and Pigozzi (1989), and when $\mathcal{L}$ is implicative, $\mathsf{Alg}\mathcal{L}$ coincides with the class of $\mathcal{L}$-algebras as defined by Rasiowa (1974).

A simple construction that plays an important rôle in the paper is the following. If $\mathcal{C}$ is a closure system, for each $F \in \mathcal{C}$ we consider the closure system $\mathcal{C}^F := \{G \in \mathcal{C} : F \subseteq G\}$. Hence, each g-model $\langle \mathbf{A}, \mathcal{C} \rangle$ of a logic $\mathcal{L}$ gives rise to a family of g-models of the form $\langle \mathbf{A}, \mathcal{C}^F \rangle$, one for each for $F \in \mathcal{C}$. In particular, from the largest g-model $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{L}}\mathbf{A} \rangle$ we obtain a g-model of the form $\langle \mathbf{A}, (\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F \rangle$ for each $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$.

If $\langle \mathbf{A}, \mathcal{C} \rangle$ is a g-matrix and $F \in \mathcal{C}$, the ***Suszko congruence*** of F (relative to $\mathcal{C}$) is $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{C}} F := \widetilde{\Omega}^{\mathbf{A}} \mathcal{C}^F = \bigcap \{ \Omega^{\mathbf{A}} G : G \in \mathcal{C}, F \subseteq G \}$. This notion was formally introduced[3] by Czelakowski (2003), in the special case where $\mathcal{C} = \mathcal{F}i_{\mathcal{L}}\mathbf{A}$; in this case, since the relativization is actually determined by $\mathcal{L}$, it makes sense to use the symbol $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}} F$ instead of the more complicated $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{F}i_{\mathcal{L}}\mathbf{A}} F$, when $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, and therefore

$$\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}} F := \widetilde{\Omega}^{\mathbf{A}} (\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F = \bigcap \{ \Omega^{\mathbf{A}} G : G \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}, F \subseteq G \}. \tag{1}$$

A model $\langle \mathbf{A}, F \rangle$ of a logic $\mathcal{L}$ is ***Suszko-reduced*** when its Suszko congruence $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}} F$ relative to $\mathcal{L}$ is the identity relation. The class of Suszko-reduced models of $\mathcal{L}$ is denoted by $\mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$. The class of algebraic reducts of the matrices in $\mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$ turns out to be the class $\mathsf{Alg}\mathcal{L}$; this fact reinforces the relevance of this class as a universal algebraic counterpart of a logic.

The map given by $F \longmapsto \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}} F$ defined on $\mathcal{F}i_{\mathcal{L}}\mathbf{A}$ is called ***the Suszko operator*** (relative to $\mathcal{L}$) on $\mathbf{A}$. A logic is protoalgebraic if and only if the Suszko operator (relative to it) and the Leibniz operator, both on the formula algebra, coincide on its theories (Czelakowski, 2001, Theorem 1.5.4); or, equivalently, if and only if the two operators coincide on the filters of the logic on arbitrary algebras (Czelakowski, 2003, Theorem 1.10). Thus, it seems that the specific properties of the Suszko operator should be particularly relevant for algebraic studies of logics where protoalgebraicity is not assumed; for instance, it is one of the key tools in Raftery's study of truth-equational logics (Raftery, 2006). The paper by Albuquerque et al. (2016) studies a common framework that encompasses both the Leibniz and the Suszko operators, and obtains characterizations of several classes in the Leibniz hierarchy in terms of properties of the Suszko operator.

Each closure system $\mathcal{C}$ on a set A has an associated closure operator C over A, defined as $CX := \bigcap \{ F \in \mathcal{C} : X \subseteq F \}$ for all $X \subseteq A$. Using it we can define ***the Frege relation*** of a closure system $\mathcal{C}$ as $\Lambda\mathcal{C} := \big\{ \langle a, b \rangle \in A \times A : C\{a\} = C\{b\} \big\}$; notice that $\langle a, b \rangle \in \Lambda\mathcal{C}$ if and only if a and b belong to the same members of $\mathcal{C}$. This defines ***the Frege operator*** (relative to $\mathcal{C}$) as the map $F \longmapsto \Lambda_{\mathcal{C}} F := \Lambda\mathcal{C}^F$, for $F \in \mathcal{C}$. The relations $\Lambda\mathcal{C}$ and $\Lambda_{\mathcal{C}} F$ are equivalence relations, but not necessarily congruences; it turns out that the

[3] Czelakowski attributes its invention and first characterization to Suszko, in unpublished lectures.

largest congruence of $\mathbf{A}$ below the Frege relation $\Lambda\mathcal{C}$ is the Tarski congruence $\widetilde{\Omega}^{\mathbf{A}}\mathcal{C}$, and the largest congruence of $\mathbf{A}$ below $\Lambda_{\mathcal{C}}F$ is the Suszko congruence $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{C}}F$.

The set $Th\mathcal{L}$ of theories of a logic $\mathcal{L}$ is a closure system, and so we can always view a logic as the g-matrix $\langle \mathbf{Fm}, Th\mathcal{L}\rangle$; the associated closure operator will be denoted by $C_{\mathcal{L}}$. Moreover, $\mathcal{F}i_{\mathcal{L}}\mathbf{Fm} = Th\mathcal{L}$. Thus, all the above definitions and constructions given for arbitrary g-matrices can in particular be given for a logic. In this case, the superscript that would correspond to the formula algebra will be omitted; thus, on the set of theories of $\mathcal{L}$ we have the Leibniz operator Ω and the Suszko operator $\widetilde{\Omega}_{\mathcal{L}}$, and we can also consider the Tarski operator $\widetilde{\Omega}$ on closure systems of theories. The Frege relation and operator relative to $Th\mathcal{L}$ are denoted by $\Lambda\mathcal{L}$ and $\Lambda_{\mathcal{L}}$ instead of $\Lambda Th\mathcal{L}$ and $\Lambda_{Th\mathcal{L}}$, respectively. The relation $\Lambda\mathcal{L}$ is also denoted by $\dashv\vdash_{\mathcal{L}}$, as it is simply the relation of interderivability with respect to the logic $\mathcal{L}$. As established in general, note that $\widetilde{\Omega}\mathcal{L}$ is the largest congruence below $\Lambda\mathcal{L}$.

So far we have recalled the definition of two classes of algebras associated with each logic $\mathcal{L}$, namely $\mathsf{Alg}^*\mathcal{L}$ and $\mathsf{Alg}\mathcal{L}$. A third class it is useful to consider, called the ***intrinsic variety*** of $\mathcal{L}$, is defined as $\mathbb{V}\mathcal{L} := \mathbb{V}(\mathbf{Fm}/\widetilde{\Omega}\mathcal{L})$, where for a class K of algebras the variety it generates is denoted by $\mathbb{V}(\mathsf{K})$. Since the congruence $\widetilde{\Omega}\mathcal{L}$ is fully invariant, it follows that $\mathbb{V}\mathcal{L} \vDash \alpha \approx \beta$ if and only if $\langle \alpha, \beta\rangle \in \widetilde{\Omega}\mathcal{L}$. The following facts about the three classes will be relevant to the paper:

$$\mathsf{Alg}^*\mathcal{L} \subseteq \mathsf{Alg}\mathcal{L} \subseteq \mathbb{V}\mathcal{L} \qquad \mathbb{V}(\mathsf{Alg}^*\mathcal{L}) = \mathbb{V}(\mathsf{Alg}\mathcal{L}) = \mathbb{V}\mathcal{L}.$$

An interesting fact we will need is the following.

Lemma 1. *Let $\mathcal{L}$ be a logic complete with respect to a class of (g-)matrices with the class K of algebras as algebraic reducts. For all $\alpha, \beta \in Fm$, if $\mathsf{K} \vDash \alpha \approx \beta$, then $\alpha \dashv\vdash_{\mathcal{L}} \beta$. As a consequence, $\mathbb{V}\mathcal{L} \subseteq \mathbb{V}(\mathsf{K})$, and hence both $\mathsf{Alg}^*\mathcal{L}$ and $\mathsf{Alg}\mathcal{L}$ are included in the variety generated by K.*

Proof. Assume that $\mathsf{K} \vDash \alpha \approx \beta$; this means that for any $\mathbf{A} \in \mathsf{K}$ and any $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $h\alpha = h\beta$. In particular, for any matrix $\langle \mathbf{A}, F\rangle$ in the class, $h\alpha \in F$ if and only if $h\beta \in F$. The completeness of $\mathcal{L}$ with respect to the class of matrices implies that $\alpha \dashv\vdash_{\mathcal{L}} \beta$. The case of g-matrices is proved similarly. That is, $\big\{\langle \alpha, \beta\rangle \in Fm \times Fm : \mathsf{K} \vDash \alpha \approx \beta\big\} \subseteq \Lambda\mathcal{L}$. But since the set is clearly a congruence of the formula algebra, this fact implies that $\big\{\langle \alpha, \beta\rangle \in Fm \times Fm : \mathsf{K} \vDash \alpha \approx \beta\big\} \subseteq \widetilde{\Omega}\mathcal{L}$. This shows that $\mathbb{V}\mathcal{L} \subseteq \mathbb{V}(\mathsf{K})$, and the fact that $\mathsf{Alg}^*\mathcal{L}$ and $\mathsf{Alg}\mathcal{L}$ generate the variety $\mathbb{V}\mathcal{L}$ proves the final assertion. ◻

In practice, this may give interesting and workable information for a logic that is *defined* from a single (g-)matrix, or a small set of (g-)matrices: the equations that hold in the algebraic reducts of the defining (g-)matrices also hold in the three classes of algebras associated with the logic (see Example 23 for an application).

3 Assertional logics and truth-equational logics

Several classes of logics, of different strength, can be defined by considering how *the truth filter* in their matrix models (i.e., the set F of the matrices $\langle \mathbf{A}, F \rangle$ that are models of $\mathcal{L}$) is determined, and by their relation to the relative equational consequences of classes of algebras. To introduce them we need some further notation and terminology.

Equations are identified with pairs of formulas, which are conventionally denoted by $\alpha \approx \beta$ instead of $\langle \alpha, \beta \rangle$. Any set $\boldsymbol{\tau}(x)$ of equations in at most one variable x induces a map, denoted also by $\boldsymbol{\tau}$, that transforms (sets of) formulas into sets of equations; it is defined by putting $\boldsymbol{\tau}\varphi \coloneqq \boldsymbol{\tau}(\varphi)$ for any $\varphi \in Fm$, and $\boldsymbol{\tau}\Gamma \coloneqq \bigcup \{ \boldsymbol{\tau}\varphi : \varphi \in \Gamma \}$ for any $\Gamma \subseteq Fm$. Then, for any algebra $\mathbf{A}$ we consider the set of "solutions" of the equations in $\boldsymbol{\tau}(x)$,

$$\begin{aligned} \boldsymbol{\tau}\mathbf{A} &\coloneqq \{ a \in A : \mathbf{A} \vDash \boldsymbol{\tau}(x) \; [\![a]\!] \} \\ &= \{ a \in A : \delta^{\mathbf{A}}(a) = \varepsilon^{\mathbf{A}}(a) \ \ \text{for all } \delta \approx \varepsilon \in \boldsymbol{\tau}(x) \}, \end{aligned}$$

and for each $a \in A$ we put

$$\boldsymbol{\tau}^{\mathbf{A}}(a) \coloneqq \{ \langle \delta^{\mathbf{A}}(a), \varepsilon^{\mathbf{A}}(a) \rangle : \delta \approx \varepsilon \in \boldsymbol{\tau}(x) \} \subseteq A \times A.$$

It is interesting to notice that $a \in \boldsymbol{\tau}\mathbf{A}$ if and only if $\boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq \mathrm{Id}_A$, the identity relation on A.

Definition 2. Let $\langle \mathbf{A}, F \rangle$ be a matrix, M a class of matrices, and $\boldsymbol{\tau}(x)$ a set of equations.

- $\boldsymbol{\tau}$ ***defines the set*** F, or ***defines truth in*** $\langle \mathbf{A}, F \rangle$, when $F = \boldsymbol{\tau}\mathbf{A}$; i.e., when for any $a \in A$, $a \in F$ if and only if $\mathbf{A} \vDash \boldsymbol{\tau}(x) \; [\![a]\!]$, i.e., if and only if $\boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq \mathrm{Id}_A$.
- $\boldsymbol{\tau}$ ***defines truth in*** M when it defines truth in all the matrices in M.
- ***Truth is equationally definable in*** M when there is a set of equations $\boldsymbol{\tau}(x)$ that defines truth in M.

In all these cases, the equations in the set $\boldsymbol{\tau}(x)$ are called the ***defining equations***.

Note that when this happens, for each algebra $\mathbf{A}$ there can be at most one subset F of A such that $\langle \mathbf{A}, F \rangle \in \mathsf{M}$; this (in general weaker) property is called in the literature the ***implicit definability*** of truth in M.

We are particularly interested in logics that have a complete matrix semantics where truth is equationally definable. These logics can be alternatively (and more intuitively) described with the help of the ***equational consequence relative to a class of algebras*** K. This is a closure relation $\vDash_{\mathsf{K}}$ on the set of equations, defined as follows. For any set Θ of equations and any equation $\delta \approx \varepsilon$,

$$\begin{aligned} \Theta \vDash_{\mathsf{K}} \delta \approx \varepsilon \iff & \text{ for every } \mathbf{A} \in \mathsf{K} \text{ and every } h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}), \\ & \text{ if } \mathbf{A} \vDash \alpha \approx \beta \, [\![h]\!] \text{ for all } \alpha \approx \beta \in \Theta, \text{ then } \mathbf{A} \vDash \delta \approx \varepsilon \, [\![h]\!]. \end{aligned}$$

With this definition, the following fact is easy to prove.

Lemma 3. *A logic $\mathcal{L}$ is complete with respect to some class M of matrices where truth is equationally definable if and only if there is a class K of algebras and a set of equations $\boldsymbol{\tau}(x)$ such that for all $\Gamma \cup \{\varphi\} \subseteq Fm$,*

$$\Gamma \vdash_{\mathcal{L}} \varphi \iff \boldsymbol{\tau}\Gamma \vDash_{\mathsf{K}} \boldsymbol{\tau}\varphi.$$

Proof. For one direction, take K as the class of algebraic reducts of M; for the other, take $\mathsf{M} := \big\{\langle \mathbf{A}, \boldsymbol{\tau}\mathbf{A}\rangle : \mathbf{A} \in \mathsf{K}\big\}$. □

When the situation is as in the lemma, the class K is called an ***algebraic semantics*** for the logic $\mathcal{L}$.

A special kind of logics having an algebraic semantics correspond to those where there is a single defining equation with a particular and simple form, which we proceed to describe. A class of algebras is ***pointed*** when there is a term that is constant in the class. This term, usually denoted by $\top$, can be a primitive constant of the language, or be made up from primitive constants, or be a term with variables such that in the algebras of the class, all interpretations give it the same value; in this second case, it can safely be assumed that the term has only the variable x. So, we assume that $\top$ is a term with at most the variable x, and will occasionally write $\top(x)$ to emphasize this fact.

Now we can introduce the first main concept studied in the paper.

Definition 4. Let K be a pointed class of algebras, with $\top$ as the corresponding constant term. The ***assertional logic of*** K is the logic $\mathcal{L}$ determined by the following condition: for all $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash_{\mathcal{L}} \varphi \iff \{\gamma \approx \top : \gamma \in \Gamma\} \vDash_{\mathsf{K}} \varphi \approx \top.$$

A logic $\mathcal{L}$ is an ***assertional logic*** when it is the assertional logic of some pointed class of algebras.

In other words, $\mathcal{L}$ is the assertional logic of K if and only if $\mathcal{L}$ has K as an algebraic semantics with $x \approx \top$ as defining equation, and if and only if $\mathcal{L}$ is complete with respect to the class of matrices $\big\{\langle \mathbf{A}, \{\top^{\mathbf{A}}\}\rangle : \mathbf{A} \in \mathsf{K}\big\}$. We will see that the simplicity of the equation entails strong properties, not shared by logics having algebraic semantics with arbitrary defining equations.[4] Note that the constant term $\top$ must be a theorem of any assertional logic, because $\top \approx \top$ obviously holds in K. Note also that if a class of algebras is pointed,

[4] A logic having K as algebraic semantics with defining equations $\boldsymbol{\tau}$ is also called "the $\boldsymbol{\tau}$-assertional logic of K" in the literature; in such a case, the term "1-assertional" is used for our "assertional". In the present paper we will not need this more general terminology.

then the variety it generates is pointed as well. Using Lemma 1, we can obtain the following fact, which is also of interest.

Lemma 5. *If $\mathcal{L}$ is the assertional logic of a pointed class of algebras* K*, then* $\mathbb{V}\mathcal{L} \subseteq \mathbb{V}(\mathsf{K})$*. As a consequence, the classes* $\mathsf{Alg}^*\mathcal{L}$ *and* $\mathsf{Alg}\mathcal{L}$ *are also pointed, with the same constant term as* K*.* □

Now we can introduce the second main concept studied in the paper.

Definition 6. A logic $\mathcal{L}$ is ***truth-equational*** when truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$.

The notion of truth-equational logic was studied by Raftery (2006); the above definition, which is more convenient for the present paper, is actually an equivalent characterization, which follows from Theorem 25 of Raftery (2006). Raftery proved that truth-equational logics need not be protoalgebraic, but nevertheless they can be characterized by properties of the Leibniz operator (see Theorem 18 below), and hence they belong in the Leibniz hierarchy.

It is clear from the definition, by the completeness of $\mathcal{L}$ with respect to the class $\mathsf{Mod}^*\mathcal{L}$, that if $\mathcal{L}$ is truth-equational, then it has an algebraic semantics, namely the class $\mathsf{Alg}^*\mathcal{L}$. Note that the converse is not true, as witnessed by Example 1 of Raftery (2006), but it is so when $\boldsymbol{\tau}$ has the form $x \approx \top$, with $\top$ a constant term of $\mathsf{Alg}^*\mathcal{L}$; this fact is contained in the following characterization, essentially due to Raftery, of the assertional logics as a subclass of truth-equational logics.

Theorem 7. *For any logic $\mathcal{L}$ the following conditions are equivalent:*

- **(i)** *$\mathcal{L}$ is an assertional logic.*
- **(ii)** *$\mathcal{L}$ is truth-equational, with a truth definition of the form $x \approx \top$, where $\top$ is a constant term of* $\mathsf{Alg}^*\mathcal{L}$ *or, equivalently, of* $\mathsf{Alg}\mathcal{L}$*.*
- **(iii)** *$\mathcal{L}$ has* $\mathsf{Alg}^*\mathcal{L}$ *as an algebraic semantics with $x \approx \top$ as defining equation, where $\top$ is a constant term of* $\mathsf{Alg}^*\mathcal{L}$*.*
- **(iv)** *$\mathcal{L}$ has* $\mathsf{Alg}\mathcal{L}$ *as an algebraic semantics with $x \approx \top$ as defining equation, where $\top$ is a constant term of* $\mathsf{Alg}\mathcal{L}$*.*

Proof. Assertional logics satisfiy the conditions established in Corollary 40 of Raftery (2006) for a logic to be truth-equational, in this case with a truth definition of the form $x \approx \top$ where $\top$ is a theorem of the logic. Moreover, by Lemma 5, we know that $\top$ will be a constant of $\mathsf{Alg}^*\mathcal{L}$ and of $\mathsf{Alg}\mathcal{L}$. This shows that (i) implies (ii). It has already been observed as a general property that (ii) implies (iii). Similarly, (ii) implies (iv), because a logic is truth-equational if and only if truth is equationally definable in $\mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$ (Raftery, 2006, Theorem 28), and the class of algebraic reducts of $\mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$ is $\mathsf{Alg}\mathcal{L}$. Finally, each of (iii) and (iv) implies (i), simply by the involved definitions. □

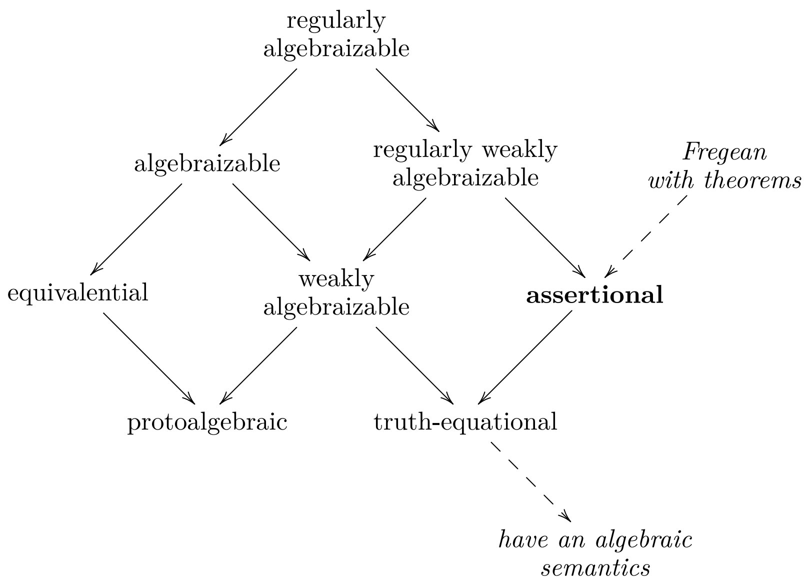


Fig. 1 The classes of logics in the fragment of the Leibniz hierarchy relevant to this paper, including the newly added class (in boldface), and showing (in italics) two related classes not belonging to it. Arrows indicate class inclusion.

Thus, all assertional logics are truth-equational; the class of the latter lies between the class of the former and that of the logics having an algebraic semantics (see Figure 1). This brings back into the Leibniz hierarchy many non-protoalgebraic logics that previously had seemed excluded from it. For instance, the $\langle \wedge, \vee, \top, \bot \rangle$-fragment of classical logic, which is the assertional logic of the variety of bounded distributive lattices; Visser's "basic logic" $BPL^\star$, shown to be non-protoalgebraic by Suzuki et al. (1998, Theorem 14); the implication-less fragment IPC^* of intuitionistic logic, proven to be non-protoalgebraic by Blok and Pigozzi (1989, § 5.2.5); and its denumerably many axiomatic extensions considered by Rebagliato and Verdú (1993). IPC^* and its extensions are examples where the constant term is not made up from primitive constants of the language; indeed, there $\top \coloneqq \neg(x \wedge \neg x)$.

Observe that by Theorem 7, if $\mathcal{L}$ is the assertional logic of some class K of algebras, then it is the assertional logic of the class $\mathsf{Alg}^*\mathcal{L}$, and also of the class $\mathsf{Alg}\mathcal{L}$.

Assertional logics can be characterized in an independent way through the notion of a ***unital*** class of matrices, i.e., a class of matrices where all the filters are one-element sets:

Theorem 8. *For any logic, $\mathcal{L}$ the following conditions are equivalent:*

(i) *$\mathcal{L}$ is an assertional logic.*
(ii) *The class of matrices $\mathsf{Mod}^*\mathcal{L}$ is unital.*
(iii) *The class of matrices $\mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$ is unital.*
(iv) *$\mathcal{L}$ has theorems and is complete with respect to a unital class of matrices.*

Proof. To show that (i) implies (ii) we use the characterizations of being assertional in Theorem 7. Thus, the assumption implies that $\mathsf{Mod}^*\mathcal{L} = \big\{\langle \mathbf{A}, \{\top^{\mathbf{A}}\}\rangle : \mathbf{A} \in \mathsf{Alg}^*\mathcal{L}\big\}$ and that this is a unital class. (i) implies (iii) for the same reason, applied to the class $\mathsf{Mod}^{\mathrm{Su}}\mathcal{L} = \big\{\langle \mathbf{A}, \{\top^{\mathbf{A}}\}\rangle : \mathbf{A} \in \mathsf{Alg}\mathcal{L}\big\}$. Trivially, each of (ii) and (iii) implies, separately, the second assertion of (iv), as a consequence of the completeness of $\mathcal{L}$ with respect to $\mathsf{Mod}^*\mathcal{L}$ and $\mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$, respectively. Now observe that if $\mathcal{L}$ has no theorems, then for any algebra $\mathbf{A}$, the matrix $\langle \mathbf{A}, \emptyset\rangle$ is a model of $\mathcal{L}$. But, in particular, for a trivial (i.e., one-element) algebra $\mathbf{A}$, the matrix $\langle \mathbf{A}, \emptyset\rangle$ is always reduced and Suszko-reduced, because then $\varOmega^{\mathbf{A}}\{\emptyset\} = \widetilde{\varOmega}^{\mathbf{A}}_{\mathcal{L}}\{\emptyset\} = A \times A = \mathrm{Id}_A$. Thus, we would have that $\langle \mathbf{A}, \emptyset\rangle \in \mathsf{Mod}^*\mathcal{L}$ and $\langle \mathbf{A}, \emptyset\rangle \in \mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$, respectively, against the assumption that the respective class is unital. This shows that $\mathcal{L}$ has theorems and completes the proof of (iv). Finally, in order to show that (iv) implies (i), let M be the unital class of matrices with respect to which $\mathcal{L}$ is complete, and let K be the class of their algebraic reducts. Observe that since $\mathcal{L}$ has theorems, all $\mathcal{L}$-filters are non-empty. Therefore, since the intersection of two $\mathcal{L}$-filters is always an $\mathcal{L}$-filter, and it cannot be empty, there can be at most one one-element $\mathcal{L}$-filter in each (arbitrary) algebra. The assumption that M is unital means that algebras in K have indeed one such $\mathcal{L}$-filter, and it is the only one on the algebra making the matrix reduced. Let $\top$ be a theorem of $\mathcal{L}$ in at most the variable x (which exists by the first assumption), and let $\mathbf{A} \in \mathsf{K}$. Since $\top$ is a theorem, for every $a \in A$ the point $\top^{\mathbf{A}}(a)$ must belong to the mentioned $\mathcal{L}$-filter, therefore this $\mathcal{L}$-filter must be exactly the set $\{\top^{\mathbf{A}}(a)\}$, for any $a \in A$. This also implies that $\top^{\mathbf{A}}(a) = \top^{\mathbf{A}}(b)$ for all $a, b \in A$. Therefore, $\top$ is a constant term of K, that is, the class K is pointed, and $\mathsf{M} = \big\{\langle \mathbf{A}, \{\top^{\mathbf{A}}\} : \mathbf{A} \in \mathsf{K}\rangle\big\}$. After this, the completeness of $\mathcal{L}$ with respect to M means that $\mathcal{L}$ is the assertional logic of K. ◻

The fact that assertional logics have a unital class of reduced models has the following, seldom noticed consequence:

Corollary 9. *If $\mathcal{L}$ is an assertional logic, then the class of algebras $\mathsf{Alg}^*\mathcal{L}$ is relatively point-regular.*

Proof. Let $\top$ be the constant term of $\mathsf{Alg}^*\mathcal{L}$ witnessing that $\mathcal{L}$ is assertional, as in the previous results. Let $\mathbf{A} \in \mathsf{Alg}^*\mathcal{L}$ and let $\theta, \theta' \in \mathrm{Co}_{\mathsf{Alg}^*\mathcal{L}}\mathbf{A}$ such that $\top^{\mathbf{A}}/\theta = \top^{\mathbf{A}}/\theta'$. Since $\mathbf{A}/\theta \in \mathsf{Alg}^*\mathcal{L}$, by Theorem 8, $\langle \mathbf{A}/\theta, \{\top^{\mathbf{A}/\theta}\}\rangle \in \mathsf{Mod}^*\mathcal{L}$. Now, if $\pi\colon \mathbf{A} \to \mathbf{A}/\theta$ is the canonical projection, we have that

$$\theta = \pi^{-1}\mathrm{Id}_{A/\theta} = \pi^{-1}\varOmega^{\mathbf{A}/\theta}\{\top^{\mathbf{A}/\theta}\} = \varOmega^{\mathbf{A}}\pi^{-1}\{\top^{\mathbf{A}/\theta}\} = \varOmega^{\mathbf{A}}\big(\top^{\mathbf{A}}/\theta\big).$$

The same argument for θ' shows that $\theta' = \Omega^{\mathbf{A}}\big(\top^{\mathbf{A}}/\theta'\big)$. The assumption that $\top^{\mathbf{A}}/\theta = \top^{\mathbf{A}}/\theta'$ implies that $\theta = \theta'$. ☐

The following characterization of assertional logics (if defined as in Theorem 8) is essentially due to Suszko (in unpublished lectures), according to Czelakowski (1981); the name "Suszko rules" was coined by Rautenberg (1993).

Theorem 10. *For any logic $\mathcal{L}$, the following conditions are equivalent:*

(i) *$\mathcal{L}$ is an assertional logic.*
(ii) *$\mathcal{L}$ has theorems and satisfies the so-called "Suszko rules":*

$$x, y, \varphi(x, \vec{z}) \vdash_{\mathcal{L}} \varphi(y, \vec{z}), \tag{2}$$

for all $\varphi(x, \vec{z}) \in Fm$.
(iii) *$\mathcal{L}$ has theorems and satisfies that $\langle x, y\rangle \in \widetilde{\Omega}_{\mathcal{L}} C_{\mathcal{L}}\{x, y\}$.*
(iv) *$\mathcal{L}$ has theorems and satisfies that for every algebra $\mathbf{A}$ and every $a, b \in A$, $\langle a, b\rangle \in \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}} Fi^{\mathbf{A}}_{\mathcal{L}}\{a, b\}$*

Proof. (i)⇒(ii) We know all assertional logics have theorems. Completeness of $\mathcal{L}$ with respect to some unital class of matrices, which Theorem 8 guarantees, directly implies the Suszko rules.

(ii)⇒(iii) and (iv) Let $\Gamma \in Th\mathcal{L}$ be such that $C_{\mathcal{L}}\{x, y\} \subseteq \Gamma$, that is, $x, y \in \Gamma$. Then by the Suszko rules, $\varphi(x, \vec{z}) \in \Gamma$ if and only if $\varphi(y, \vec{z}) \in \Gamma$, for all $\varphi(x, \vec{z}) \in Fm$. This means that $\langle x, y\rangle \in \Omega\Gamma$. Therefore, $\langle x, y\rangle \in \widetilde{\Omega}_{\mathcal{L}} C_{\mathcal{L}}\{x, y\}$, which proves (iii). Point (iv) is proved in the same way, but working on the $\mathcal{L}$-filters of an arbitrary algebra.

(iii)⇒(ii) follows by the same argument as the preceding implication; as a matter of fact, that the Suszko rules hold is equivalent to the condition that $\langle x, y\rangle \in \widetilde{\Omega}_{\mathcal{L}} C_{\mathcal{L}}\{x, y\}$.

(ii)⇒(i) Let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*\mathcal{L}$. Since $\mathcal{L}$ has theorems, $F \neq \emptyset$. Then the Suszko rules imply that F is a one-element set: If $a, b \in F$, then for every $\vec{c} \in A^n$, $\varphi^{\mathbf{A}}(a, \vec{c}) \in F$ if and only if $\varphi^{\mathbf{A}}(b, \vec{c}) \in F$, that is, $\langle a, b\rangle \in \Omega^{\mathbf{A}} F$; since the matrix is reduced, this implies that $a = b$. Thus, all the reduced models of $\mathcal{L}$ are unital, and by Theorem 8 this fact implies that $\mathcal{L}$ is an assertional logic.

(iv)⇒(iii) because the latter is a particular case of the former. ☐

After the preceding results, we think it becomes clear that the class of assertional logics should be counted among those in the Leibniz hierarchy, as it can be defined by conditions on the Leibniz congruence: notice that the second conditions in points (iii) and (iv) of Theorem 10 can be paraphrased as "$\langle x, y\rangle \in \Omega\Gamma$ for all $\Gamma \in Th\mathcal{L}$ such that $x, y \in \Gamma$" and "$\langle a, b\rangle \in \Omega^{\mathbf{A}} F$ for all $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$ such that $a, b \in F$", respectively. For protoalgebraic logics, these can be simplified to "$\langle x, y\rangle \in \Omega C_{\mathcal{L}}\{x, y\}$" and "$\langle a, b\rangle \in \Omega^{\mathbf{A}} Fi^{\mathbf{A}}_{\mathcal{L}}\{a, b\}$", respectively.

It is well known that inside protoalgebraic logics, any of these conditions, or the equivalent ones found in Theorems 7 and 8, determine the classes of *regularly weakly algebraizable* logics; and inside equivalential logics, they produce the *regularly algebraizable* logics. These classes are usually considered as belonging to the Leibniz hierarchy, and by the same reason should the class of assertional logics be considered in it; its location in the hierarchy is parallel to the former ones, as Figure 1 on page 10 shows.

This new member of the hierarchy is different from the existing ones, and its location is really as shown in Figure 1, as the following examples confirm.

- *There are truth-equational logics that are not assertional.* Examples of this are all algebraizable logics that are not regularly algebraizable, such as all substructural logics associated with a variety of non-integral residuated lattices, described by Galatos et al. (2007); among the best known members of this class we find Relevance Logic (with and without the "Mingle" axiom; i.e., R and RM) and the multiplicative-additive fragment of Linear Logic $MALL$. Raftery (2006, Example 9) provides a non-protoalgebraic example: The logic in the language $\langle \to, \neg \rangle$ defined by taking as algebraic semantics the variety generated by the Sobociński three-element algebra, with $x \approx x \to x$ as defining equation. This logic, which has the same theorems as (but does not coincide with) the implication-negation fragment of RM, is neither protoalgebraic nor assertional, but is truth-equational. Notice that, although the term $x \to x$ is a theorem of the logic, it is not a constant term of the class of algebras.
- *There are assertional logics that are not (regularly) weakly algebraizable.* Examples of this will be all assertional logics that are not protoalgebraic, some of which are mentioned after Theorem 7.

From our Theorem 8, using Theorem 5.6.3 of Czelakowski (2001), it follows that the class of regularly weakly algebraizable logics is the intersection of the class of protoalgebraic logics and of assertional logics, and hence also the intersection of the classes of weakly algebraizable logics and of assertional logics; Figure 1 on page 10 shows these facts. In particular, a logic $\mathcal{L}$ is regularly weakly algebraizable if and only if it is protoalgebraic and $\mathsf{Mod}^*\mathcal{L}$ is unital; it is interesting to notice that the two just mentioned conditions can be formulated by making reference only to the class $\mathsf{Alg}^*\mathcal{L}$:

Corollary 11. *A logic $\mathcal{L}$ is regularly weakly algebraizable if and only if it has $\mathsf{Alg}^*\mathcal{L}$ as an algebraic semantics with $x \approx \top$ as defining equation, where $\top$ is a constant term of $\mathsf{Alg}^*\mathcal{L}$, and $\mathsf{Alg}^*\mathcal{L}$ is closed under subdirect products.*

Proof. Putting Theorems 7 and 8 together, we see that the condition that $\mathsf{Mod}^*\mathcal{L}$ is unital can be equivalently formulated in terms of $\mathsf{Alg}^*\mathcal{L}$ as stated. On the other hand, it is well-known (Czelakowski, 2001, Thm. 1.3.7) that a logic is protoalgebraic if and only if the class of matrices $\mathsf{Mod}^*\mathcal{L}$ is closed under subdirect products; but by the same theorems, the condition that $\mathsf{Mod}^*\mathcal{L}$ is unital implies that the logic is truth-equational, which implies that the truth filter of the matrices in $\mathsf{Mod}^*\mathcal{L}$ is unique, and therefore in our situation $\mathsf{Mod}^*\mathcal{L}$ is closed under subdirect products if and only if the class of algebras $\mathsf{Alg}^*\mathcal{L}$ is closed under subdirect products. □

4 Full generalized models, and the Frege hierarchy

The reduction construction allows to introduce a special class of g-models of a logic. A ***basic full g-model*** of a logic $\mathcal{L}$ is one of the form $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{L}}\mathbf{A} \rangle$, for some algebra $\mathbf{A}$. A ***full g-model*** of $\mathcal{L}$ is one whose reduction is a basic full g-model; that is, a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ such that $\mathcal{C}/\widetilde{\Omega}^{\mathbf{A}}\mathcal{C} = \mathcal{F}i_{\mathcal{L}}(\mathbf{A}/\widetilde{\Omega}^{\mathbf{A}}\mathcal{C})$. Note that a logic, viewed as a g-matrix, is a full g-model of itself, and indeed the largest one on the formula algebra. It turns out that $\mathsf{Alg}\mathcal{L}$ is also the class of algebraic reducts of the reduced full g-models of $\mathcal{L}$; in fact, the reduced full g-models of $\mathcal{L}$ are exacly those of the form $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{L}}\mathbf{A} \rangle$ with $\mathbf{A} \in \mathsf{Alg}\mathcal{L}$. The notion of a full g-model of a logic, introduced by Font and Jansana (1996) and further studied in Font et al. (2006) and other papers, has allowed to develop a very general approach to the algebraic study of sentential logics, and in particular is instrumental in the following definitions.

The Frege hierarchy is a classification of logics according to what kind of replacement properties they (and their full g-models) satisfy. In abstract terms, replacement properties are defined algebraically as concerning congruences. A g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ has ***the property of congruence*** when its Frege relation is a congruence of $\mathbf{A}$, i.e., when $\Lambda\mathcal{C} = \widetilde{\Omega}^{\mathbf{A}}\mathcal{C}$. A g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ has ***the strong property of congruence*** when for any $F \in \mathcal{C}$, the g-matrix $\langle \mathbf{A}, \mathcal{C}^F \rangle$ has the property of congruence, i.e., when $\Lambda_{\mathcal{C}} F = \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{C}} F$ for all $F \in \mathcal{C}$; note that this means that the Frege and the Suszko operators relative to $\mathcal{C}$ coincide. These two properties of congruence are preserved by reductions. Since the relation $\Lambda Th\mathcal{L}$ for a sentential logic $\mathcal{L}$ is its interderivability relation $\dashv\vdash_{\mathcal{L}}$, these two properties when formulated for a sentential logic amount to natural *replacement properties* of the interderivability relation.

These two properties originate the four classes of logics in the Frege hierarchy.

Definition 12. Let $\mathcal{L}$ be a logic.

- $\mathcal{L}$ is ***selfextensional*** when, viewed as the g-matrix $\langle \mathbf{Fm}, Th\mathcal{L} \rangle$, it has the property of congruence; i.e., when the interderivability relation $\dashv\vdash_{\mathcal{L}}$ is a congruence of $\mathbf{Fm}$.
- $\mathcal{L}$ is ***Fregean*** when, viewed as the g-matrix $\langle \mathbf{Fm}, Th\mathcal{L} \rangle$, it has the strong property of congruence; i.e., when for each $\Gamma \in Th\mathcal{L}$, the interderivability relation modulo Γ (i.e., the relation $\Lambda_{\mathcal{L}}\Gamma$) is a congruence of $\mathbf{Fm}$.
- $\mathcal{L}$ is ***fully selfextensional*** when all its full g-models have the property of congruence.

- $\mathcal{L}$ is ***fully Fregean*** when all its full g-models have the strong property of congruence.

The notion of a selfextensional logic is due to Wójcicki (1979); see also Wójcicki (1988, Chapter 5). The notion of a Fregean logic was introduced, in a slightly restricted form, by Pigozzi (1991) and Czelakowski (1992), and independently and as given here, by Font (1993). The other two classes of logics were introduced by Font and Jansana (1996); the hierarchy as such was first considered by Font (2003), and named after Frege in Font (2006). Observe that a logic is Fregean if and only if the Suszko and the Frege operators (relative to it) coincide on the theories of the logic. In Font and Jansana (1996, Proposition 2.40) it is shown that $\mathcal{L}$ is fully selfextensional if and only if for any algebra $\mathbf{A}$, the basic full g-model $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{L}}\mathbf{A}\rangle$ has the property of congruence, and if and only if for every $\mathbf{A} \in \mathsf{Alg}\mathcal{L}$ the relation $\varLambda\mathcal{F}i_{\mathcal{L}}\mathbf{A}$ is the identity relation; that is, if and only if in the algebras in $\mathsf{Alg}\mathcal{L}$, different points generate different $\mathcal{L}$-filters ("$\mathcal{L}$-filters separate points"). This characterization is the clue to some of the interesting applications of fully selfextensional logics to the development of an abstract duality theory (Gehrke et al., 2010); it will be used in Theorem 15.

Some obvious relations hold between the four classes (taking into account that a logic is always a full g-model of itself), and are depicted in Figure 2. Two questions this graph naturally rises is whether the top (smallest) class of fully Fregean logics is the intersection of the two middle classes of Fregean logics and of fully selfextensional logics, and whether the lowest (largest) class of selfextensional logics is their union. These questions were posed as open problems in Font (2003, § 6.2) and Font (2006, p. 202); the first one is answered affirmatively in the present paper for logics with theorems (Theorem 26), and the second one is answered negatively, even for logics with very strong properties (Example 23).

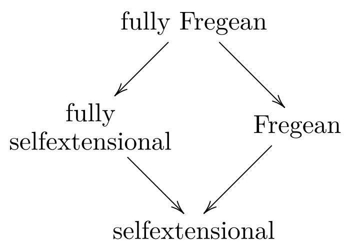


Fig. 2 The classes of logics in the Frege hierarchy. Arrows indicate class inclusion.

In order to find relations between the class of truth-equational logics and the classes in the Frege hierarchy, we start from the following observation.

Lemma 13. *Let $\mathcal{L}$ be a Fregean logic.*

1. *$\mathcal{L}$ satisfies the "Suszko rules"* (2) *displayed in Theorem* 10.
2. *If $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*\mathcal{L}$, then F is either empty or a one-element set.*

Proof. 1. Trivially, for any logic $\mathcal{L}$ it holds that $\langle x,y\rangle \in \varLambda_{\mathcal{L}} C_{\mathcal{L}}\{x,y\}$. If $\mathcal{L}$ is Fregean, the relation $\varLambda_{\mathcal{L}} C_{\mathcal{L}}\{x,y\}$ will be a congruence, which implies that for all $\varphi(x,\vec{z}) \in Fm$, $\big\langle \varphi(x,\vec{z}), \varphi(y,\vec{z})\big\rangle \in \varLambda_{\mathcal{L}} C_{\mathcal{L}}\{x,y\}$. That is, $C_{\mathcal{L}}\{x,y,\varphi(x,\vec{z})\} = C_{\mathcal{L}}\{x,y,\varphi(y,\vec{z})\}$, which amounts to the rules (2).
2. Assume that F is non-empty and take any $a,b \in F$. Since F is an $\mathcal{L}$-filter, from rules (2) it follows that for any $\varphi(x,\vec{z}) \in Fm$ and any $\vec{c} \in A^n$, $\varphi^{\mathbf{A}}(a,\vec{c}) \in F$ if and only if $\varphi^{\mathbf{A}}(b,\vec{c}) \in F$. By the classical characterization of Czelakowski (2001, Theorem 0.5.3), this says that $\langle a,b\rangle \in \varOmega^{\mathbf{A}} F$, and since the matrix is reduced this implies that $a = b$. Thus, F is a one-element set. ☐

Since, in general, the filter of a reduced matrix can be empty only when the algebra is trivial, we see that reduced models of Fregean logics on non-trivial algebras must be unital. This may be a practical criterion to disprove that a certain logic, (some of) whose reduced models are known, is Fregean. For instance, this shows that Belnap-Dunn's well-known four-valued logic is not Fregean, because it has reduced models on a nontrivial algebra with two-element designated sets, for instance those given by the two prime filters of the four-element De Morgan lattice (usually called $FOUR$) that defines the logic, which is a simple algebra; see Font (1997, p. 427).

One consequence of Lemma 13 (together with a result to be reviewed in the next section) is the following characterization:

Theorem 14. *Let $\mathcal{L}$ be a Fregean logic. The following conditions are equivalent:*

(i) *$\mathcal{L}$ has theorems.*
(ii) *$\mathcal{L}$ is assertional.*
(iii) *$\mathcal{L}$ is truth-equational.*
(iv) *The Leibniz operator is injective over the $\mathcal{L}$-filters of arbitrary algebras.*
(v) *The Leibniz operator is injective over the theories of $\mathcal{L}$.*

Proof. By Lemma 13, if a Fregean logic has theorems, then its class of reduced models is unital, and therefore by Theorem 8, the logic is assertional; this proves that (i) implies (ii). That (ii) implies (iii) is contained in Theorem 7. Now, that (iii) implies (iv) follows from Theorem 28 of Raftery (2006), a result that you will find here as Theorem 18, because being completely order reflecting implies being injective. Clearly, (v) follows from (iv) as a particular case. Finally, (v) implies (i) because, if a logic has no theorems, then $\emptyset$ and Fm are both theories of the logic, and always $\varOmega\emptyset = \varOmega Fm = Fm \times Fm$, thus breaking injectivity of the Leibniz operator on the theories of the logic. ☐

We thus see that Fregean logics with theorems are assertional, and hence truth-equational; the situation is that depicted in Figure 1 on page 10. The following examples confirm that there are no other relations.

- *There are assertional logics that are not Fregean.* We find many examples of this situation even among the regularly algebraizable logics, such as the global consequences of the usual normal modal logics ($K, T, S4, S5$, etc.), or Łukasiewicz's many-valued logics, or, more generally, the logics associated in Galatos et al. (2007) with any variety of integral residuated lattices that is not a variety of generalized Heyting algebras. In all these examples, the defining equation of the algebraization is of the form $x \approx \top$ for a constant $\top$, so that each is the assertional logic of the corresponding algebraic counterpart (a variety of normal modal algebras, or the corresponding variety of residuated lattices, respectively). But they are not selfextensional (the modal cases are easily shown by using Kripke models, and the second group is shown by Bou et al. (2009, Theorem 4.12)), hence *a fortiori* they are not Fregean.
- *There are Fregean logics with theorems that are not regularly weakly algebraizable.* Examples are the already mentioned logic IPC^* and its axiomatic extensions, which are not protoalgebraic, hence in particular not regularly weakly algebraizable. That IPC^* is Fregean is proved by Font and Jansana (1996, § 5.1.4), and all axiomatic extensions of a Fregean logic are Fregean as well; and all these logics have theorems (indeed, they are assertional).

Thus, the class of assertional logics is the smallest class in the Leibniz hierarchy containing the Fregean logics with theorems, as shown in Figure 1.

As a final application of Theorem 8, we obtain a (weakened) version of Theorem 14 for fully selfextensional logics.

Theorem 15. *A fully selfextensional logic is assertional if and only if it is truth-equational.*

Proof. By Theorem 7, all assertional logics are truth-equational. So let $\mathcal{L}$ be a fully selfextensional and truth-equational logic, and let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$. Since $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}}(\bigcap \mathcal{F}i_{\mathcal{L}}\mathbf{A}) \subseteq \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}} F = \mathrm{Id}_A$, it follows that $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}}(\bigcap \mathcal{F}i_{\mathcal{L}}\mathbf{A}) = \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}} F = \mathrm{Id}_A$. One of the basic characterizations of truth-equational logics (Raftery, 2006, Theorem 28) is that the Suszko operator is injective on their filters, therefore $F = \bigcap \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, that is, F is the smallest $\mathcal{L}$-filter of A. Thus, if $a, b \in F$, a and b belong to the same $\mathcal{L}$-filters (namely: all). But $\mathbf{A} \in \mathsf{Alg}\mathcal{L}$ and for these algebras, $\mathcal{L}$-filters separate points, because $\mathcal{L}$ is fully selfextensional (as commented on page 15 after Definition 12). Therefore, $a = b$. We have shown that $\mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$ is unital, and by Theorem 8 this implies that $\mathcal{L}$ is assertional. $\square$

Some of the non-protoalgebraic examples of assertional logics mentioned before are fully selfextensional; for instance, Visser's logic BPL^*, as shown in Bou (2001), or the $\langle \wedge, \vee, \top, \bot \rangle$-fragment of classical logic, as follows from Theorem 4.28 of Font and Jansana (1996).

Notice that, unlike in the Fregean case (Theorem 14), the condition that $\mathcal{L}$ has theorems cannot be added as an equivalent one to those in Theorem 15; the logics that preserve degrees of truth with respect to certain varieties of commutative, integral residuated lattices (those that are not varieties of generalized Heyting algebras) provide an infinity of counterexamples: all these logics are fully selfextensional and have theorems but are not assertional; these properties are shown in, or follow from, Corollary 4.2, Lemma 2.6, Corollary 3.6 and Theorem 4.12 of Bou et al. (2009).

5 The full generalized models of truth-equational logics

The key characterization of truth-equational logics uses the following property.

Definition 16. Let $\mathcal{L}$ be a logic, and let $\mathbf{A}$ be an algebra. The Leibniz operator $\Omega^{\mathbf{A}}$ is ***completely order-reflecting*** over $\mathcal{F}i_{\mathcal{L}}\mathbf{A}$ when for all $\mathcal{F} \cup \{G\} \subseteq \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, if $\bigcap\limits_{F \in \mathcal{F}} \Omega^{\mathbf{A}} F \subseteq \Omega^{\mathbf{A}} G$ then $\bigcap \mathcal{F} \subseteq G$.

The following reformulation in terms of the Suszko operator, whose proof is an easy exercise, is very convenient:

Lemma 17. *Let $\mathcal{L}$ be a logic, and let $\mathbf{A}$ be an algebra. The Leibniz operator $\Omega^{\mathbf{A}}$ is completely order-reflecting over $\mathcal{F}i_{\mathcal{L}}\mathbf{A}$ if and only if for all $F, G \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, if $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{L}} F \subseteq \Omega^{\mathbf{A}} G$, then $F \subseteq G$.* □

The main result placing the class of truth-equational logics in the Leibniz hierarchy, due to Raftery (2006, Theorem 28), is the following.

Theorem 18. *For any logic $\mathcal{L}$, the following conditions are equivalent:*

- **(i)** *$\mathcal{L}$ is truth-equational.*
- **(ii)** *The Leibniz operator is completely order-reflecting over the $\mathcal{L}$-filters of arbitrary algebras.*
- **(iii)** *The Leibniz operator is completely order-reflecting over the theories of $\mathcal{L}$.* □

In particular this implies that the Leibniz operator is order-reflecting, and hence injective, on the theories of $\mathcal{L}$ (and on the $\mathcal{L}$-filters of any algebra).

This characterization can be used to obtain an alternative proof of the truth-equationality of Fregean logics with theorems (Theorem 14), which needs not use assertional logics. To this end we show that the Leibniz operator is completely order-reflecting on the theories of $\mathcal{L}$, by using Lemma 17 over the formula algebra. Let $\Gamma, \Gamma' \in Th\mathcal{L}$ be such that $\widetilde{\Omega}_{\mathcal{L}} \Gamma \subseteq \Omega \Gamma'$. We have to show that $\mathit{\Gamma} \subseteq \mathit{\Gamma}'$, so let $\varphi \in \mathit{\Gamma}$. Take now any theorem ψ of $\mathcal{L}$, which exists by assumption; then in particular $\psi \in \mathit{\Gamma}$, and this implies that $C_{\mathcal{L}}(\mathit{\Gamma},\varphi) = \mathit{\Gamma} = C_{\mathcal{L}}(\mathit{\Gamma},\psi)$, that is, $\langle \varphi,\psi \rangle \in \mathit{\Lambda}_{\mathcal{L}}\mathit{\Gamma}$. But $\mathcal{L}$ is Fregean, which means that $\mathit{\Lambda}_{\mathcal{L}}\mathit{\Gamma} = \widetilde{\mathit{\Omega}}_{\mathcal{L}}\mathit{\Gamma}$. Therefore, by the assumption, $\langle \varphi,\psi \rangle \in \mathit{\Omega}\mathit{\Gamma}'$. Since also $\psi \in \mathit{\Gamma}'$, by compatibility it follows that $\varphi \in \mathit{\Gamma}'$, as desired.

The following technical but important property will allow us to obtain some characterizations of truth-equationality in terms of the full g-models of the logic.

Lemma 19. *Let $\mathcal{L}$ be any logic, $\mathbf{A}$ any algebra, and $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$. The following conditions are equivalent:*

- **(i)** *The g-matrix $\big\langle \mathbf{A}, (\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F \big\rangle$ is a full g-model of $\mathcal{L}$.*
- **(ii)** *For all $G \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, if $\widetilde{\mathit{\Omega}}^{\mathbf{A}}_{\mathcal{L}} F \subseteq \mathit{\Omega}^{\mathbf{A}} G$, then $F \subseteq G$.*

Proof. One of the central characterizations of the notion of full g-model of a logic (Font and Jansana, 1996, Theorem 2.14) is that a g-matrix $\langle \mathbf{A},\mathcal{C} \rangle$ is a full g-model of $\mathcal{L}$ if and only if $\mathcal{C} = \{G \in \mathcal{F}i_{\mathcal{L}}\mathbf{A} : \widetilde{\mathit{\Omega}}^{\mathbf{A}}\mathcal{C} \subseteq \mathit{\Omega}^{\mathbf{A}}G\}$. Since by (1) on page 5, $\widetilde{\mathit{\Omega}}^{\mathbf{A}}(\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F = \widetilde{\mathit{\Omega}}^{\mathbf{A}}_{\mathcal{L}} F$, in particular a g-matrix of the form $\big\langle \mathbf{A}, (\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F \big\rangle$, for some $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, is a full g-model of $\mathcal{L}$ if and only if $(\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F = \{G \in \mathcal{F}i_{\mathcal{L}}\mathbf{A} : \widetilde{\mathit{\Omega}}^{\mathbf{A}}_{\mathcal{L}} F \subseteq \mathit{\Omega}^{\mathbf{A}}G\}$. But the direct inclusion holds by (1), and the reverse inclusion is exactly condition (ii). $\square$

This allows us to obtain our first characterization of truth-equational logics in terms of the form of their full g-models: a logic is truth-equational if and only if each of its filters determines a full g-model of the logic; we see that the same property, limited to the theories of the logic, is also sufficient to characterize truth-equationality.

Theorem 20. *For any logic $\mathcal{L}$, the following conditions are equivalent:*

- **(i)** *$\mathcal{L}$ is truth-equational.*
- **(ii)** *For every $\mathbf{A}$ and every $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, the g-matrix $\big\langle \mathbf{A}, (\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F \big\rangle$ is a full g-model of $\mathcal{L}$.*
- **(iii)** *For every $\mathit{\Gamma} \in Th\mathcal{L}$, the g-matrix $\big\langle \mathbf{Fm}, (Th\mathcal{L})^{\mathit{\Gamma}} \big\rangle$ is a full g-model of $\mathcal{L}$.*

Proof. In order to prove that (i) implies (ii), assume that $\mathcal{L}$ is truth-equational. Then, by Theorem 18 and Lemma 17, we see that for any $\mathbf{A}$, any $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$ satisfies condition (ii) in Lemma 19, therefore its condition (i) yields the present condition (ii). Clearly, (iii) is a particular case of (ii). And finally from (iii) we can prove (i): By applying Lemma 19 to the formula algebra, we see that (iii) amounts to saying that for every $\mathit{\Gamma},\mathit{\Gamma}' \in Th\mathcal{L}$, if $\widetilde{\mathit{\Omega}}_{\mathcal{L}}\mathit{\Gamma} \subseteq \mathit{\Omega}\mathit{\Gamma}'$, then $\mathit{\Gamma} \subseteq \mathit{\Gamma}'$. But by Lemma 17 applied also to the formula algebra, this is to say that the Leibniz operator is completely order-reflecting over the theories of $\mathcal{L}$, and by Theorem 18, this implies that $\mathcal{L}$ is truth-equational. $\square$

The equivalence between (i) and (ii) is obtained in Albuquerque et al. (2016) as a by-product of a more general study of compatibility operators in abstract algebraic logic, of which the Suszko operator is a paradigmatic example; here we have given a direct proof. It is interesting to relate this characterization to that of protoalgebraic logics found in Font and Jansana (1996, Theorem 3.4): A logic $\mathcal{L}$ is protoalgebraic if and only if every full g-model of $\mathcal{L}$ is of the form $\left\langle \mathbf{A}, (\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F \right\rangle$ for some algebra $\mathbf{A}$ and some $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$. This is in some sense "dual" to the characterization of truth-equational logics in Theorem 20. As a consequence, a logic is weakly algebraizable if and only if its full g-models are exactly the g-matrices of the form $\left\langle \mathbf{A}, (\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F \right\rangle$ for some $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$; this was already obtained in Font and Jansana (1996, Theorem 3.8), and in fact this characterization of weakly algebraizable logics lies at the roots of the very definition of this class of logics in Font and Jansana (1996).

We are also interested in the following extension of the previous characterization: A logic is truth-equational if and only if the class of its full g-models is so-to-speak closed under the operation $\mathcal{C} \mapsto \mathcal{C}^F$; and again it is enough to require this property for the full g-models over the formula algebra.

Theorem 21. *For any logic $\mathcal{L}$, the following conditions are equivalent:*

- **(i)** *$\mathcal{L}$ is truth-equational.*
- **(ii)** *For every full g-model $\langle \mathbf{A}, \mathcal{C} \rangle$ of $\mathcal{L}$ and every $F \in \mathcal{C}$, the g-matrix $\langle \mathbf{A}, \mathcal{C}^F \rangle$ is a full g-model of $\mathcal{L}$.*
- **(iii)** *For every full g-model $\langle \mathbf{Fm}, \mathcal{C} \rangle$ of $\mathcal{L}$ over the formula algebra and every $\Gamma \in \mathcal{C}$, the g-matrix $\langle \mathbf{Fm}, \mathcal{C}^{\Gamma} \rangle$ is also a full g-model of $\mathcal{L}$.*

Proof. (i)⇒(ii) It is a general property of the theory of full g-models that the intersection of (the closure systems of) two full g-models of a logic produces another full g-model of the same logic; this is commented just before Theorem 1.20 of Font et al. (2006), and is also proved in Proposition 5.96 of Font (2016). If $\mathcal{L}$ is truth-equational, by Theorem 20, for every $F \in \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, the g-matrix $\left\langle \mathbf{A}, (\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F \right\rangle$ is a full g-model of $\mathcal{L}$. Now, if $\langle \mathbf{A}, \mathcal{C} \rangle$ is a full g-model of $\mathcal{L}$, $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{L}}\mathbf{A}$, and hence clearly $\mathcal{C} \cap (\mathcal{F}i_{\mathcal{L}}\mathbf{A})^F = \mathcal{C}^F$. Therefore, by the mentioned general property, the g-matrix $\langle \mathbf{A}, \mathcal{C}^F \rangle$ is a full g-model of $\mathcal{L}$, as desired.

(iii) is a particular case of (ii), and the implication (iii)⇒(i) is trivial because the g-matrix $\langle \mathbf{Fm}, Th\mathcal{L} \rangle$ is always full (indeed, it is a basic full g-model, by definition), therefore our (iii) implies the condition in Theorem 20(iii) as a particular case, and hence implies that $\mathcal{L}$ is truth-equational. □

6 Applications to the hierarchies

The preceding characterization of the full g-models of truth-equational logics allows us to refine the Frege hierarchy inside this class.

Theorem 22. *A truth-equational logic is fully selfextensional if and only if it is fully Fregean.*

Proof. Every fully Fregean logic is fully selfextensional, so in one direction there is nothing to prove. Now assume that $\mathcal{L}$ is a truth-equational and fully selfextensional logic, and let $\langle \mathbf{A}, \mathcal{C} \rangle$ be any full g-model of $\mathcal{L}$ and $F \in \mathcal{C}$. Since $\mathcal{L}$ is truth-equational, by Theorem 21 the g-matrix $\langle \mathbf{A}, \mathcal{C}^F \rangle$ is also a full g-model of $\mathcal{L}$. Then, since $\mathcal{L}$ is fully selfextensional, the g-matrix has the property of congruence. This shows that all the full g-models of $\mathcal{L}$ have the strong property of congruence, that is, that $\mathcal{L}$ is fully Fregean. □

Thus, for truth-equational logics (hence, in a large part of the Leibniz hierarchy) the Frege hierarchy reduces to three classes:

$$\begin{array}{c} \text{fully Fregean (and truth-equational)} \\ \downarrow \\ \text{Fregean (and truth-equational)} \\ \downarrow \\ \text{selfextensional (and truth-equational)} \end{array}$$

These three classes are different, and the lowest one is still a proper subclass of that of all truth-equational logics, as the following considerations show.

- The last mentioned fact (i.e., that not all truth-equational logics are selfextensional) is witnessed by the many algebraizable logics that are not selfextensional, as already mentioned after Theorem 14.
- Babyonyshev (2003) has constructed a Fregean logic that is not fully Fregean. This logic, which has a proof-theoretic definition, has theorems, therefore by Theorem 14 it is assertional, and hence truth-equational.
- Next we construct a truth-equational and non-protoalgebraic logic that is selfextensional but not Fregean:

Example 23. Consider the algebra $\mathbf{A} = \langle \{0,1,2\}, \Box, \neg \rangle$, where $\Box$ and $\neg$ are two 1-ary operations, defined as follows:

$$\neg 1 = \Box 1 = \Box 0 = 0 \qquad \neg 0 = \Box 2 = 1 \qquad \neg 2 = 2,$$

and consider the logic $\mathcal{L}$ in the language $\langle \Box, \neg \rangle$ determined by the matrix $\langle \mathbf{A}, \{1\} \rangle$.

Fact 1. $\mathcal{L}$ is assertional, and hence truth-equational: To see this, note that in $\mathbf{A}$, $\neg\Box\Box a = 1$ for all $a \in A$, so that $\neg\Box\Box x$ is a constant term of $\mathbf{A}$, and $\mathcal{L}$ is the assertional logic of the class $\{\mathbf{A}\}$ with $\top := \neg\Box\Box x$. Therefore, by Theorem 7, $\mathcal{L}$ is truth-equational, with $\{x \approx \neg\Box\Box x\}$ as defining equation.

Fact 2. $\mathcal{L}$ is not Fregean: To see this, it is enough to check, from the definition, that

$$\Box x, \neg\Box x \dashv\vdash_{\mathcal{L}} \Box x, \neg x$$

(because no evaluation makes the two premises on either side simultaneously equal to 1), and that

$$\Box x\,, \neg\neg\Box x \nvdash_{\mathcal{L}} \neg\neg x$$

(just evaluate x to 2). Therefore $\Lambda_{\mathcal{L}} C_{\mathcal{L}}\{\Box x\}$ is not a congruence with respect to the operation $\neg$.

FACT 3. If $\varphi \dashv\vdash_{\mathcal{L}} \psi$, then $\mathbf{A} \vDash \varphi \approx \psi$: Since all connectives are unary, there are two variables x and y such that $Var\{\varphi,\psi\} \subseteq \{x,y\}$.

We first construct all the terms in the variables x and y up to equivalence in $\mathbf{A}$. Observe that this is equivalent to ask for a set of representatives of the congruence classes that form the universe of the free algebra $\mathbf{Fm_A}\{x,y\}$ over the variety generated by $\mathbf{A}$ with two free generators. We reason as follows. The set A^{A^2} of binary functions on A can be given naturally the structure of an algebra $\mathbf{A}^{A^2}$. Then let $\pi_i \colon A^2 \to A$ be the projection map on the i-th component, for $i \in \{1,2\}$, and let $\mathbf{C}$ be the subalgebra of $\mathbf{A}^{A^2}$ generated by $\{\pi_1,\pi_2\}$. We claim that

$$C = \big\{\pi_1, \pi_2, \Box\pi_1, \Box\pi_2, \neg\pi_1, \neg\pi_2, \Box\Box\pi_1, \neg\Box\pi_1, \neg\Box\pi_2, \neg\Box\Box\pi_1\big\}.$$

The inclusion from right to left follows from the fact that $\mathbf{C}$ is a subalgebra of $\mathbf{A}^{A^2}$, whereas the other one is a consequence of the fact that the identities $\neg\neg x \approx x$, $\Box\Box x \approx \Box\Box y$, $\Box\Box\Box x \approx \Box\Box x$ and $\Box\neg x \approx \Box x$ hold in $\mathbf{A}$.

Now, recall that the free algebra $\mathbf{Fm_A}\{x,y\}$ is isomorphic to $\mathbf{C}$ via the map sending the equivalence classes of x and y to π_1 and π_2 respectively; see for instance (Bergman, 2011, Theorem 4.9). Applying this fact to our claim, we conclude that

$$T(x,y) \coloneqq \big\{x, y, \Box x, \Box y, \neg x, \neg y, \Box\Box x, \neg\Box x, \neg\Box y, \neg\Box\Box x\big\}$$

is the set of terms in two variables up to equivalence in $\mathbf{A}$.

Since $Var\{\varphi,\psi\} \subseteq \{x,y\}$, there are $\varphi',\psi' \in T(x,y)$ such that $\mathbf{A} \vDash \varphi \approx \varphi'$ and $\mathbf{A} \vDash \psi \approx \psi'$. By Lemma 1, $\varphi \dashv\vdash_{\mathcal{L}} \varphi'$ and $\psi \dashv\vdash_{\mathcal{L}} \psi'$, and since by assumption $\varphi \dashv\vdash_{\mathcal{L}} \psi$, it follows that $\varphi' \dashv\vdash_{\mathcal{L}} \psi'$. But it is easy to check, working case by case, that no two distinct terms in $T(x,y)$ are interderivable in $\mathcal{L}$. Therefore, we conclude that $\varphi' = \psi'$. This implies that $\mathbf{A} \vDash \varphi \approx \psi$, as required.

FACT 4. $\mathcal{L}$ is selfextensional: As a consequence of Fact 3 and Lemma 1, $\alpha \dashv\vdash_{\mathcal{L}} \beta$ if and only if $\mathbf{A} \vDash \alpha \approx \beta$. But this last relation is clearly a congruence. Therefore, $\Lambda\mathcal{L} = \dashv\vdash_{\mathcal{L}}$ is a congruence, that is, $\mathcal{L}$ is selfextensional.

FACT 5. $\mathcal{L}$ is not protoalgebraic: This is because its language contains only unary connectives. As a direct consequence of the characterization in Theorem 1.1.3 of Czelakowski (2001), a protoalgebraic logic in such a language should be trivial, which $\mathcal{L}$ is not. ☐

This example also answers, in the negative, an open problem on the structure of the Frege hierarchy, posed in Font (2003, p. 78) and Font (2006, p. 202): that of whether the class of selfextensional logics is the union of the class of Fregean logics and the class of fully selfextensional logics. The logic constructed in Example 23 is selfextensional but not Fregean, hence not fully Fregean, and this in turn implies (by Theorem 22, since it is truth-equational) that it is not fully selfextensional either.

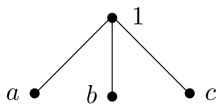
For *finitary* logics the result in Theorem 22 produces another refinement of the Frege hierarchy.

Corollary 24. *A finitary and weakly algebraizable logic is fully selfextensional if and only if it is Fregean, and if and only if it is fully Fregean.*

Proof. By Corollary 80 of Czelakowski and Pigozzi (2004), a finitary protoalgebraic logic is Fregean if and only if it is fully Fregean. Since weakly algebraizable logics are protoalgebraic, this applies to them, and since they are also truth-equational, merging this with Theorem 22 we obtain the stated result. □

Thus, for finitary weakly algebraizable logics (hence, *a fortiori*, for finitary algebraizable logics), the Frege hierarchy reduces to only two classes, the selfextensional and the Fregean. That in this case these two classes are indeed different is shown by the following construction.

Example 25. Consider the language $\langle \to, \Box, \mathbf{a}, \mathbf{b}, \mathbf{c}, \top \rangle$ of type $\langle 2,1,0,0,0,0 \rangle$, and the set $A \coloneqq \{a,b,c,1\}$ with the order structure given by the following graph:



We equip it with the structure of an algebra $\mathbf{A} = \langle A, \to, \Box, a, b, c, 1 \rangle$ of the above similarity type, where the four constants are interpreted in the obvious way, and for every $x, y \in A$,

$$x \to y \coloneqq \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{otherwise,} \end{cases} \qquad \Box x \coloneqq \begin{cases} b & \text{if } x \in \{1,a,c\}, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that the implicative reduct of $\mathbf{A}$ is a Hilbert algebra.

Let $\mathcal{L}$ be the logic determined by the g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$, where

$$\mathcal{C} \coloneqq \big\{ \{1\}, \{a,1\}, \{c,1\}, A \big\}.$$

Observe that all the members of $\mathcal{C}$ are implicative filters.

Fact 1. $\mathcal{L}$ is finitary: It is well known that any logic defined by a finite set of finite matrices (hence, in particular, by a finite g-matrix) is finitary.

Fact 2. $\mathcal{L}$ is a finitely regularly algebraizable logic: The implicative fragment of $\mathcal{L}$ is a logic defined by a family of implicative filters of a Hilbert algebra, and is therefore an implicative logic in the sense of Rasiowa (1974). Moreover, it is easy to check that

$$x \to y, y \to x \vdash_{\mathcal{L}} \Box x \to \Box y.$$

As a consequence, $\mathcal{L}$ itself is an implicative logic, hence a finitely regularly algebraizable logic (Blok and Pigozzi, 1989, § 5.2).

FACT 3. $\mathcal{L}$ is selfextensional: Observe that the closure system $\mathcal{C}$ separates points in A, therefore $\varLambda\mathcal{C} = \mathrm{Id}_A$, and hence $\widetilde{\varOmega}^{\mathbf{A}}\mathcal{C} = \mathrm{Id}_A$, that is, the g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ has the property of congruence (and is reduced). This easily implies (Czelakowski and Pigozzi, 2004, Theorem 82) that $\langle \mathbf{Fm}, \mathcal{T}h\mathcal{L} \rangle$ has the property of congruence, that is, the logic $\mathcal{L}$ is selfextensional.

FACT 4. $\mathcal{L}$ is not fully Fregean: It is easy to see that the following deductions hold

$$\emptyset \vdash_{\mathcal{L}} \top \qquad \mathbf{a}, \mathbf{c} \vdash_{\mathcal{L}} x \qquad \mathbf{b} \vdash_{\mathcal{L}} x,$$

and that this implies that $\mathcal{F}i_{\mathcal{L}}\mathbf{A} = \mathcal{C}$. Therefore, $\langle \mathbf{A}, \mathcal{C} \rangle$ is a full g-model of $\mathcal{L}$. Now, consider the closure system $\mathcal{C}^{\{a,1\}} = \big\{\{a,1\}, A\big\}$. It is clear that $\langle c, b \rangle \in \varLambda\mathcal{C}^{\{a,1\}}$, because c and b belong to the same members of $\mathcal{C}^{\{a,1\}}$, and that $\langle \Box c, \Box b \rangle \notin \varLambda\mathcal{C}^{\{a,1\}}$, because $\Box c = b \notin \{a,1\}$ while $\Box b = 1 \in \{a,1\}$. Hence the g-model $\langle \mathbf{A}, \mathcal{C}^{\{a,1\}} \rangle$ does not have the property of congruence, which is to say that the full g-model $\langle \mathbf{A}, \mathcal{C} \rangle$ has not the strong property of congruence. We conclude that $\mathcal{L}$ is not fully Fregean.

FACT 5. $\mathcal{L}$ is neither Fregean nor fully selfextensional: This follows from Fact 4 and Corollary 24, taking into account that $\mathcal{L}$ is finitary (Fact 1) and weakly algebraizable (Fact 2). □

This example also solves an old open problem in abstract algebraic logic: that of whether, for protoalgebraic logics, to be selfextensional implies to be fully selfextensional; the general case was solved by Babyonyshev (2003). Example 25 shows that this is not the case, even for logics with much stronger properties, namely for finitely regularly algebraizable logics.

The reader may have noticed that Example 25 solves the issues addressed by Example 23 as well. However, the latter has the additional interest, over the former, of being in some sense "minimal" as an example of a matrix-determined non-Fregean logic, because it is defined by a 3-element matrix, and, trivially, all the logics determined by 2-element (g-)matrices are Fregean.

Finally, by combining several of the previous results, we obtain a result that clarifies the structure of the Frege hierarchy alone (although our proof goes through a class in the Leibniz hierarchy): for logics with theorems the top (smallest) class of the Frege hierarchy is actually the intersection of its two middle classes.

Theorem 26. *A logic with theorems is fully Fregean if and only if it is both Fregean and fully selfextensional.*

Proof. Trivially, if a logic is fully Fregean, then it is both Fregean and fully selfextensional. For the converse, suppose that a logic has these two properties. By Theorem 14, the logic will be truth-equational, and then we can apply Theorem 22, which tells us that, since it is assumed to be fully selfextensional, it is in fact fully Fregean. ◻

This gives a partial, positive answer (for logics with theorems) to another of the open problems formulated in Font (2003, § 6.2) and Font (2006, p. 202). Now, it becomes an Open Problem whether the assumption that the logic has theorems can be dispensed with in this result.

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References

Albuquerque, H., Font, J. M. and Jansana, R. (201x). Compatibility operators in abstract algebraic logic, *The Journal of Symbolic Logic*, 81, 417–462.

Babyonyshev, S. (2003). Fully Fregean logics, *Reports on Mathematical Logic*, 37, 59–78.

Bergman, C. (2011). *Universal Algebra. Fundamentals and Selected Topics*, CRC Press.

Blok, W. and Pigozzi, D. (1986). Protoalgebraic logics, *Studia Logica*, 45, 337–369.

Blok, W. and Pigozzi, D. (1989). *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.*, A.M.S., Providence, January, Out of print. Reprinted in the *Classic Reprints* series, Avanced Reasoning Forum, 2014.

Blok, W. and Pigozzi, D. (1992). Algebraic semantics for universal Horn logic without equality, in A. Romanowska and J. D. H. Smith, editors, *Universal Algebra and Quasigroup Theory*, pages 1–56. Heldermann, Berlin.

Bou, F. (2001). *Strict implication and subintuitionistic logics*, Master Thesis, University of Barcelona, In Spanish.

Bou, F., Esteva, F., Font, J. M., Gil, A. J., Godo, Ll., Torrens, A. and Verdú, V. (2009). Logics preserving degrees of truth from varieties of residuated lattices, *Journal of Logic and Computation*, 19, 1031–1069.

Czelakowski, J. (1981). Equivalential logics, I, II, *Studia Logica*, 40, 227–236 and 355–372.

Czelakowski, J. (1992). *Consequence operations: Foundational studies*. Reports of the research project *Theories, models, cognitive schemata*, Institute of Philosophy and Sociology, Polish Academy of Sciences, Warszawa.

Czelakowski, J. (2001). *Protoalgebraic logics*, volume 10 of *Trends in Logic - Studia Logica Library*, Kluwer Academic Publishers, Dordrecht.

Czelakowski, J. (2003). The Suszko operator. Part I, *Studia Logica (Special issue on Abstract Algebraic Logic, Part II)*, 74, 181–231.

Czelakowski, J. and Pigozzi, D. (2004). Fregean logics, *Annals of Pure and Applied Logic*, 127, 17–76.

Font, J. M. (1993). On the Leibniz congruences, In C. Rauszer, editor, *Algebraic Methods in Logic and in Computer Science*, volume 28 of *Banach Center Publications*, pages 17–36. Polish Academy of Sciences, Warszawa.

Font, J. M. (1997). Belnap's four-valued logic and De Morgan lattices, *Logic Journal of the IGPL*, 5, 413–440.

Font, J. M. (2003). Generalized matrices in abstract algebraic logic, in V. F. Hendriks and J. Malinowski, editors, *Trends in Logic. 50 years of Studia Logica*, volume 21 of *Trends in Logic - Studia Logica Library*, pages 57–86. Kluwer Academic Publishers, Dordrecht.

Font, J. M. (2006). Beyond Rasiowa's algebraic approach to non-classical logics, *Studia Logica*, 82, 172–209.

Font, J. M. (2015). Abstract algebraic logic – an introductory chapter, in N. Galatos and K. Terui, editors, *Hiroakira Ono on Residuated Lattices and Substructural Logics*, Outstanding Contributions to Logic, Springer, 68 p, to appear.

Font, J. M. (2016). *Abstract Algebraic Logic. An Introductory Textbook*, volume 60 of *Studies in Logic – Mathematical Logic and Foundations*, College Publications, London.

Font, J. M. and Jansana, R. (1996). *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*, Springer-Verlag, Out of print, *Second revised edition* published in 2009 by the Association for Symbolic Logic, and reprinted by Cambridge University Press (2017).

Font, J. M., Jansana, R. and Pigozzi, D. (2003). A survey of abstract algebraic logic, *Studia Logica (Special issue on Abstract Algebraic Logic, Part II)*, 74, 13–97, with an update in 91, 125–130, 2009.

Font, J. M., Jansana, R. and Pigozzi, D. (2006). On the closure properties of the class of full g-models of a deductive system, *Studia Logica (Special issue in memory of Willem Blok)*, 83, 215–278.

Galatos, N., Jipsen, P., Kowalski, T. and Ono, H. (2007). *Residuated lattices: an algebraic glimpse at substructural logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*, Elsevier, Amsterdam.

Gehrke, M., Jansana, R. and Palmigiano, A. (2010). Canonical extensions for congruential logics with the deduction theorem, *Annals of Pure and Applied Logic*, 161, 1502–1519.

Henkin, L., Monk, J. D. and Tarski, A. (1985). *Cylindric algebras, part II*, North-Holland, Amsterdam.

Pigozzi, D. (1991). Fregean algebraic logic. In H. Andréka, J. D. Monk, and I. Németi, editors, *Algebraic Logic*, volume 54 of *Colloquia mathematica Societatis János Bolyai*, pages 473–502, North-Holland, Amsterdam.

Raftery, J. (2006). The equational definability of truth predicates, *Reports on Mathematical Logic (Special issue in memory of Willem Blok)*, 41, 95–149.

Rasiowa, H. (1974). *An algebraic approach to non-classical logics*, volume 78 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam.

Rautenberg, W. (1993). On reduced matrices, *Studia Logica*, 52, 63–72.

Rebagliato, J. and Verdú, V. (1993). On the algebraization of some Gentzen systems, *Fundamenta Informaticae (Special issue on Algebraic Logic and its Applications)*, 18, 319–338.

Suzuki, Y., Wolter, F. and Zacharyaschev, M. (1998). Speaking about transitive frames in propositional languages, *Journal of Logic, Language and Information*, 7, 317–339.

Wójcicki, R. (1979). Referential matrix semantics for propositional calculi, *Bulletin of the Section of Logic*, 8, 170–176.

Wójcicki, R. (1988). *Theory of logical calculi. Basic theory of consequence operations*, volume 199 of *Synthèse Library*, Reidel, Dordrecht.