THE SEMANTIC ISOMORPHISM THEOREM IN ABSTRACT ALGEBRAIC LOGIC

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Abstract. One of the most interesting aspects of Blok and Pigozzi's algebraizability theory is that the notion of algebraizable logic $\mathcal{L}$ can be characterised by means of Syntactic and Semantic Isomorphism Theorems. While the Syntactic Isomorphism Theorem concerns the relation between the theories of the algebraizable logic $\mathcal{L}$ and those of the equational consequence relative to its equivalent algebraic semantics K, the Semantic Isomorphism Theorem describes the interplay between the filters of $\mathcal{L}$ on an arbitrary algebra $\boldsymbol{A}$ and the congruences of $\boldsymbol{A}$ relative to K. The pioneering insight of Blok and Jónsson, and the further generalizations by Galatos, Tsinakis, Gil-Férez and Russo, showed that the concept of algebraizability was not intrinsic to the connection between a logic and an equational consequence, thus inaugurating the abstract theory of equivalence between structural closure operators. However all these works focus only on the Syntactic Isomorphism Theorem, disregarding the semantic aspects present in the original theory. In this paper we fill this gap by introducing the notion of compositional lattice, which acts on a category of evaluational frames. In this new framework the non-linguistic flavour of the Semantic Isomorphism Theorem can be naturally recovered. In particular, we solve the problem of finding sufficient and necessary conditions for transferring a purely syntactic equivalence to the semantic level as in the Semantic Isomorphism Theorem.

1. Introduction

The theory of algebraizability was introduced by Blok and Pigozzi [5] as a common mathematical framework to describe the relations that hold between a logic and its algebraic semantics. In order to review its basic concepts, we recall that a class of algebras K is *generalized quasi-variety* if it can be axiomatized by a set of generalized quasi-equations, i.e. quasi-equations whose antecedent is a possibly infinite set of equations written with a *countable* set of variables. It is well known [4, Theorem 8.1] (see also [18]) that generalized quasi-varieties are exactly the classes of algebras closed under isomorphic copies, subalgebras, direct products and the class operator $\mathbb{U}$ defined for every class of algebras K as follows:

$$\mathbb{U}(\mathsf{K}) \coloneqq \{\boldsymbol{A} : \text{ the countably generated subalgebras of } \boldsymbol{A} \text{ belong to } \mathsf{K}\}.$$

Accordingly a logic $\mathcal{L}$ (formulated in a countable set of variables) is called *algebraizable* with respect to a generalized quasi-variety K if there are two structural transformers, of formulas into (sets of) equations and vice-versa, which allow to interpret the consequence of $\mathcal{L}$ into the equational consequence $\vDash_{\mathsf{K}}$ relative to K and which are one inverse to the other modulo the inter-derivability relation $=\!\mid\vDash_{\mathsf{K}}$. One of the central results in the theory asserts that a logic $\mathcal{L}$ is algebraizable with respect to the generalized quasi-variety K if and only if the lattices of *theories* of $\mathcal{L}$ and $\vDash_{\mathsf{K}}$ are isomorphic, once they are expanded with the actions of the monoid of *substitutions*. In abstract algebraic logic this characterization of algebraizability is known as the Syntactic Isomorphism Theorem, since it makes reference only to the common structure shared by some linguistic objects, i.e. the theories of $\mathcal{L}$ and of those of $\vDash_{\mathsf{K}}$. But the strength of algebraizability comes also from the fact that the notion of syntactic equivalence, which is implicit in its definition, transfers to the semantic level. In fact, a logic $\mathcal{L}$ is algebraizable with respect to the generalized quasi-variety K if and only if for every algebra $\boldsymbol{A}$ the lattice of *filters* of $\mathcal{L}$ on $\boldsymbol{A}$ and the lattice of *congruences* relative to K are isomorphic, when expanded with the actions of the monoid of *endomorphisms* of $\boldsymbol{A}$. This result is known as the Semantic Isomorphism Theorem, since it moves the attention to the interplay between the models of $\mathcal{L}$ and of $\vDash_{\mathsf{K}}$. Precise statements of the two results will be given in Theorems 3.1 and 3.7 below.

As soon as it was recognized that algebraizability was a synonym of the special kind of deductive equivalence between $\mathcal{L}$ and $\vDash_{\mathsf{K}}$ expressed in the Syntactic Isomorphism Theorem, it became clear that a suitable generalisation of this notion would provide a framework to describe deductive equivalences between arbitrary structural closure operators, not necessarily defined on the formulas or equations of a given algebraic language. This intuition led Blok and Jónsson [4] to give the first abstract formulation of algebraizability within the context of $\mathcal{M}$-sets, where $\mathcal{M}$ is a monoid (whose elements play the role of the ordinary substitutions) acting on a set (which can be thought of as the set of formulas over which the consequence operator is defined, equipped with the corresponding substitutions). In this setting, emphasis shifted from the linguistic aspect of the equivalence (given by the existence of transformers from formulas into equations and vice versa) to its lattice-theoretic aspect, as expressed in the Syntactic Isomorphism Theorem. Accordingly, they defined two structural closure operators on different $\mathcal{M}$-sets to be equivalent if there is an isomorphism between the complete lattices of their closed sets expanded by the actions of $\mathcal{M}$, which induce unary operations on these lattices after closing under the corresponding closure operators.

In the last years the history of the abstract version of the Syntactic Isomorphism Theorem has gone very far. One of the first steps was done by Blok and Jónsson themselves, by proving that the "only if" part of the Syntactic Isomorphism Theorem holds in the context of $\mathcal{M}$-sets too. This could have lead to optimistic expectations, but unfortunately Gil-Férez provided a counterexample to its "if" part in [16] (see also [17]). Therefore two characterisations of algebraizability, which were equivalent in the original setting, turned out not to be so when moved to the more abstract context of $\mathcal{M}$-sets. From then on, the research focused on the problem of finding necessary and sufficient conditions under which the Syntactic Isomorphism Theorem could be recovered even in its abstract version.

The next step of the abstraction process was due to Galatos and Tsinakis in [15], which moved the study of algebraizability to the even more general context of modules over complete residuated lattices. Their idea is to split the equivalences in two halves, which they call structural representations. Then the quest became that of finding sufficient and necessary conditions (on a module over a complete residuated lattice) under which every structural representation is induced by a module morphism. The cornerstone of their work was the intuition that this problem could be elegantly solved with categorical tools: equipping the modules over a fixed complete residuated lattice with a categorical structure, they characterised the desired objects as the projective ones[1]. Galatos and Tsinakis' successful approach suggested further research in this direction in even more abstract contexts by Galatos and Gil-Férez [13], and by Russo [23]. To come full circle, Font and Moraschini applied in [12] those general results in order to recover a solution intrinsic to the motivating setting of $\mathcal{M}$-sets.

However, all these abstractions focused on the study of the Syntactic Isomorphism Theorem. As a side effect an interesting feature of the original theory, namely the non-linguistic flavour of algebraizability expressed in the Semantic Isomorphism Theorem, has been disregarded. In this paper we fill this gap by introducing the new framework of categories of evaluational frames (over a fixed compositional lattice), whose objects consist in a suitable generalization of the pairs made up by the set of formulas of an algebraic language and one of its algebraic models. This double nature of evaluational frames makes it possible to restore the interplay between the models of two equivalent structural closure operators. Unfortunately, as it happened in the case of $\mathcal{M}$-sets, some evaluational frames may fail to enjoy the abstract version of the Semantic Isomorphism Theorem. Therefore our main goal will be that of characterizing the evaluational frames for which the Semantic Isomorphism Theorem can be recovered. We do this and prove that every evaluational frame that comes from the concrete example of the set of formulas (or the set of equations, sequents, hypersequents etc.) is of this kind. As a consequence we will obtain a uniform way of establishing the Semantic Isomorphism Theorem for almost every concrete kind of algebraizability known in the literature, e.g., for sentential logics, k-deductive systems, Gentzen systems, hypersequents etc.

The structure of the paper is as follows. In Section 2 we recall some basic concepts and notations from residuation theory and universal algebra we make use of along the paper, and in Section 3 we sketch a brief history of the work done until now, in order to recall all the definitions and results we shall need. Then in Section 4 we build a conceptual setting, in which it is possible

[1] In fact Galatos and Tsinakis characterised these objects as the onto-projective ones, but Gil-Férez proved in [16] that, in categories of modules over complete residuated lattices, epis coincide with surjective morphisms.

to recover the semantic aspects of algebraizability. Our approach takes inspiration from the fact that the key feature of the Semantic Isomorphism Theorem is the description of the interplay between the set of formulas, over which a logic is formulated, and its algebraic models. In the original theory, this interplay is mainly made through substitutions on the set of formulas, endomorphisms on the models and evaluations from the set of formulas to the models, on which the notion of deductive filter relies. For this reason we introduce a category of new objects, called evaluational frames over a compositional residuated lattice, which are pairs made up by a syntactic and a semantic component, which behave respectively as a set of formulas (of an algebraic language) and one of its algebraic models. Even if in this general setting we loose the inner difference between these two components, it is still possible to recover their interplay thanks to the action of the compositional residuated lattice, which is a structure intended to incorporate suitable generalisations of substitutions, endomorphisms and evaluations.

In Section 5 we show that the presence of mappings that play the role of substitutions and evaluations in the motivating case allows to introduce, respectively, abstract notions of logic and of deductive filter in this new framework. As we recovered the possibility of describing the interplay between the set of formulas and its models in terms of logics and deductive filters, this concludes the building of the conceptual setting in which the study of an abstract version of the Semantic Isomorphism Theorem can be carried on.

The central point for passing from the Syntactic to the Semantic Isomorphism Theorem depends on the possibility of interpreting purely syntactical transformers between formulas and equations into transformers between models. Even if in the original theory this possibility is ensured by interpreting formulas and equations, respectively, as elements and pairs of elements of the models, in the more general framework of evaluational frames there is no standard procedure for making this interpretation. In fact, we will provide an example of a syntactic transformer between evaluational frames that completely lacks an interpretation (Example 8.11). Hence it is clear that, within the context of evaluational frames, the analysis of the semantic aspects of algebraizability should consist in the study of the conditions under which such an interpretation can be recovered.

According to this intuition, in Section 6 we show that an evaluational frame enjoys one half of the Semantic Isomorphism Theorem if and only if it interprets every syntactic transformer in an unique way (Theorem 6.2). We call INT this last property. Therefore the quest for sufficient and necessary conditions under which the Semantic Isomorphism Theorem can be recovered coincides with the possibility of characterising the evaluational frames with the INT. First we characterise them in terms of their inner structure (Theorem 6.7). Then we use this result in order to describe the relation, which holds between a given category of evaluational frames and that of modules which play the role of their syntactic components, as a categorical adjunction (Theorem 6.10). In Section 7 we introduce a weaker interpretability condition, called WINT, by requiring only existence (but not uniqueness) of interpretations of syntactic transformers. Then we characterise categorically the evaluational frames that enjoy it (Theorem 7.4). Quite remarkably, it turns out that projective evaluational frames, which arise naturally from the abstract study of the Syntactic Isomorphism Theorem, satisfy the WINT (Theorem 7.5).

In Section 8 we focus on evaluational frames that enjoy the full algebraizability property FAL, i.e., evaluational frames for which both the Syntactic and the Semantic Isomorphism Theorem hold. Even if their general characterization is purely categorical (Theorem 8.4), we are able to describe their inner structure at least in the well-behaved case of cyclic and projective ones (Theorem 8.9). We end the paper characterising the evaluational frames with the FAL that arise from structures similar to $\mathcal{M}$-sets in a way similar to the one presented in [12]. Interestingly enough, these last evaluational frames turn out to possess a very general kind of variable, which relates them to the motivating examples of algebraic languages equipped with their algebraic models.

2. Preliminaries

In this section we fix some terminology and notation in residuation theory and universal algebra we will make use of; for standard background on these topics we refer the reader to [14]. For information on abstract algebraic logic and category theory we refer the reader to [8, 9, 10, 11] and [1, 2, 19] respectively. The symbol "$\mathcal{P}(\cdot)$" always denotes the power set construction. We write $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$ ... for posets, in particular for lattices. All along the paper we denote universes by italic capital letters and whole structures with other typefaces: for example, in the case of posets, we will write $\mathcal{Q} = \langle Q, \leqslant \rangle$. Moreover, we will reduce the use of parentheses in unary functions to a reasonable minimum. A ***closure operator*** on a poset $\mathcal{R}$ is a monotone function $\gamma\colon R \to R$ such that $x \leqslant \gamma x = \gamma\gamma x$ for all $x \in R$. Given two closure operators γ and δ on $\mathcal{R}$ we will write $\gamma \leqslant \delta$ to denote that $\gamma x \leqslant \delta x$ for every $x \in R$.

A map $f\colon \mathcal{Q} \to \mathcal{R}$ is ***residuated*** when there is another map $f^+\colon \mathcal{R} \to \mathcal{Q}$ such that for all $x \in Q$, $y \in R$ the following holds:

$$fx \leqslant y \iff x \leqslant f^+ y.$$

In this case we say that f^+ is the ***residuum*** of f. It can be easily proved that if $f\colon \mathcal{Q} \to \mathcal{R}$ is residuated, then its residuum f^+ is uniquely determined and is defined as follows: for every $y \in R$,

$$f^+ y = \max\{x \in Q : fx \leqslant y\}.$$

Since we will deal only with complete lattices we will make use of a nice characterisation result, namely that the residuated maps between complete lattices coincide with the functions that preserve arbitrary suprema. This means that in the particular case of power set lattices, a map $f\colon \mathcal{P}(Q) \to \mathcal{P}(R)$ is residuated if and only if it is determined by its restriction to unitary subsets, i.e., $fX = \bigcup\big\{ f\{x\} : x \in X \big\}$ for all $X \subseteq Q$; then the residuum can be defined as $f^{+}Y = \big\{ x \in R : f\{x\} \subseteq Y \big\}$ for all $Y \subseteq R$. In the (very common) case where f has been actually defined from a point function from Q to R, then the residuum f^{+} is just the ordinary "inverse image" function.

The notion of residuation can be applied to binary functions by fixing one argument; the resulting notions will provide a fundamental tool in our analysis of structures abstracting sets of formulas and models of a given type. More precisely, we say that a map $\cdot\colon \mathcal{Q} \times \mathcal{R} \to \mathcal{S}$ is ***residuated*** when there are two functions $\backslash\!.\colon \mathcal{Q} \times \mathcal{S} \to \mathcal{R}$ and $/\!.\colon \mathcal{S} \times \mathcal{R} \to \mathcal{Q}$ such that for every $x \in Q, y \in R$ and $z \in S$, the following holds:

$$x \cdot y \leqslant z \iff y \leqslant x \backslash\!. \, z \iff x \leqslant z \,/\!. \, y.$$

In this case we say that $\backslash\!.$ and $/\!.$ are respectively the ***left*** and ***right residuum*** of $\cdot$. The residua are uniquely determined and can be defined as follows: for every $x \in Q, y \in R$ and $z \in S$

$$x \backslash\!. \, z = \max\{r \in R : x \cdot r \leqslant z\} \qquad z \,/\!. \, y = \max\{q \in Q : q \cdot y \leqslant z\}.$$

Also in this case we get a nicer characterization for complete structures: if $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ are complete lattices, then residuated maps $\cdot\colon \mathcal{Q} \times \mathcal{R} \to \mathcal{S}$ coincide with functions that preserve arbitrary suprema in both coordinates.

Then an algebra $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \backslash\!., /\!., 1\rangle$ is a ***residuated lattice*** if $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid and $\cdot\colon A \times A \to A$ is a residuated mapping (with respect to the lattice order of $\langle A, \wedge, \vee\rangle$) with residua $\backslash\!.$ and $/\!.$. We say that $\mathcal{A}$ is ***complete*** if its lattice reduct $\langle A, \wedge, \vee\rangle$ is complete. An alternative usual name for these structures is "unital quantale".

Let us now briefly fix some category-theoretic notations. Let $\mathcal{C}$ be a category. Then, given two objects A and B, we denote by $\mathcal{H}om_{\mathcal{C}}(\mathsf{A}, \mathsf{B})$ the collections of arrows in $\mathcal{C}$ from A to B. Moreover 1_{A} will be the identity arrow on A. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An ***adjunction*** is a tuple $\langle \mathcal{F}, \mathcal{G}, \eta, \varepsilon\rangle$ such that $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ are functors and $\eta\colon 1_{\mathcal{C}} \to \mathcal{G}\mathcal{F}$ and $\varepsilon\colon \mathcal{F}\mathcal{G} \to 1_{\mathcal{D}}$ natural transformations such that $\varepsilon_{\mathcal{F}(\mathsf{A})} \circ \mathcal{F}(\eta_{\mathsf{A}}) = 1_{\mathcal{F}(\mathsf{A})}$ and $\mathcal{G}(\varepsilon_{\mathsf{B}}) \circ \eta_{\mathcal{G}(\mathsf{B})} = 1_{\mathcal{G}(\mathsf{B})}$ for every object A of $\mathcal{C}$ and B of $\mathcal{D}$. In this case we say that $\mathcal{F}$ is a ***left adjoint*** to $\mathcal{G}$ and that $\mathcal{G}$ is a ***right adjoint*** to $\mathcal{F}$. It is worth remarking that left adjoint functors preserve colimits. We denote by $\mathcal{F} \dashv \mathcal{G}$ the fact that there is an adjunction between $\mathcal{C}$ and $\mathcal{D}$ where $\mathcal{F}$ is left adjoint to $\mathcal{G}$ and $\mathcal{G}$ is a right adjoint to $\mathcal{F}$.

Finally let us briefly sketch the few concepts and definitions of universal algebra we will make use of. We will assume all along the paper that the algebras $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}\dots$, we are working with, are of the same similarity type. Given two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we let $\mathcal{H}om(\boldsymbol{A}, \boldsymbol{B})$ be the set of homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$ and $\mathcal{E}nd(\boldsymbol{A}) \coloneqq \mathcal{H}om(\boldsymbol{A}, \boldsymbol{A})$ the set of endomorphisms of $\boldsymbol{A}$. As usual, $\mathrm{Con}\boldsymbol{A}$ will denote the complete lattice of congruences of $\boldsymbol{A}$. Moreover, Fm will be the set of formulas of the fixed type built up from countably many variables and $\boldsymbol{Fm}$ the associated formula algebra. Accordingly, we denote by Eq the set of equations built up from Fm. Formally speaking, equations are just pairs of formulas, i.e., $Eq := Fm \times Fm$, and sometimes it will be useful to think of them in this way, but we shall write them also in the more suggestive notation $\alpha \approx \beta$.

Along the paper we will be interested in ***generalised quasi-varieties***, which are classes of algebras axiomatized by generalised quasi-equations, i.e., quasi-equations where the antecedent is the conjunction of a possibly infinite set of equations. Let K be a generalised quasi-variety and $\boldsymbol{A}$ an arbitrary algebra. We denote by $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ set of congruences of $\boldsymbol{A}$ which yield a quotient in K and refer to them as congruences of ***$\boldsymbol{A}$ relative to*** K. When K is a generalised quasi-variety it is possible to prove that $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is a complete lattice whose arbitrary meets coincide with those of $\mathrm{Con}\boldsymbol{A}$. Therefore it is safe to define a closure operator $\mathit{\Theta}^{\boldsymbol{A}}_{\mathsf{K}} \colon \mathcal{P}(A \times A) \to \mathcal{P}(A \times A)$ of generation of the congruences of $\boldsymbol{A}$ relative to K, that is, $\mathit{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(X)$ is the smallest congruence of $\boldsymbol{A}$ relative to K which contains X, for every $X \subseteq A \times A$. Moreover, the ***equational consequence relative to*** K is the consequence relation over the set of equations defined, for every $\mathit{\Sigma} \cup \{\alpha \approx \beta\} \subseteq Eq$, as follows:

$$\mathit{\Sigma} \vDash_{\mathsf{K}} \alpha \approx \beta \iff \text{ for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } h \in \mathcal{H}om(\boldsymbol{Fm}, \boldsymbol{A})$$
$$\text{if } h\varepsilon = h\delta \text{ for every } \varepsilon \approx \delta \in \mathit{\Sigma}, \text{ then } h\alpha = h\beta.$$

When K is a a generalised quasi-variety, it turns out that $\mathit{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(\cdot)$ coincides with the generation of deductive filters over $\boldsymbol{A}$ of the equational consequence relative to K.

Finally, we will denote restrictions of a given map to subsets of its domain as the original one, since no confusion shall occur.

3. Deductive equivalence

In the late 80's the analysis of the relations that hold between a logic, i.e., a structural closure operator over the poset $\langle \mathcal{P}(Fm), \subseteq \rangle$, and its associated algebraic semantics culminated with the introduction of the theory of algebraizability, due to Wim Blok and Don Pigozzi [5] and later refined by Herrmann, Czelakowski and Jansana. Its key point is the usage of what are called, in the present terminology, structural transformers to establish a deductive equivalence between the consequence of the logic and the equational consequence relative to the class of algebras which plays the role of its algebraic semantics.

More precisely, a map $\tau \colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ is a ***structural transformer*** (from formulas to equations) when there is a set $E(x)$ of equations in a single variable x such that for all $\mathit{\Gamma} \subseteq Fm$,

$$\tau(\mathit{\Gamma}) = \left\{ \sigma_\varphi \alpha \approx \sigma_\varphi \beta : \alpha \approx \beta \in E(x) \, , \varphi \in \mathit{\Gamma} \right\}$$

where $\sigma_\varphi \colon Fm \to Fm$ is any substitution sending the variable x to φ. It is easy to see that this is equivalent to requiring that $\tau \colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ is a map that commutes with unions and with substitutions, that is, it is residuated and commutes with substitutions. By claiming that τ commutes with substitutions, we mean just that $\tau\sigma(\Gamma) = \{\langle \sigma\alpha, \sigma\beta \rangle : \langle \alpha, \beta \rangle \in \tau(\Gamma)\}$ for every $\Gamma \subseteq Fm$ and every substitution σ. Dually, we say that a map $\rho \colon \mathcal{P}(Eq) \to \mathcal{P}(Fm)$ is a structural transformer (from equations into formulas) when it is residuated and commutes with substitutions.

Omitting several finiteness assumptions, which play no relevant role in the formulation of the subsequent results, Blok and Pigozzi's original definition amounts to the followings: a logic $\mathcal{L}$ is ***algebraizable*** with equivalent algebraic semantics the generalised quasi-variety K, when there are two structural transformers $\tau \colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ and $\rho \colon \mathcal{P}(Eq) \to \mathcal{P}(Fm)$ satisfying the following conditions:

A1. $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if $\tau\Gamma \vDash_{\mathsf{K}} \tau\varphi$; and
A2. $x \approx y \dashv\vDash_{\mathsf{K}} \tau\rho(x \approx y)$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$ and $x \approx y \in Eq$. For example, the intuitionistic propositional calculus is algebraizable with equivalent algebraic semantics the variety of Heyting algebras through the structural transformers $\tau(x) \coloneqq \{x \approx 1\}$ and $\rho(x, y) \coloneqq \{x \to y, y \to x\}$.

One of the strengths of the notion of algebraizability is that it can be equivalently characterised from several points of view. In particular, one of the most important results of Blok and Pigozzi is the so called Syntactic Isomorphism Theorem, which characterises algebraizability in terms of the existence of an isomorphism $\Phi \colon \mathcal{T}h\mathcal{L} \to \mathcal{T}h{\vDash_{\mathsf{K}}}$ between the expanded lattices of theories $\mathcal{T}h\mathcal{L}$ and $\mathcal{T}h{\vDash_{\mathsf{K}}}$ of both consequences; the expansion is given by the endomorphisms of the formula algebra, $\mathcal{E}nd(\boldsymbol{Fm})$, which induce unary operations on these lattices after closing under the corresponding closure operators, which are denoted respectively by $C_{\mathcal{L}}$ and C_{K}. To be more precise, they prove the following.

Theorem 3.1 (Syntactic Isomorphism)**.** *Let $\mathcal{L}$ be a logic and K a generalised quasi-variety. $\mathcal{L}$ is algebraizable with equivalent algebraic semantics K if and only if there is an isomorphism $\Phi \colon \mathcal{T}h\mathcal{L} \to \mathcal{T}h{\vDash_{\mathsf{K}}}$ such that $\Phi C_{\mathcal{L}}\sigma = C_{\mathsf{K}}\sigma\Phi$ over $\mathcal{T}h\mathcal{L}$ for every $\sigma \in \mathcal{E}nd(\boldsymbol{Fm})$.*

After the introduction of the notion of algebraizable logic, it was soon understood that the idea of algebraizability is not intrinsic to the connection between two structural consequences, one defined over formulas and another over equations. This led to the extension of the theory to k-dimensional systems by Blok and Pigozzi themselves [6], and independently to Gentzen systems, studied first by Rebagliato and Verdú [22] and then by Raftery [20]. These investigations suggested that what lies behind the idea of algebraizability is simply a notion of equivalence between two structural closure operators, but it was only with the study of this notion in the much more abstract context of $\mathcal{M}$-sets by Blok and Jónsson [4] that an appropriate mathematical framework to formulate such intuitions was found.

In fact $\mathcal{M}$-sets abstract the notion of a collection of syntactic objects (formulas, equations, sequents, hypersequents, etc.) built up from a fixed algebraic language, equipped with the natural action of ordinary substitutions on them. More precisely, let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be a monoid; then $\mathsf{R} = \langle R, \star\rangle$ is a (left) $\mathcal{M}$***-set*** when R is a non-empty set and $\star \colon M \times R \to R$ is a map, called the ***action*** of the monoid on R, satisfying the following conditions:

M1. $(\sigma \cdot \sigma') \star x = \sigma \star (\sigma' \star x)$; and
M2. $1 \star x = x$,

for every $\sigma, \sigma' \in M$ and $x \in R$. The notion of right $\mathcal{M}$-set is defined in a dual way. Since we will work mainly with left $\mathcal{M}$-sets, we will simply talk about "$\mathcal{M}$-sets", assuming that they are left ones unless explicitly stated otherwise. The next example confirms the claim that $\mathcal{M}$-sets can be used to describe collections of syntactic objects equipped with substitutions.

Example 3.2. Consider an algebra $\boldsymbol{A}$. Let $\mathcal{M}(\boldsymbol{A}) \coloneqq \langle \mathcal{E}nd(\boldsymbol{A}), \circ, 1_A\rangle$ be the monoid of its endomorphisms. Moreover put $\mathrm{Eq}(A) \coloneqq A \times A$ and $\mathrm{Seq}(A) \coloneqq F(A) \times F(A)$, where $F(A)$ is the set of finite sequences of elements of A. Then consider the mappings

$$\begin{aligned} \star_{\mathsf{A}} &\colon \mathcal{E}nd(\boldsymbol{A}) \times A \to A \\ \star_{\mathrm{Eq}(\mathsf{A})} &\colon \mathcal{E}nd(\boldsymbol{A}) \times \mathrm{Eq}(A) \to \mathrm{Eq}(A) \\ \star_{\mathrm{Seq}(\mathsf{A})} &\colon \mathcal{E}nd(\boldsymbol{A}) \times \mathrm{Seq}(A) \to \mathrm{Seq}(A) \end{aligned}$$

defined respectively as $\sigma \star_{\mathsf{A}} a \coloneqq \sigma a$, $\sigma \star_{\mathrm{Eq}(\mathsf{A})} \langle a, c\rangle \coloneqq \langle \sigma a, \sigma c\rangle$ and $\sigma \star_{\mathrm{Seq}(\mathsf{A})} \langle \overline{a}, \overline{c}\rangle \coloneqq \langle \sigma\overline{a}, \sigma\overline{c}\rangle$ for every $\sigma \in \mathcal{E}nd(\boldsymbol{A})$, $a \in A$, $\langle a, c\rangle \in \mathrm{Eq}(A)$ and $\langle \overline{a}, \overline{c}\rangle \in \mathrm{Seq}(A)$. It is easy to prove that $\mathsf{A} \coloneqq \langle A, \star_{\mathsf{A}}\rangle$, $\mathrm{Eq}(\mathsf{A}) \coloneqq \langle \mathrm{Eq}(A), \star_{\mathrm{Eq}(\mathsf{A})}\rangle$ and $\mathrm{Seq}(\mathsf{A}) \coloneqq \langle \mathrm{Seq}(A), \star_{\mathrm{Seq}(\mathsf{A})}\rangle$ are $\mathcal{M}(\boldsymbol{A})$-sets.

In the particular case of the monoid $\mathcal{M}(\boldsymbol{Fm})$, whose elements are the ordinary substitutions, we put $\mathsf{Eq} \coloneqq \mathrm{Eq}(\mathsf{Fm})$ and $\mathsf{Seq} \coloneqq \mathrm{Seq}(\mathsf{Fm})$. Moreover we refer to Fm, Eq and Seq respectively as to the $\mathcal{M}(\boldsymbol{Fm})$-sets of ***formulas***, ***equations*** and ***sequents***. ⊠

As we mentioned before, two structural closure operators on different $\mathcal{M}$-sets are said to be equivalent if there is an isomorphism between the complete lattices of their closed sets expanded by actions of $\mathcal{M}$. Therefore the notion of equivalence, within the context of $\mathcal{M}$-sets, abstracts the aspects of algebraizability expressed in the second condition of Theorem 3.1. Unfortunately it is not possible to obtain an abstract version of the whole Syntactic Isomorphism Theorem in the context of $\mathcal{M}$-sets, and actually a counterexample to its "if" part was found in [17]. Therefore the subsequent investigation focused on the problem of finding sufficient and necessary conditions under which the whole Syntactic Isomorphism Theorem could be recovered in this abstract setting.

This quest led Galatos and Tsinakis to move the problem to the yet more abstract setting of categories of modules over complete residuated lattices [15]. In order to do this they "lift" to power sets all the constructions done until now (see Example 3.3). More precisely, given a complete residuated lattice $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$, we say that $\mathbb{R} = \langle R, \wedge, \vee, *_{\mathbb{R}}\rangle$ is a (left) ***module***

over $\mathcal{A}$, or a (left) $\mathcal{A}$***-module***, when $\langle R, \wedge, \vee \rangle$ is a complete lattice and $*_{\mathbb{R}} \colon A \times R \to R$ a residuated mapping satisfying the following conditions:

R1. $(\sigma \cdot \sigma') *_{\mathbb{R}} x = \sigma *_{\mathbb{R}} (\sigma' *_{\mathbb{R}} x)$; and

R2. $1 *_{\mathbb{R}} x = x$,

for all $\sigma, \sigma' \in A$ and $x \in R$. We will use letters $\mathbb{R}$, $\mathbb{S}$, $\mathbb{T}\dots$ to denote (left) $\mathcal{A}$-modules. The notion of right $\mathcal{A}$-module is defined dually. Since we will work mainly with left $\mathcal{A}$-modules, we will simply talk about "$\mathcal{A}$-modules", assuming that they are left ones unless explicitly stated otherwise.

Example 3.3. Let $\mathcal{M} = \langle M, \cdot^{\mathcal{M}}, 1^{\mathcal{M}} \rangle$ be a monoid. The algebra

$$\mathcal{P}(\mathcal{M}) \coloneqq \langle \mathcal{P}(M), \cap, \cup, \cdot, \backslash\cdot, /\cdot, 1 \rangle,$$

where $\cdot \colon \mathcal{P}(M) \times \mathcal{P}(M) \to \mathcal{P}(M)$ is the operation obtained lifting the product $\cdot^{\mathcal{M}}$ of the monoid to the powerset, $\backslash\cdot$ and $/\cdot$ are its residua and $1 = \{1^{\mathcal{M}}\}$, is a complete residuated lattice. Moreover, given an $\mathcal{M}$-set $\mathsf{R} = \langle R, \star \rangle$, the structure $\mathcal{P}(\mathsf{R}) \coloneqq \langle \mathcal{P}(R), \cap, \cup, \star_{\mathcal{P}(\mathsf{R})} \rangle$, where $\star_{\mathcal{P}(\mathsf{R})}$ is the function obtained by lifting the action $\star$ to the power set, is a $\mathcal{P}(\mathcal{M})$-module.

Let us now simplify the notation in the case $\mathcal{M} = \mathcal{M}(\boldsymbol{A})$ for some algebra $\boldsymbol{A}$. We put $\mathbb{A} \coloneqq \mathcal{P}(\mathsf{A})$, $\mathrm{Eq}(\mathbb{A}) \coloneqq \mathcal{P}(\mathrm{Eq}(\mathsf{A}))$ and $\mathrm{Seq}(\mathbb{A}) \coloneqq \mathcal{P}(\mathrm{Seq}(\mathsf{A}))$. Clearly they are $\mathcal{P}(\mathcal{M}(\boldsymbol{A}))$-modules. If $\mathcal{M} = \mathcal{M}(\boldsymbol{Fm})$, we put $\mathbb{E}\mathrm{q} \coloneqq \mathrm{Eq}(\mathbb{F}\mathrm{m})$ and $\mathbb{S}\mathrm{eq} \coloneqq \mathrm{Seq}(\mathbb{F}\mathrm{m})$. Moreover we refer to $\mathbb{F}\mathrm{m}$, $\mathbb{E}\mathrm{q}$ and $\mathbb{S}\mathrm{eq}$ respectively as to the $\mathcal{P}(\mathcal{M}(\boldsymbol{Fm}))$-modules of ***formulas***, ***equations*** and ***sequents***. $\boxtimes$

In order to explain how it is possible to state a result analogous to Theorem 3.1 in the context of $\mathcal{A}$-modules, let us recall some basic concepts. A ***module morphism*** $\tau \colon \mathbb{R} \to \mathbb{S}$ from $\mathbb{R}$ to $\mathbb{S}$ is a residuated mapping $\tau \colon R \to S$ such that $\tau(\sigma *_{\mathbb{R}} x) = \sigma *_{\mathbb{S}} \tau x$ for every $\sigma \in A$ and $x \in R$. $\mathcal{A}$-modules and module morphisms between them form a category which we denote by $\mathcal{A}$-Mod. We say that a function $\gamma \colon R \to R$ is a ***structural closure operator*** on $\mathbb{R}$ when it is a closure operator on $\langle R, \wedge, \vee \rangle$ such that for every $\sigma \in A$ and $x \in R$,

$$\sigma *_{\mathbb{R}} \gamma x \leqslant \gamma(\sigma *_{\mathbb{R}} x).$$

Given a structural closure operator γ on $\mathbb{R}$ we put $\mathbb{R}_\gamma \coloneqq \langle \gamma[R], \wedge^{\mathbb{R}}, \gamma\vee^{\mathbb{R}}, \gamma*_{\mathbb{R}} \rangle$, where $\gamma\vee^{\mathbb{R}}$ is the function defined as $\gamma\vee^{\mathbb{R}}(x, y) = \gamma(x \vee^{\mathbb{R}} y)$ for every $x, y \in \gamma[R]$ and similarly for $\gamma*_{\mathbb{R}}$. It is easy to prove that $\mathbb{R}_\gamma$ is still a module over $\mathcal{A}$ and that $\gamma \colon \mathbb{R} \to \mathbb{R}_\gamma$ is a surjective module morphism.

The central idea of Galatos and Tsinakis is that of shifting the attention from the whole isomorphism to one of its symmetrical halves. More precisely, given two structural closure operators γ and δ on $\mathbb{R}$ and $\mathbb{S}$ respectively, we say that $\Phi \colon \mathbb{R}_\gamma \to \mathbb{S}_\delta$ is a ***structural representation*** (of γ into δ) when it is a module morphism such that

$$x \leqslant y \Longleftrightarrow \Phi x \leqslant \Phi y$$

for every $x, y \in R$. Then one might say that γ and δ are deductively equivalent when there are two structural representations, one of γ into δ and one of δ into γ, that are mutually inverse. The recovery of the Syntactic

Isomorphism Theorem here would be to know when such an isomorphism arises from linguistic transformers between the universes of $\mathbb{R}$ and $\mathbb{S}$ as in the original situation.

Accordingly, we say that a representation $\Phi\colon \mathbb{R}_\gamma \to \mathbb{S}_\delta$ of γ into δ is ***induced*** by a module morphism $\tau\colon \mathbb{R} \to \mathbb{S}$ from $\mathbb{R}$ to $\mathbb{S}$ when the following diagram commutes.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tau"] \arrow[d, two heads, "\gamma"'] & \mathbb{S} \arrow[d, two heads, "\delta"] \\ \mathbb{R}_\gamma \arrow[r, "\Phi"'] & \mathbb{S}_\delta \end{tikzcd}$$

We say that $\mathbb{R}$ has the REP, i.e., the ***the representation property***, when for every $\mathbb{S}$ and every γ and δ structural closure operators on $\mathbb{R}$ and $\mathbb{S}$ respectively, every structural representation of γ into δ is induced by a structural transformer from $\mathbb{R}$ to $\mathbb{S}$. Thus, the (one half of the) original problem consists in finding a necessary and sufficient condition for an $\mathcal{A}$-module $\mathbb{R}$ to have the REP.

Gil-Férez proved in [16, Proposition 4.21] that epimorphisms in the category $\mathcal{A}$-Mod coincide with surjective module morphisms. Hence we can express the solution of the representation problem in the context of $\mathcal{A}$-modules given by Galatos and Tsinakis in [15, Theorem 5.1] as follows:

Theorem 3.4. *An $\mathcal{A}$-module has the* REP *if and only if it is projective.*

Therefore the quest for the understanding of the representation problem in this new framework turned out to coincide with the study of projective objects in $\mathcal{A}$-Mod, which were characterised by Gil-Férez in a very abstract way [16, Theorem 4.51]. However, in order to get more information about their inner structure we shall restrict to a particular class of well-behaved $\mathcal{A}$-modules, which have a special element which plays a role somehow similar to the one of a variable in the formula algebra. More precisely we say that $\mathbb{R}$ is ***cyclic*** when there is an element $v \in R$ such that $\{\sigma \star_{\mathbb{R}} v : \sigma \in A\} = R$. In this case we say that v is a ***generator*** of $\mathbb{R}$. Moreover, observe that the complete residuated lattice $\mathcal{A}$ is an $\mathcal{A}$-module itself, which we denote by $\mathbb{A}$.

Theorem 3.5. *Let $\mathbb{R}$ be an $\mathcal{A}$-module. The following conditions are equivalent:*

(i) *$\mathbb{R}$ is cyclic and projective.*
(ii) *$\mathbb{R}$ is a retract of $\mathbb{A}$.*
(iii) *There are $u \in A$ and a generator v of $\mathbb{R}$ such that $u \star v = v$ and $((a \star v) /_{\star} v) \cdot u = a \cdot u$ for every $a \in A$.*

Proof. The equivalence between (i) and (ii) follows from [12, Theorem 8] and that between (i) and (iii) from [15, Theorem 5.7]. ⊠

Putting $v \coloneqq \{x\}$ for some variable x and $u \coloneqq \{\sigma\}$, where σ is the substitution sending all variables to x, it is easy to prove that $\mathbb{F}\mathrm{m}$ satisfies condition (iii) of Theorem 3.5. The same happens for $\mathbb{E}\mathrm{q}$, if we put $v \coloneqq \{x \approx y\}$ for two different variables x and y and $u \coloneqq \{\sigma\}$, where σ is the substitution sending x to x and every variable different from x to y. Therefore from Theorem 3.5 it follows that $\mathbb{F}\mathrm{m}$ and $\mathbb{E}\mathrm{q}$ are cyclic and projective [15, Corollary 5.9]. By Theorem 3.4 we conclude that they enjoy the REP. From this it follows that, in the case of formulas and equations, all deductive equivalences are induced by transformers, that is, they correspond to algebraizability in the original sense. This fact somehow justifies that the abstract generalisation matches the original concepts.

In order to prove that $\mathbb{S}\mathrm{eq}$ has the REP too, we need to introduce some new concept. In [15, Lemma 5.11] it is shown that $\mathcal{A}$-Mod has coproducts. More precisely, the coproduct $\coprod_{i \in I} \mathbb{R}_i$ of a family of $\mathcal{A}$-modules $\{\mathbb{R}_i\}_{i \in I}$ is the $\mathcal{A}$-module whose universe it the cartesian product of $\{R_i\}_{i \in I}$, with lattice operations and scalar multiplication defined component-wise, equipped with the family of module morphisms $\{\pi_i \colon \mathbb{R}_i \to \coprod_{i \in I} \mathbb{R}_i\}_{i \in I}$ defined for every $i \in I$ as

$$\pi_i(x)(j) = \begin{cases} x & \text{if } j = i \\ \bot^{\mathbb{R}_j} & \text{otherwise} \end{cases}$$

for every $x \in R_i$ and $j \in I$. Since $\mathbb{S}\mathrm{eq}$ is the coproduct of a family of cyclic and projective $\mathcal{P}(\mathcal{M}(\boldsymbol{Fm}))$-modules [15, Theorem 5.13] and projectivity is preserved by coproducts, we conclude that $\mathbb{S}\mathrm{eq}$ has the REP too.

Finally, thanks to the work of Galatos and Tsinakis in the more general context of modules over complete residuated lattices, Font and Moraschini characterised in [12] the $\mathcal{M}$-sets for which an abstract version of the Syntactic Isomorphism Theorem can be recovered. For the present aim, it will be useful to state this result in the following way.

Theorem 3.6. *Let R be an $\mathcal{M}$-set. $\mathcal{P}(\mathsf{R})$ has the REP if and only if it it is the coproduct of a family of cyclic and projective $\mathcal{P}(\mathcal{M})$-modules.*

Proof. This is a direct consequence of Theorems 15 and 28 of [12]. ⊠

However, a side effect of all these abstractions is that the unified and abstract treatment of any kind of structural closure operators blurs the distinction between formula algebras and arbitrary algebras. As a consequence an interesting feature of the original theory is disregarded, namely the interplay that exists in the concrete cases between the theory lattice on the formula algebra and the filter lattices on arbitrary algebras. In particular this relation reflects in the fact, also discovered by Blok and Pigozzi, that, given an arbitrary algebra $\boldsymbol{A}$ of the similarity type under consideration, the isomorphism of Theorem 3.1 can be extended to a new one between its expanded lattice of deductive filters $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ of $\mathcal{L}$ over $\boldsymbol{A}$ and that of congruences $\mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$ of $\boldsymbol{A}$ relative to K; the expansion is given by endomorphisms $\mathcal{E}nd(\boldsymbol{A})$, which induce unary operations on these lattices after closing respectively under the closure operators $Fi^{\boldsymbol{A}}_{\mathcal{L}}$ of generation of deductive filters of $\mathcal{L}$ over $\boldsymbol{A}$ and $\varTheta^{\boldsymbol{A}}_{\mathsf{K}}$ respectively. This semantic isomorphism is achieved by interpreting the structural transformers in a standard way. For the present aim, it is useful to express the result essentially due to Blok and Pigozzi in the following form.

Theorem 3.7 (Semantic Isomorphism)**.** *Let $\mathcal{L}$ be a logic,* K *a generalised quasi-variety and* $\boldsymbol{A}$ *an arbitrary algebra. If there is an isomorphism* $\Phi\colon \mathcal{T}h\mathcal{L} \to \mathcal{T}h \vDash_{\mathsf{K}}$ *such that* $\Phi C_{\mathcal{L}}\sigma = C_{\mathsf{K}}\sigma\Phi$ *over* $\mathcal{T}h\mathcal{L}$ *for every* $\sigma \in \mathcal{E}nd(\boldsymbol{Fm})$*, then there is an isomorphism* $\Phi^{\boldsymbol{A}}\colon \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ *such that* $\Phi^{\boldsymbol{A}} Fi^{\boldsymbol{A}}_{\mathcal{L}}\sigma = \Theta^{\boldsymbol{A}}_{\mathsf{K}}\sigma\Phi^{\boldsymbol{A}}$ *over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ *for every* $\sigma \in \mathcal{E}nd(\boldsymbol{A})$ *and* $\Phi^{\boldsymbol{A}} Fi^{\boldsymbol{A}}_{\mathcal{L}}h = \Theta^{\boldsymbol{A}}_{\mathsf{K}}h\Phi$ *over* $\mathcal{T}h\mathcal{L}$ *for every* $h \in \mathcal{H}om(\boldsymbol{Fm}, \boldsymbol{A})$*.*

This result has been celebrated at least for two reasons. First because it provides a unification of the well-known correspondence between the lattices of certain subsets of many algebraic structures and those of their congruences: think for example of the correspondence between the congruences of a group and its normal subgroups [3] or of that between the congruences of a Boolean or a Heyting algebra and its lattice filters [21]. Secondly because, coupled with Theorem 3.1, it provides a readily falsifiable characterisation of algebraizability. This is because it makes no reference to any particular structural transformer and allows to construct small counterexamples, since it works for an arbitrary algebras (and, in particular, for the simple finite ones).

Even if the study of an abstract and categorical version of the Syntactic Isomorphism Theorem has gone so far, nothing of that kind exists for its semantic counterpart, expressed in Theorem 3.7. The main aim of this paper is to recover this semantical landscape by describing it in the abstract setting and to investigate whether the mentioned extension of the isomorphism theorem is preserved and under which conditions.

4. Evaluational frames

As we mentioned before, we are interested in structures which behave like the pair made up by a collection of linguistic objects and one of its algebraic models. Since the interplay of between formulas and models depends mainly on the action of substitutions on the formulas, endomorphisms on the models and evaluations from the formulas to the models, our structures should be equipped with suitable generalisations of these mappings. In order to do this we introduce some objects, called compositional lattices, which contain three kinds of elements and whose actions on our structures yield, respectively, generalisations of substitutions, endomorphisms on the models and evaluations. It should be noticed that an important feature of the behaviour of evaluations is that they can be composed with substitutions and with endomorphisms on the models, yielding in both cases new well-defined evaluations. Therefore our compositional lattices should internalise two operations of composition: one that describes the composition between evaluations and substitutions and another that describes the composition between endomorphisms on the models and evaluations.

Definition 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two complete residuated lattices. Then $\mathcal{H} = \langle H, \wedge, \vee, \oplus, \otimes \rangle$ is an $\langle \mathcal{A}, \mathcal{B} \rangle$-***compositional lattice*** if the following conditions hold:

(i) $\langle H, \wedge, \vee, \oplus \rangle$ is right $\mathcal{A}$-module;
(ii) $\langle H, \wedge, \vee, \otimes \rangle$ is left $\mathcal{B}$-module; and
(iii) $(b \otimes h) \oplus a = b \otimes (h \oplus a)$ for every $a \in A$, $h \in H$ and $b \in B$.

When no confusion occurs we will talk simply of compositional lattices, forgetting of the prefix $\langle \mathcal{A}, \mathcal{B} \rangle$. The next example shows how compositional lattices can be faithfully used to describe the behaviour under composition of evaluations from the formulas (of a given algebraic language) to one of its algebraic models, with respect to substitutions and endomorphisms on the model.

Example 4.2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two algebras of the same similarity type. Then consider the functions

$$\oplus \colon \mathcal{P}(\mathcal{H}om(\boldsymbol{A}, \boldsymbol{B})) \times \mathcal{P}(\mathcal{E}nd(\boldsymbol{A})) \to \mathcal{P}(\mathcal{H}om(\boldsymbol{A}, \boldsymbol{B}))$$

$$\otimes \colon \mathcal{P}(\mathcal{E}nd(\boldsymbol{B})) \times \mathcal{P}(\mathcal{H}om(\boldsymbol{A}, \boldsymbol{B})) \to \mathcal{P}(\mathcal{H}om(\boldsymbol{A}, \boldsymbol{B}))$$

obtained by lifting the usual composition of maps to the power sets, that is putting $H \oplus X \coloneqq \{h \circ \sigma : h \in H \text{ and } \sigma \in X\}$ and $Y \otimes H \coloneqq \{g \circ h : g \in Y \text{ and } h \in H\}$ for every $X \subseteq \mathcal{E}nd(\boldsymbol{A}), H \subseteq \mathcal{H}om(\boldsymbol{A}, \boldsymbol{B})$ and $Y \subseteq \mathcal{E}nd(\boldsymbol{B})$. Now, recall from Example 3.3 that $\mathcal{P}(\mathcal{M}(\boldsymbol{A}))$ and $\mathcal{P}(\mathcal{M}(\boldsymbol{B}))$ are complete residuated lattices. Keeping this in mind, it is easy to prove that

$$\mathcal{H}(\boldsymbol{A}, \boldsymbol{B}) \coloneqq \langle \mathcal{P}(\mathcal{H}om(\boldsymbol{A}, \boldsymbol{B})), \cap, \cup, \oplus, \otimes \rangle$$

is a $\langle \mathcal{P}(\mathcal{M}(\boldsymbol{A})), \mathcal{P}(\mathcal{M}(\boldsymbol{B})) \rangle$-compositional lattice. As a particular illustration, this means that for every $\sigma \in \mathcal{E}nd(\boldsymbol{A}), h \in \mathcal{H}om(\boldsymbol{A}, \boldsymbol{B})$ and $g \in \mathcal{E}nd(\boldsymbol{B})$ the following diagram commutes (observe that the two outer arrows of the diagram coincide).

$$\begin{tikzcd}
\boldsymbol{A} \arrow[r, "\{\sigma\}"'] \arrow[dr, "\{h\}\oplus\{\sigma\}"'] \arrow[drr, bend left=60, "\{g\}\otimes(\{h\}\oplus\{\sigma\})"] \arrow[drr, bend right=60, "(\{g\}\otimes\{h\})\oplus\{\sigma\}"'] & \boldsymbol{A} \arrow[d, "\{h\}"] \arrow[dr, "\{g\}\otimes\{h\}"] & \\
& \boldsymbol{B} \arrow[r, "\{g\}"] & \boldsymbol{B}
\end{tikzcd}$$

In the case where $\boldsymbol{A}$ coincides with the formula algebra $\boldsymbol{Fm}$, we have that $\mathcal{H}om(\boldsymbol{A}, \boldsymbol{B})$ and $\mathcal{E}nd(\boldsymbol{A})$ are, respectively, the collection of evaluations from $\boldsymbol{Fm}$ into $\boldsymbol{B}$ and the collection of substitutions. This will be the motivating example all along the paper, since it arises naturally from the consideration of the Semantic Isomorphism Theorem. ⊠

Then we turn to introduce our desired objects, which should be enriched by the action of a compositional lattice.

Definition 4.3. Let $\mathcal{H}$ be an $\langle\mathcal{A},\mathcal{B}\rangle$-compositional lattice. $\boldsymbol{R} = \langle\mathbb{R},\mathbb{R}',\langle\cdot,\cdot\rangle_{\boldsymbol{R}}\rangle$ is an ***$\mathcal{H}$-evaluational frame*** if the following conditions hold:

(i) $\mathbb{R}$ is a left $\mathcal{A}$-module with action $\star_{\boldsymbol{R}}$;
(ii) $\mathbb{R}'$ is a left $\mathcal{B}$-module with action $*_{\boldsymbol{R}}$;
(iii) $\langle\cdot,\cdot\rangle_{\boldsymbol{R}}\colon H\times R\to R'$ is a residuated map;
(iv) $\langle h\oplus a,x\rangle_{\boldsymbol{R}} = \langle h,a\star_{\boldsymbol{R}} x\rangle_{\boldsymbol{R}}$ for every $h\in H$, $a\in A$ and $x\in R$; and
(v) $\langle b\otimes h,x\rangle_{\boldsymbol{R}} = b *_{\boldsymbol{R}} \langle h,x\rangle_{\boldsymbol{R}}$ for every $b\in B$, $h\in H$ and $x\in R$.

We will refer to $\mathbb{R}$ as to the ***syntactic component*** of $\boldsymbol{R}$ and to $\mathbb{R}'$ as to its ***semantic component***, according to the intuition that $\mathbb{R}$ should represent a collection of linguistic objects and $\mathbb{R}'$ one of its algebraic models, as it is made clear at the end of Example 4.4. Moreover, we will denote $\mathcal{H}$-evaluational frames by italic boldface capital letters $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$... As we will show in the next example, the idea of evaluational frames is that of abstracting the behaviour of two algebras with respect to their endomorphisms and homomorphisms from one to the other.

Example 4.4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two algebras of the same similarity type. Then consider the functions

$$\begin{aligned}\langle\cdot,\cdot\rangle_{\boldsymbol{R}(\mathbb{A},\mathbb{B})}&\colon \mathcal{P}(\mathcal{H}om(\boldsymbol{A},\boldsymbol{B}))\times\mathcal{P}(A)\to\mathcal{P}(B)\\ \langle\cdot,\cdot\rangle_{\boldsymbol{Eq}(\mathbb{A},\mathbb{B})}&\colon \mathcal{P}(\mathcal{H}om(\boldsymbol{A},\boldsymbol{B}))\times\mathcal{P}(\mathrm{Eq}(A))\to\mathcal{P}(\mathrm{Eq}(B))\\ \langle\cdot,\cdot\rangle_{\boldsymbol{Seq}(\mathbb{A},\mathbb{B})}&\colon \mathcal{P}(\mathcal{H}om(\boldsymbol{A},\boldsymbol{B}))\times\mathcal{P}(\mathrm{Seq}(A))\to\mathcal{P}(\mathrm{Seq}(B))\end{aligned}$$

defined as $\langle H,X\rangle_{\boldsymbol{R}(\mathbb{A},\mathbb{B})} \coloneqq \{ha\in B : h\in H \text{ and } a\in X\}$, $\langle H,Y\rangle_{\boldsymbol{Eq}(\mathbb{A},\mathbb{B})} \coloneqq \{\langle ha,hc\rangle\in\mathrm{Eq}(B) : h\in H \text{ and } \langle a,c\rangle\in Y\}$, and $\langle H,Z\rangle_{\boldsymbol{Seq}(\mathbb{A},\mathbb{B})} \coloneqq \{\langle h\overline{a},h\overline{c}\rangle\in\mathrm{Seq}(B) : h\in H \text{ and } \langle\overline{a},\overline{c}\rangle\in Z\}$ for every $H\in\mathcal{P}(\mathcal{H}om(\boldsymbol{A},\boldsymbol{B}))$, $X\in\mathcal{P}(A)$, $Y\in\mathcal{P}(\mathrm{Eq}(A))$ and $Z\in\mathcal{P}(\mathrm{Seq}(A))$. It is easy to prove that the structures

$$\begin{aligned}\boldsymbol{R}(\mathbb{A},\mathbb{B}) &\coloneqq \langle\mathbb{A},\mathbb{B},\langle\cdot,\cdot\rangle_{\boldsymbol{R}(\mathbb{A},\mathbb{B})}\rangle\\ \boldsymbol{Eq}(\mathbb{A},\mathbb{B}) &\coloneqq \langle\mathrm{Eq}(\mathbb{A}),\mathrm{Eq}(\mathbb{B}),\langle\cdot,\cdot\rangle_{\boldsymbol{Eq}(\mathbb{A},\mathbb{B})}\rangle\\ \boldsymbol{Seq}(\mathbb{A},\mathbb{B}) &\coloneqq \langle\mathrm{Seq}(\mathbb{A}),\mathrm{Seq}(\mathbb{B}),\langle\cdot,\cdot\rangle_{\boldsymbol{Seq}(\mathbb{A},\mathbb{B})}\rangle\end{aligned}$$

are $\mathcal{H}(\boldsymbol{A},\boldsymbol{B})$-evaluational frames.

In the particular case of a compositional lattice of the form $\mathcal{H}(\boldsymbol{Fm},\boldsymbol{A})$, we put $\boldsymbol{Fm}(\mathbb{A}) \coloneqq \boldsymbol{R}(\mathbb{F}\mathrm{m},\mathbb{A})$, $\boldsymbol{Eq}(\mathbb{A}) \coloneqq \boldsymbol{Eq}(\mathbb{F}\mathrm{m},\mathbb{A})$ and $\boldsymbol{Seq}(\mathbb{A}) \coloneqq \boldsymbol{Seq}(\mathbb{F}\mathrm{m},\mathbb{A})$. We will refer to $\boldsymbol{Fm}(\mathbb{A})$, $\boldsymbol{Eq}(\mathbb{A})$ and $\boldsymbol{Seq}(\mathbb{A})$ respectively as to the $\mathcal{H}(\boldsymbol{Fm},\boldsymbol{A})$-evaluational frames of ***formulas***, ***equations*** and ***sequents***. ⊠

A fundamental example of $\mathcal{H}$-evaluational frame, that will play a central role along our analysis, is given by the compositional lattice $\mathcal{H}$ itself. In order to clarify this point, recall that $\mathcal{H}$ is an $\langle\mathcal{A},\mathcal{B}\rangle$-compositional lattice. Then we put $\boldsymbol{H} \coloneqq \langle\mathbb{A},\mathbb{H},\oplus\rangle$, where $\mathbb{A}$ is the complete residuated lattice

$\mathcal{A}$ seen as an $\mathcal{A}$-module and $\mathbb{H}$ the $\mathcal{B}$-module $\langle H, \wedge, \vee, \otimes\rangle$ (see point (ii) of Definition 4.1).

Lemma 4.5. $\boldsymbol{H}$ *is an* $\mathcal{H}$*-evaluational frame.*

Proof. We know that $\mathbb{A}$ and $\mathbb{H}$ are respectively an $\mathcal{A}$-module and a $\mathcal{B}$-module. Moreover the function $\oplus\colon H \times A \to H$ is residuated by condition (i) of Definition 4.1. It only remains to prove that conditions (iv) and (v) of Definition 4.3 hold, but this follows respectively from conditions (i) and (iii) of Definition 4.1. ⊠

Now we turn to define arrows between evaluational frames. Since evaluational frames are pair of modules equipped with some more structure it is natural to think that they will be pair of module morphisms which satisfy some additional condition. This is indeed the case.

Definition 4.6. Let $\boldsymbol{R}$ and $\boldsymbol{S}$ be two $\mathcal{H}$-evaluational frames. A pair $\boldsymbol{\tau} = \langle \tau, \tau'\rangle$ is a ***frame morphism*** from $\boldsymbol{R}$ to $\boldsymbol{S}$ if $\tau\colon \mathbb{R} \to \mathbb{S}$ and $\tau'\colon \mathbb{R}' \to \mathbb{S}'$ are respectively $\mathcal{A}$ and $\mathcal{B}$-module morphisms and the following diagram commutes.

$$\begin{array}{ccc} H \times R & \xrightarrow{\langle \cdot, \cdot \rangle_{\boldsymbol{R}}} & R' \\ {\scriptstyle id}\big\downarrow \;\; \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \tau'} \\ H \times S & \xrightarrow[\langle \cdot, \cdot \rangle_{\boldsymbol{S}}]{} & S' \end{array}$$

We write $\boldsymbol{\tau}\colon \boldsymbol{R} \to \boldsymbol{S}$ if $\boldsymbol{\tau}$ is a frame morphism from $\boldsymbol{R}$ to $\boldsymbol{S}$. The idea of frame morphisms comes from the study of the Semantic Isomorphism Theorem: the semantic isomorphism $\Phi^{\boldsymbol{A}}$ which appears in it (see Theorem 3.7) is obtained by interpreting the syntactic structural transformers which yields the algebraizability and closing under $\mathcal{F}i^{\boldsymbol{A}}_{\mathcal{L}}$ and $\Theta^{\boldsymbol{A}}_{\mathsf{K}}$ respectively. The definition of frame morphism $\boldsymbol{\tau}\colon \boldsymbol{R} \to \boldsymbol{S}$ is intended to reflect this phenomenon since $\boldsymbol{\tau}$ is a pair $\langle \tau, \tau'\rangle$, where $\tau\colon \mathbb{R} \to \mathbb{S}$ plays the role of a syntactic structural transformer and $\tau'\colon \mathbb{R}' \to \mathbb{S}'$ that of its interpretation.

Example 4.7. Consider any structural transformer $\tau\colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ and any algebra $\boldsymbol{A}$ of the same similarity type of $\boldsymbol{Fm}$. We let $\tau'\colon \mathcal{P}(A) \to \mathcal{P}(\mathrm{Eq}(A))$ be the function defined as $\tau'(B) = \{\langle \alpha^{\boldsymbol{A}}(a), \beta^{\boldsymbol{A}}(a)\rangle \in A \times A : \alpha(x) \approx \beta(x) \in \tau(x) \text{ and } a \in B\}$ for every $B \in \mathcal{P}(A)$. It is easy to prove that $\langle \tau, \tau'\rangle\colon \boldsymbol{Fm}(\mathbb{A}) \to \boldsymbol{Eq}(\mathbb{A})$ is an $\mathcal{H}(\boldsymbol{Fm}, \boldsymbol{A})$-frame morphism. ⊠

A direct consequence of the definition of frame morphism is that the image of an evaluational frame under a frame morphism is still an evaluational frame. In order to explain this fact observe that, given a module morphism $\tau\colon \mathbb{R} \to \mathbb{S}$, the structure $\tau[\mathbb{R}] \coloneqq \langle \tau[R], \wedge, \vee, \star_{\mathbb{S}}\rangle$, is still a module whose suprema coincide with those of $\mathbb{S}$ [16, Example 4.2 (d)]. Then, given a frame morphism $\boldsymbol{\tau}\colon \boldsymbol{R} \to \boldsymbol{S}$ between two $\mathcal{H}$-evaluational frames $\boldsymbol{R}$ and $\boldsymbol{S}$, we put $\boldsymbol{\tau}[\boldsymbol{R}] \coloneqq \langle \tau[\mathbb{R}], \tau'[\mathbb{R}'], \langle \cdot, \cdot \rangle_{\boldsymbol{S}}\rangle$.

Lemma 4.8. *If $\boldsymbol{\tau}\colon \boldsymbol{R} \to \boldsymbol{S}$ is a frame morphism between two $\mathcal{H}$-evaluational frames, then $\boldsymbol{\tau}[\boldsymbol{R}]$ is an $\mathcal{H}$-evaluational frame too.*

It is easy to prove that, if we restrict to evaluational frames whose actions is given by a fixed compositional lattice, frame morphisms give evaluational frames the structure of a category, as we record below.

Lemma 4.9. *$\mathcal{H}$-evaluational frames with frame morphisms between them form a category, which we denote by $\mathcal{H}$-Fra.*

Proof. Take pairs of identity module morphisms as identity frame morphisms and define composition component-wise. ⊠

From now on we will work within the context of $\mathcal{H}$-evaluational frames for a fixed $\langle\mathcal{A},\mathcal{B}\rangle$-compositional lattice $\mathcal{H}$. Doing this, we will most of the times refer simply to evaluational frames, forgetting of the prefixes $\mathcal{H}$ and $\langle\mathcal{A},\mathcal{B}\rangle$ in our definitions and results although $\mathcal{H}$, $\mathcal{A}$ and $\mathcal{B}$ will be used in their proofs. Then we define a functor $\mathcal{S}\colon \mathcal{H}\text{-}\mathsf{Fra} \to \mathcal{A}\text{-}\mathsf{Mod}$, called the ***syntax*** functor. Given an $\mathcal{H}$-evaluational frame $\boldsymbol{R}$, we put $\mathcal{S}(\boldsymbol{R}) \coloneqq \mathbb{R}$ and, given a frame morphism $\boldsymbol{\tau}\colon \boldsymbol{R} \to \boldsymbol{S}$, we put $\mathcal{S}(\boldsymbol{\tau}) \coloneqq \tau\colon \mathbb{R} \to \mathbb{S}$.

Lemma 4.10. *$\mathcal{S}$ is a functor.*

We conclude this section by proving that $\mathcal{H}$-Fra has coproducts, since they will play a central role in several parts of the paper. In order to do this, given a family of evaluational frames $\{\boldsymbol{R}_i\}_{i\in I}$, we put

$$\coprod_{i\in I}\boldsymbol{R}_i \coloneqq \langle \coprod_{i\in I}\mathbb{R}_i, \coprod_{i\in I}\mathbb{R}'_i, \langle\cdot,\cdot\rangle_{\coprod\limits_{i\in I}\boldsymbol{R}_i}\rangle$$

where the binary function $\langle\cdot,\cdot\rangle_{\coprod_{i\in I}\boldsymbol{R}_i}\colon H \times \coprod_{i\in I} R_i \to \coprod_{i\in I} R'_i$ is defined as $\langle h,x\rangle_{\coprod_{i\in I}\boldsymbol{R}_i}(i) = \langle h, x(i)\rangle_{\boldsymbol{R}_i}$ for every $i \in I$, $h \in H$ and $x \in \coprod_{i\in I} R_i$. Then for every $i \in I$ we put $\boldsymbol{\pi}_i \coloneqq \langle \pi_i, \pi'_i\rangle$, where π_i and π'_i are the module morphisms associated to the coproducts $\coprod_{i\in I}\mathbb{R}_i$ and $\coprod_{i\in I}\mathbb{R}'_i$, computed respectively in the categories $\mathcal{A}$-Mod and $\mathcal{B}$-Mod. It is easy to check that $\coprod_{i\in I}\boldsymbol{R}_i$ is an evaluational frame and that $\boldsymbol{\pi}_i\colon \boldsymbol{R}_i \to \coprod_{i\in I}\boldsymbol{R}_i$ is a frame morphism for every $i \in I$.

Theorem 4.11. *$\mathcal{H}$-Fra has coproducts: If $\{\boldsymbol{R}_i\}_{i\in I}$ is a family of evaluational frames, then $\coprod_{i\in I}\boldsymbol{R}_i$ equipped with the frame morphisms $\{\boldsymbol{\pi}_i\}_{i\in I}$ is a coproduct for it.*

Proof. Suppose we are given an evaluational frame $\boldsymbol{S}$ together with a frame morphism $\boldsymbol{\tau}_i\colon \boldsymbol{R}_i \to \boldsymbol{S}$ for every $i \in I$. Since $\coprod_{i\in I}\mathbb{R}_i$ and $\coprod_{i\in I}\mathbb{R}'_i$ are the coproducts of $\{\mathbb{R}_i\}_{i\in I}$ and $\{\mathbb{R}'_i\}_{i\in I}$ respectively, we know that there are two unique frame morphisms $\rho\colon \coprod_{i\in I}\mathbb{R}_i \to \mathbb{S}$ and $\rho'\colon \coprod_{i\in I}\mathbb{R}'_i \to \mathbb{S}'$ which make the following diagrams commute.

$$\begin{tikzcd} \coprod\limits_{i\in I}\mathbb{R}_i \arrow[r, "\rho"] & \mathbb{S} \\ \mathbb{R}_i \arrow[u, "\pi_i"] \arrow[ur, "\tau_i"'] & \end{tikzcd} \qquad \begin{tikzcd} \coprod\limits_{i\in I}\mathbb{R}'_i \arrow[r, "\rho'"] & \mathbb{S}' \\ \mathbb{R}'_i \arrow[u, "\pi'_i"] \arrow[ur, "\tau'_i"'] & \end{tikzcd}$$

Therefore it only remains to prove that $\boldsymbol{\rho} \coloneqq \langle \rho, \rho' \rangle$ is a frame morphism. Since ρ and ρ' are module morphisms, it will be enough to check the commutativity condition of the definition of frame morphisms. Observe that

$$\rho\overline{x} = \bigvee_{i\in I}^{\mathbb{S}} \tau_i \overline{x}(i) \text{ and } \rho'\overline{y} = \bigvee_{i\in I}^{\mathbb{S}'} \tau'_i \overline{y}(i)$$

for every $\overline{x} \in \coprod_{i\in I} R_i$ and $\overline{y} \in \coprod_{i\in I} R'_i$ [15, Lemma 5.11]. Then letting $h \in H$ and $\overline{x} \in \coprod_{i\in I} R_i$, we have that $\rho'\langle h, \overline{x}\rangle_{\coprod_{i\in I}\boldsymbol{R}_i} = \bigvee_{i\in I}^{\mathbb{S}'} \tau'_i\langle h, \overline{x}(i)\rangle_{\boldsymbol{R}_i} = \bigvee_{i\in I}^{\mathbb{S}'}\langle h, \tau_i\overline{x}(i)\rangle_{\boldsymbol{S}} = \langle h, \bigvee_{i\in I}^{\mathbb{S}} \tau_i\overline{x}(i)\rangle_{\boldsymbol{S}} = \langle h, \rho\overline{x}\rangle_{\boldsymbol{S}}$ for every $i \in I$ and therefore we are done. ⊠

5. Biclosure operators

Since evaluational frames with frame morphisms abstract the behaviour of formulas and models in the theory of algebraizable logics (see Example 4.4 and 4.7), it is natural to think of a structural closure operator in the context of evaluational frames as of a pair of closure operators given by a logic and one of its structural generalised models [10].

Definition 5.1. Let $\boldsymbol{R}$ be an evaluational frame. A pair $\boldsymbol{\delta} = \langle \delta, \delta' \rangle$ is a ***biclosure operator*** over $\boldsymbol{R}$ if δ and δ' are structural closure operators, respectively on $\mathbb{R}$ and $\mathbb{R}'$, and if $\langle h, x\rangle_{\boldsymbol{R}} \leqslant y$, then $\langle h, \delta x\rangle_{\boldsymbol{R}} \leqslant \delta' y$ for every $h \in H$, $x \in R$ and $y \in R'$.

We will denote biclosure operators by $\boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\varepsilon} \dots$ The most well-known structural generalised model of a certain logic over a given algebra is the one given by the whole family of its deductive filters. The next example clarifies how this idea transfers to the context of evaluational frames.

Example 5.2. Let $\boldsymbol{A}$ be an algebra. Then consider a logic $\mathcal{L}$ (in the same similarity type of $\boldsymbol{A}$) and let $Fi_{\mathcal{L}}^{\boldsymbol{A}} \colon \mathcal{P}(A) \to \mathcal{P}(A)$ be the closure operator which associates to a given $X \in \mathcal{P}(A)$ the smallest deductive filter of $\mathcal{L}$ over $\boldsymbol{A}$ which contains X. It turns out that $\langle C_{\mathcal{L}}, Fi_{\mathcal{L}}^{\boldsymbol{A}}\rangle$ is a biclosure operator over the $\mathcal{H}(\boldsymbol{Fm}, \boldsymbol{A})$-evaluational frame $\boldsymbol{Fm}(\mathbb{A})$. In order to prove this, observe that $C_{\mathcal{L}}$ and $Fi_{\mathcal{L}}^{\boldsymbol{A}}$ are clearly closure operators. Moreover, the fact that $C_{\mathcal{L}}$ is structural is part of the formal definition of a logic. In order to prove the structurality of $Fi_{\mathcal{L}}^{\boldsymbol{A}}$, we reason as follows. Consider an endomorphism $\sigma \colon \boldsymbol{A} \to \boldsymbol{A}$ and a set $X \subseteq A$. Clearly, we have that $\sigma[X] \subseteq Fi_{\mathcal{L}}^{\boldsymbol{A}}(\sigma[X])$ and, therefore, that $X \subseteq \sigma^{-1}[Fi_{\mathcal{L}}^{\boldsymbol{A}}(\sigma[X])]$. Now, recall that inverse images of a deductive filter under a homomorphism is still a deductive filter. In particular, this implies that $Fi^{\boldsymbol{A}}_{\mathcal{L}}(X) \subseteq Fi^{\boldsymbol{A}}_{\mathcal{L}}(\sigma^{-1}[Fi^{\boldsymbol{A}}_{\mathcal{L}}(\sigma[X])]) = \sigma^{-1}[Fi^{\boldsymbol{A}}_{\mathcal{L}}(\sigma[X])]$. Hence we conclude that $\sigma[Fi^{\boldsymbol{A}}_{\mathcal{L}}(X)] \subseteq Fi^{\boldsymbol{A}}_{\mathcal{L}}(\sigma[X])$, thus establishing that $Fi^{\boldsymbol{A}}_{\mathcal{L}}$ is structural. Finally, consider a set H of homomorphisms from $\boldsymbol{Fm}$ to $\boldsymbol{A}$ and a sets $\Gamma \subseteq Fm$ and $X \subseteq A$. Suppose that $h(\gamma) \in X$ for all $h \in H$ and $\gamma \in \Gamma$. Then for every a formula $\varphi \in C_{\mathcal{L}}(\Gamma)$ and $h \in H$, we have that $h(\varphi) \in Fi^{\boldsymbol{A}}_{\mathcal{L}}(X)$. This means that $\langle H, C_{\mathcal{L}}(\Gamma)\rangle_{\boldsymbol{Fm}(\mathbb{A})} \subseteq Fi^{\boldsymbol{A}}_{\mathcal{L}}(X)$. This establishes that $\langle C_{\mathcal{L}}, Fi^{\boldsymbol{A}}_{\mathcal{L}}\rangle$ is a biclosure operator, as desired.

An analogous result can be obtained in the equational case. Let K be a generalised quasi-variety and C_{K} the equational consequence relative to it. Then recall that, since K is a generalised quasi-variety, $\Theta^{\boldsymbol{A}}_{\mathsf{K}} \colon \mathcal{P}(A \times A) \to \mathcal{P}(A \times A)$ is a well-defined closure operator, which coincides with the generation of deductive filter over $\boldsymbol{A}$ of the consequence C_{K}. Keeping this in mind, it is easy to prove that $\langle C_{\mathsf{K}}, \Theta^{\boldsymbol{A}}_{\mathsf{K}}\rangle$ is a biclosure operator over the $\mathcal{H}(\boldsymbol{Fm}, \boldsymbol{A})$-evaluational frame $\boldsymbol{Eq}(\mathbb{A})$. ⊠

Let $\mathcal{L}$ be a logic and $\boldsymbol{A}$ an algebra. It is well-known that $F \subseteq A$ is a deductive filter of $\mathcal{L}$ if and only if $h^{-1}[F]$ is a theory of $\mathcal{L}$ for every homomorphism $h \colon \boldsymbol{Fm} \to \boldsymbol{A}$. This fact reflects in the behaviour of biclosure operators, as we remark in the following result.

Lemma 5.3. *Let $\boldsymbol{R}$ be an evaluational frame and $\boldsymbol{\delta} = \langle \delta, \delta'\rangle$ a pair of structural closure operators, respectively over $\mathbb{R}$ and $\mathbb{R}'$. The following conditions are equivalent:*

(i) *$\boldsymbol{\delta}$ is a biclosure operator.*
(ii) *$h \setminus_{\langle\cdot,\cdot\rangle_{\boldsymbol{R}}} y \in \delta[R]$ for every $h \in H$ and $y \in \delta'[R']$.*

Proof. We begin by proving direction (i)⇒(ii). Let $h \in H$ and $y \in \delta'[R']$. By residuation we always have that $\langle h, h \setminus_{\langle\cdot,\cdot\rangle_{\boldsymbol{R}}} y\rangle_{\boldsymbol{R}} \leqslant y$. Since $\boldsymbol{\delta}$ is a biclosure operator, this yields that $\langle h, \delta(h \setminus_{\langle\cdot,\cdot\rangle_{\boldsymbol{R}}} y)\rangle_{\boldsymbol{R}} \leqslant \delta'(y) = y$. By residuation we conclude that $\delta(h \setminus_{\langle\cdot,\cdot\rangle_{\boldsymbol{R}}} y) \leqslant h \setminus_{\langle\cdot,\cdot\rangle_{\boldsymbol{R}}} y$ and therefore we are done.

Then we turn to prove direction (ii)⇒(i). Let $h \in H$, $x \in R$ and $y \in R'$ such that $\langle h, x\rangle_{\boldsymbol{R}} \leqslant y$. Clearly $\langle h, x\rangle_{\boldsymbol{R}} \leqslant \delta' y$. By residuation this is equivalent to $x \leqslant h \setminus_{\langle\cdot,\cdot\rangle_{\boldsymbol{R}}} \delta' y$. Then by assumption we have that $\delta x \leqslant \delta(h \setminus_{\langle\cdot,\cdot\rangle_{\boldsymbol{R}}} \delta' y) = h \setminus_{\langle\cdot,\cdot\rangle_{\boldsymbol{R}}} \delta' y$. By residuation this implies $\langle h, \delta x\rangle_{\boldsymbol{R}} \leqslant \delta' y$ and therefore we are done. ⊠

We would like to prove that biclosure operators are frame morphisms. This can be achieved if we think of them as frame morphisms with codomains evaluational frames which play a role analogous to the one of closure systems in the context of closure operators. Let us explain this fact: given a biclosure operator $\boldsymbol{\delta}$ over $\boldsymbol{R}$, we put $\boldsymbol{R_\delta} \coloneqq \langle \mathbb{R}_\delta, \mathbb{R}'_{\delta'}, \delta'\langle\cdot,\cdot\rangle_{\boldsymbol{R}}\rangle$.

Lemma 5.4. *Let $\boldsymbol{\delta}$ be a biclosure operator over $\boldsymbol{R}$. $\boldsymbol{R_\delta}$ is an evaluational frame and $\boldsymbol{\delta} \colon \boldsymbol{R} \to \boldsymbol{R_\delta}$ a frame morphism.*

Proof. We begin by proving that $\boldsymbol{R_\delta}$ is an evaluational frame. Recall that $\mathbb{R}_\delta$ and $\mathbb{R}'_{\delta'}$ are respectively an $\mathcal{A}$ and a $\mathcal{B}$-module. Then we turn to prove that $\langle \cdot, \cdot \rangle_{\boldsymbol{R_\delta}} \colon H \times \delta[R] \to \delta'[R']$ is residuated. In order to do this, observe that

$$\langle h, x \rangle_{\boldsymbol{R}} \leqslant y \text{ if and only if } \langle h, x \rangle_{\boldsymbol{R_\delta}} \leqslant y \tag{1}$$

for every $h \in H$, $x \in \delta[R]$ and $y \in \delta'[R']$. First we construct the left residuum of $\langle \cdot, \cdot \rangle_{\boldsymbol{R_\delta}}$. Let $h \in H$, $x \in \delta[R]$ and $y \in \delta'[R']$. From (1) and the fact that $\langle \cdot, \cdot \rangle_{\boldsymbol{R}}$ is residuated, it follows that $\langle h, x \rangle_{\boldsymbol{R_\delta}} \leqslant y$ if and only if $x \leqslant h \setminus_{\langle \cdot,\cdot \rangle_{\boldsymbol{R}}} y$. By Lemma 5.3 we know that $h \setminus_{\langle \cdot,\cdot \rangle_{\boldsymbol{R}}} y \in \delta[R]$, therefore we conclude that $\setminus_{\langle \cdot,\cdot \rangle_{\boldsymbol{R}}} \colon H \times \delta'[R'] \to \delta[R]$ is the left residuum of $\langle \cdot, \cdot \rangle_{\boldsymbol{R_\delta}}$. In order to define the right residuum of $\langle \cdot, \cdot \rangle_{\boldsymbol{R_\delta}}$ it is enough to observe that from (1) and the fact that $\langle \cdot, \cdot \rangle_{\boldsymbol{R}}$ is residuated, it follows that $\langle h, x \rangle_{\boldsymbol{R_\delta}} \leqslant y$ if and only if $h \leqslant y /_{\langle \cdot,\cdot \rangle_{\boldsymbol{R}}} x$. Therefore we conclude that $/_{\langle \cdot,\cdot \rangle_{\boldsymbol{R}}} \colon \delta'[R'] \times \delta[R] \to H$ is the right residuum of $\langle \cdot, \cdot \rangle_{\boldsymbol{R_\delta}}$. Hence $\langle \cdot, \cdot \rangle_{\boldsymbol{R_\delta}}$ is residuated.

Then we turn to prove condition (iv) of Definition 4.3. First observe that

$$\delta' \langle h, \delta x \rangle_{\boldsymbol{R}} = \delta' \langle h, x \rangle_{\boldsymbol{R}} \tag{2}$$

for every $h \in H$ and $x \in R$. In order to prove this, observe that from the fact that $\boldsymbol{\delta}$ is a biclosure operator it follows that $\delta' \langle h, \delta x \rangle_{\boldsymbol{R}} \leqslant \delta' \langle h, x \rangle_{\boldsymbol{R}}$. Moreover, since $\delta' \langle \cdot, \cdot \rangle_{\boldsymbol{R}}$ is monotone in both components we obtain that $\delta' \langle h, x \rangle_{\boldsymbol{R}} \leqslant \delta' \langle h, \delta x \rangle_{\boldsymbol{R}}$ and therefore we are done. Now let $a \in A$, $h \in H$ and $x \in \delta[R]$. Applying (2) and the fact that $\delta \colon \mathbb{R} \to \mathbb{R}_\delta$ is a module morphism, we obtain that $\langle h \oplus a, x \rangle_{\boldsymbol{R_\delta}} = \delta' \langle h \oplus a, x \rangle_{\boldsymbol{R}} = \delta' \langle h, a \star_{\boldsymbol{R}} x \rangle_{\boldsymbol{R}} = \delta' \langle h, \delta(a \star_{\boldsymbol{R}} x) \rangle_{\boldsymbol{R}} = \delta' \langle h, a \star_{\boldsymbol{R_\delta}} \delta x \rangle_{\boldsymbol{R}} = \langle h, a \star_{\boldsymbol{R_\delta}} x \rangle_{\boldsymbol{R_\delta}}$.

It only remains to prove condition (v) of Definition 4.3. Recall that $\delta' \colon \mathbb{R}' \to \mathbb{R}'_{\delta'}$ is a module morphism, therefore for every $b \in B$, $h \in H$ and $x \in \delta[R]$ we have that $\langle b \otimes h, x \rangle_{\boldsymbol{R_\delta}} = \delta' \langle b \otimes h, x \rangle_{\boldsymbol{R}} = \delta'(b \ast_{\boldsymbol{R}} \langle h, x \rangle_{\boldsymbol{R}}) = b \ast_{\boldsymbol{R_\delta}} \delta' \langle h, x \rangle_{\boldsymbol{R}} = b \ast_{\boldsymbol{R_\delta}} \langle h, x \rangle_{\boldsymbol{R_\delta}}$. This concludes the proof that $\boldsymbol{R_\delta}$ is an evaluational frame.

It only remains to prove that $\boldsymbol{\delta} \colon \boldsymbol{R} \to \boldsymbol{R_\delta}$ is a frame morphism. We know that $\delta \colon \mathbb{R} \to \mathbb{R}_\delta$ and $\delta' \colon \mathbb{R}' \to \mathbb{R}'_{\delta'}$ are respectively $\mathcal{A}$ and $\mathcal{B}$-module morphisms. Moreover, that from (2) it follows that $\delta' \langle h, x \rangle_{\boldsymbol{R}} = \langle h, \delta x \rangle_{\boldsymbol{R_\delta}}$ for every $h \in H$ and $x \in R$. Therefore we are done. ⊠

Biclosure operators can be obtained by composing frame morphisms component-wise with the residua of their components. In order to be more precise, let us fix some notation. Given a frame morphism $\boldsymbol{\tau} \colon \boldsymbol{R} \to \boldsymbol{S}$, we put $\boldsymbol{\tau}^+ \coloneqq \langle \tau^+, \tau'^+ \rangle$.

Lemma 5.5. *Let $\boldsymbol{\tau} \colon \boldsymbol{R} \to \boldsymbol{S}$ be a frame morphism. $\boldsymbol{\tau}^+ \boldsymbol{\tau}$ is a biclosure operator over $\boldsymbol{R}$ such that $\boldsymbol{\tau} \colon \boldsymbol{R}_{\boldsymbol{\tau}^+\boldsymbol{\tau}} \to \boldsymbol{\tau}[\boldsymbol{R}]$ is a frame isomorphism.*

Proof. We begin by proving that $\boldsymbol{\tau}^+ \boldsymbol{\tau}$ is a biclosure operator over $\boldsymbol{R}$. First note that $\tau^+ \tau$ and $\tau'^+ \tau'$ are structural closure operators, respectively on $\mathbb{R}$ and $\mathbb{R}'$ [15, Lemma 4.1]. Then observe that

$$\tau^+(h \setminus_{\langle \cdot,\cdot \rangle_{\boldsymbol{S}}} \tau' y) = h \setminus_{\langle \cdot,\cdot \rangle_{\boldsymbol{R}}} \tau'^+ \tau' y. \tag{3}$$

for every $h \in H$ and $y \in R'$. In order to check this, consider any $x \in S$. Applying the residuation of $\tau, \tau', \langle \cdot, \cdot \rangle_{\boldsymbol{R}}$ and $\langle \cdot, \cdot \rangle_{\boldsymbol{S}}$ and the fact that $\boldsymbol{\tau}$ is a frame morphism, we have that

$$\begin{aligned} x \leqslant \tau^{+}(h \setminus_{\langle \cdot, \cdot \rangle_{\boldsymbol{S}}} \tau' y) &\iff \tau x \leqslant h \setminus_{\langle \cdot, \cdot \rangle_{\boldsymbol{S}}} \tau' y \\ &\iff \langle h, \tau x \rangle_{\boldsymbol{S}} \leqslant \tau' y \\ &\iff \tau' \langle h, x \rangle_{\boldsymbol{R}} \leqslant \tau' y \\ &\iff \langle h, x \rangle_{\boldsymbol{R}} \leqslant \tau'^{+} \tau' y \\ &\iff x \leqslant h \setminus_{\langle \cdot, \cdot \rangle_{\boldsymbol{R}}} \tau'^{+} \tau' y. \end{aligned}$$

This concludes the proof of (3). Then let $h \in H$, $x \in R$ and $y \in R'$ such that $\langle h, x \rangle_{\boldsymbol{R}} \leqslant y$. Since τ' is monotone, this implies that $\langle h, \tau x \rangle_{\boldsymbol{S}} = \tau' \langle h, x \rangle_{\boldsymbol{R}} \leqslant \tau' y$. By residuation of $\langle \cdot, \cdot \rangle_{\boldsymbol{S}}$ and monotonicity of τ^{+}, this yields $\tau^{+} \tau x \leqslant \tau^{+}(h \setminus_{\langle \cdot, \cdot \rangle_{\boldsymbol{S}}} \tau' y)$. Therefore, applying (3), we obtain that $\tau^{+} \tau x \leqslant h \setminus_{\langle \cdot, \cdot \rangle_{\boldsymbol{R}}} \tau'^{+} \tau' y$. By residuation of $\langle \cdot, \cdot \rangle_{\boldsymbol{R}}$, this is equivalent to $\langle h, \tau^{+} \tau x \rangle_{\boldsymbol{R}} \leqslant \tau'^{+} \tau' y$. This concludes the proof that $\boldsymbol{\tau}^{+} \boldsymbol{\tau}$ is a biclosure operator over $\boldsymbol{R}$.

Then we turn to prove that $\boldsymbol{\tau} \colon \boldsymbol{R}_{\boldsymbol{\tau}^{+} \boldsymbol{\tau}} \to \boldsymbol{\tau}[\boldsymbol{R}]$ is a frame isomorphism. It is straight-forward that $\tau \colon \mathbb{R}_{\tau^{+} \tau} \to \tau[\mathbb{R}]$ and $\tau' \colon \mathbb{R}'_{\tau'^{+} \tau'} \to \tau[\mathbb{R}']$ are module isomorphisms. Then let $h \in H$ and $x \in \tau^{+} \tau[R]$. We have that $\tau' \langle h, x \rangle_{\boldsymbol{R}_{\boldsymbol{\tau}^{+} \boldsymbol{\tau}}} = \tau' \tau'^{+} \tau' \langle h, x \rangle_{\boldsymbol{R}} = \tau' \langle h, x \rangle_{\boldsymbol{R}} = \langle h, \tau x \rangle_{\boldsymbol{S}}$ and therefore we are done. ⊠

This is all for what concerns biclosure operators in general. However the Semantic Isomorphism Theorem holds between a special kind of structural generalised models of the logic and of the equational consequence relative to its equivalent algebraic semantics, namely the ones of the corresponding deductive filters. Following the terminology of [10], we call the structural generalised models of deductive filters "basic". In order to abstract the behaviour of basic generalised models in the context of evaluational frames, let us introduce some new concept. Given an evaluational frame $\boldsymbol{R}$ and a structural closure operator $\delta \colon \mathbb{R} \to \mathbb{R}$, we let $\mathcal{F}(\boldsymbol{R}, \delta)' \colon R' \to R'$ be the function defined for every $y \in R'$ as follows

$$\begin{aligned} \mathcal{F}(\boldsymbol{R}, \delta)'(y) \coloneqq \bigwedge \{ z \in R' : y \leqslant z \text{ and for every } x \in R \text{ and } h \in H \\ \text{if } \langle h, x \rangle_{\boldsymbol{R}} \leqslant z, \text{ then } \langle h, \delta(x) \rangle_{\boldsymbol{R}} \leqslant z \}. \end{aligned}$$

Definition 5.6. Let δ be a structural closure operator over $\mathbb{R}$. The ***basic*** biclosure operator over $\boldsymbol{R}$ relative to δ is the pair $\mathcal{F}(\boldsymbol{R}, \delta) \coloneqq \langle \delta, \mathcal{F}(\boldsymbol{R}, \delta)' \rangle$.

Our first goal will be to prove that basic biclosure operators are indeed biclosure operators.

Lemma 5.7. *Let $\boldsymbol{R}$ be an evaluational frame and $\delta \colon \mathbb{R} \to \mathbb{R}$ a structural closure operator. $\mathcal{F}(\boldsymbol{R}, \delta)$ is a biclosure operator over $\boldsymbol{R}$.*

Proof. The fact that $\mathcal{F}(\boldsymbol{R}, \delta)'$ is a closure operator over $\mathbb{R}$ follows directly from its definition. Then we turn to check it is structural. From [15, Lemma 3.8] we know that it will be enough to check that

$$\mathcal{F}(\boldsymbol{R},\delta)'(b\setminus_{*_{\boldsymbol{R}}}\mathcal{F}(\boldsymbol{R},\delta)'y)=b\setminus_{*_{\boldsymbol{R}}}\mathcal{F}(\boldsymbol{R},\delta)'y \tag{4}$$

for every $b\in B$ and $y\in R'$. This amounts to proving that $b\setminus_{*_{\boldsymbol{R}}}\mathcal{F}(\boldsymbol{R},\delta)'y$ is a fixed point of $\mathcal{F}(\boldsymbol{R},\delta)'$. In order to do this, let $x\in R$ and $h\in H$ such that $\langle h,x\rangle_{\boldsymbol{R}}\leqslant b\setminus_{*_{\boldsymbol{R}}}\mathcal{F}(\boldsymbol{R},\delta)'y$. By residuation of $*_{\boldsymbol{R}}$, this implies that $\langle b\otimes h,x\rangle_{\boldsymbol{R}}=b*_{\boldsymbol{R}}\langle h,x\rangle_{\boldsymbol{R}}\leqslant\mathcal{F}(\boldsymbol{R},\delta)'y$. Together with the definition of $\mathcal{F}(\boldsymbol{R},\delta)'$, this yields that $b*_{\boldsymbol{R}}\langle h,\delta x\rangle_{\boldsymbol{R}}=\langle b\otimes h,\delta x\rangle_{\boldsymbol{R}}\leqslant\mathcal{F}(\boldsymbol{R},\delta)'y$. Applying again the fact that $*_{\boldsymbol{R}}$ is residuated, we obtain that $\langle h,\delta x\rangle_{\boldsymbol{R}}\leqslant b\setminus_{*_{\boldsymbol{R}}}\mathcal{F}(\boldsymbol{R},\delta)'y$. By the definition of $\mathcal{F}(\boldsymbol{R},\delta)'$ we conclude that $b\setminus_{*_{\boldsymbol{R}}}\mathcal{F}(\boldsymbol{R},\delta)'y$ is a fixed point of $\mathcal{F}(\boldsymbol{R},\delta)'$. This establishes (4) and therefore that $\mathcal{F}(\boldsymbol{R},\delta)'$ is structural.

Now, the fact that $\mathcal{F}(\boldsymbol{R},\delta)$ is a biclosure operator is an easy consequence of its definition. ⊠

Example 5.8. Let $\mathcal{L}$ be a logic and C_{K} the equational consequence relative to a generalised quasi-variety K. From Example 5.2 and the fact that $\Theta^{\boldsymbol{A}}_{\mathsf{K}}$ coincides with the generation of deductive filters of C_{K} over $\boldsymbol{A}$, it follows that

$$\mathcal{F}(\boldsymbol{Fm}(\mathbb{A}),C_{\mathcal{L}})=\langle C_{\mathcal{L}},\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{L}}\rangle \text{ and } \mathcal{F}(\boldsymbol{Eq}(\mathbb{A}),C_{\mathsf{K}})=\langle C_{\mathsf{K}},\Theta^{\boldsymbol{A}}_{\mathsf{K}}\rangle.$$

Now, suppose that there is an $\mathcal{P}(\mathcal{M}(\boldsymbol{Fm}))$-module isomorphism

$$\tau\colon \mathbb{F}\mathrm{m}_{C_{\mathcal{L}}}\to\mathbb{E}\mathrm{q}_{C_{\mathsf{K}}}.$$

From the Semantic Isomorphism Theorem it follows that for every algebra $\boldsymbol{A}$ there is an isomorphism of $\mathcal{H}(\boldsymbol{Fm},\boldsymbol{A})$-evaluational frames

$$\boldsymbol{\tau}\colon \boldsymbol{Fm}(\mathbb{A})_{\mathcal{F}(\boldsymbol{Fm}(\mathbb{A}),C_{\mathcal{L}})}\to\boldsymbol{Eq}(\mathbb{A})_{\mathcal{F}(\boldsymbol{Eq}(\mathbb{A}),C_{\mathsf{K}})}.$$

We believe that this fact justify the claim that basic biblosure operators are indeed the correct tool to model the behaviour of deductive filters in the Semantic Isomorphism Theorem. ⊠

The fact that $\mathcal{F}(\boldsymbol{R},\delta)$ is the weakest biclosure operator whose first component is δ, reflects in the arrow-theoretical property which we record below and that will play a fundamental role in the proof of Theorem 6.2.

Theorem 5.9. *Let $\boldsymbol{\gamma}$ be a biclosure operator over $\boldsymbol{R}$ and δ a structural closure operator over $\mathbb{R}$. If $\delta\leqslant\gamma$, then there is a frame morphism $\boldsymbol{\tau}\colon\boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)}\to\boldsymbol{R}_{\boldsymbol{\gamma}}$ such that $\tau=\gamma$.*

Proof. First we claim that given a module $\mathbb{S}$ over a complete residuated lattice $\mathcal{E}$ and two structural closure operators over it ε and η such that $\varepsilon\leqslant\eta$, the map $\eta\colon\mathbb{R}_\varepsilon\to\mathbb{R}_\eta$ is an $\mathcal{E}$-module morphism. In order to prove our claim, let $\{x_i\}_{i\in I}\subseteq\varepsilon[S]$. We have that

$$\eta\bigvee_{i\in I}^{\mathbb{S}_\varepsilon}x_i=\eta\,\varepsilon\bigvee_{i\in I}^{\mathbb{S}}x_i=\eta\bigvee_{i\in I}^{\mathbb{S}}x_i=\eta\bigvee_{i\in I}^{\mathbb{S}}\eta\,x_i=\bigvee_{i\in I}^{\mathbb{S}_\eta}\eta\,x_i.$$

We conclude that $\eta\colon \mathbb{R}_\varepsilon \to \mathbb{R}_\eta$ is residuated. Then we check that it is structural. Let $e \in E$ and $x \in \varepsilon[S]$. Since $\eta\colon \mathbb{S} \to \mathbb{S}_\eta$ is an $\mathcal{E}$-module morphism, we have that $\eta(e \star_{\mathbb{S}_\varepsilon} x) = \eta\varepsilon(e \star_{\mathbb{S}} x) = \eta(e \star_{\mathbb{S}} x) = e \star_{\mathbb{S}_\eta} \eta x$. This concludes the proof of our claim.

Then let $\boldsymbol{\gamma}$ be a biclosure operator over $\boldsymbol{R}$ and δ a structural closure operator over $\mathbb{R}$ such that $\delta \leqslant \gamma$. We will check that $\mathcal{F}(\boldsymbol{R}, \delta)' \leqslant \gamma'$. In order to do this, let $h \in H$, $x \in R$ and $y \in R'$ such that $\langle h, x\rangle_{\boldsymbol{R}} \leqslant \gamma' y$. Since $\boldsymbol{\gamma}$ is a biclosure operator, this yields that $\langle h, \gamma x\rangle_{\boldsymbol{R}} \leqslant \gamma' y$ and, since $\langle \cdot, \cdot\rangle_{\boldsymbol{R}}$ is monotone in both components, that $\langle h, \delta x\rangle_{\boldsymbol{R}} \leqslant \gamma' y$. Moreover $y \leqslant \gamma' y$, therefore we conclude that $\mathcal{F}(\boldsymbol{R}, \delta)' \leqslant \gamma'$.

Then, applying our claim to the fact that $\delta \leqslant \gamma$ and $\mathcal{F}(\boldsymbol{R}, \delta)' \leqslant \gamma'$, we obtain that $\gamma\colon \mathbb{R}_\delta \to \mathbb{R}_\gamma$ and $\gamma'\colon \mathbb{R}'_{\mathcal{F}(\boldsymbol{R},\delta)'} \to \mathbb{R}'_{\gamma'}$ are respectively $\mathcal{A}$ and $\mathcal{B}$-module morphisms. We put $\boldsymbol{\tau} \coloneqq \langle \gamma, \gamma'\rangle$. In order to prove that $\boldsymbol{\tau}\colon \boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)} \to \boldsymbol{R}_{\boldsymbol{\gamma}}$ is a frame morphism it only remains to prove the commutativity condition. Let $h \in H$ and $x \in \delta[R]$. Keeping in mind that $\boldsymbol{\gamma}\colon \boldsymbol{R} \to \boldsymbol{R}_{\boldsymbol{\gamma}}$ is a frame morphism, we have that $\gamma'\langle h, x\rangle_{\boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)}} = \gamma'\mathcal{F}(\boldsymbol{R}, \delta)'\langle h, x\rangle_{\boldsymbol{R}} = \gamma'\langle h, x\rangle_{\boldsymbol{R}} = \langle h, \gamma(x)\rangle_{\boldsymbol{R}_{\boldsymbol{\gamma}}}$. ⊠

6. Interpretation

As Galatos and Tsinakis's approach [15] suggests, when dealing with a generalisation of the Isomorphism Theorems typical of algebraizable logics it is useful to split the whole isomorphism into two symmetrical halves. In our case the problem will be that of interpreting a module morphism, yielding a frame morphism whose first component is the original module morphism. Recall that $\mathcal{S}\colon \mathcal{H}\text{-}\mathsf{Fra} \to \mathcal{A}\text{-}\mathsf{Mod}$ is the syntax functor, then:

Definition 6.1. $\boldsymbol{R}$ has the ***interpretation*** property (for short INT), if for every evaluational frame $\boldsymbol{S}$ the map $\mathcal{S}\colon \mathcal{H}om_{\mathcal{H}\text{-}\mathsf{Fra}}(\boldsymbol{R}, \boldsymbol{S}) \to \mathcal{H}om_{\mathcal{A}\text{-}\mathsf{Mod}}(\mathbb{R}, \mathbb{S})$ is a bijection.

In other words we say that $\boldsymbol{R}$ has the INT if for every $\boldsymbol{S}$ and every $\mathcal{A}$-module morphism $\tau\colon \mathbb{R} \to \mathbb{S}$, there is a unique frame morphism $\boldsymbol{\tau}\colon \boldsymbol{R} \to \boldsymbol{S}$ such that $\mathcal{S}(\boldsymbol{\tau}) = \tau$. In this case we put $\mathcal{I}(\tau) \coloneqq \boldsymbol{\tau}$.

Our first goal will be to prove that the INT is exactly the property we are looking for. This fact is not evident at first sight, since the Semantic Isomorphism Theorem is concerned with basic biclosure operators which are not mentioned in the definition INT. More precisely, the first half of the Semantic Isomorphism Theorem consists in condition (ii) of the next result, which turns out be be equivalent to the INT but unnecessarily complicated. This is why we prefer to work with the INT.

Theorem 6.2. *The following conditions are equivalent:*

(i) $\boldsymbol{R}$ *has that* INT.
(ii) $\boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)}$ *has the* INT *for every structural closure operator* δ *over* $\mathbb{R}$.

Proof. We begin by proving direction (i)⇒(ii). In order to do this, consider a structural closure operator δ over $\mathbb{R}$ and suppose that there is an $\mathcal{A}$-module morphism $\tau\colon \mathbb{R}_\delta \to \mathbb{S}$. Then consider the $\mathcal{A}$-module morphism $\tau\delta\colon \mathbb{R} \to \mathbb{S}$. Since $\boldsymbol{R}$ has the INT, we can consider the associated frame morphism $\mathcal{I}(\tau\delta)\colon \boldsymbol{R} \to \boldsymbol{S}$.

Recall from Lemma 5.5 that $\mathcal{I}(\tau\delta)^+\mathcal{I}(\tau\delta)$ is a biclosure operator over $\boldsymbol{R}$. It is easy to check that $\delta \leqslant (\tau\delta)^+\tau\delta$, therefore by Theorem 5.9 there is a frame morphism $\boldsymbol{\rho}\colon \boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)} \to \boldsymbol{R}_{\mathcal{I}(\tau\delta)^+\mathcal{I}(\tau\delta)}$ such that $\rho = (\tau\delta)^+\tau\delta$. Again from Lemma 5.5 we know that $\mathcal{I}(\tau\delta)\colon \boldsymbol{R}_{\mathcal{I}(\tau\delta)^+\mathcal{I}(\tau\delta)} \to \mathcal{I}(\tau\delta)[\boldsymbol{R}]$ is a frame isomorphism. Then we consider the composition $\mathcal{I}(\tau\delta)\boldsymbol{\rho}\colon \boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)} \to \mathcal{I}(\tau\delta)[\boldsymbol{R}]$. Let $x \in \delta[R]$, we have that $\tau\delta\rho(x) = \tau\delta(\tau\delta)^+\tau\delta(x) = \tau\delta(x) = \tau(x)$. Since $\mathcal{I}(\tau\delta)\colon \boldsymbol{R} \to \boldsymbol{S}$, this yields that $\mathcal{I}(\tau\delta)\boldsymbol{\rho}\colon \boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)} \to \boldsymbol{S}$ is a frame morphism such that $\mathcal{S}(\mathcal{I}(\tau\delta)\boldsymbol{\rho}) = \tau$. Therefore we conclude that the map $\mathcal{S}\colon \mathcal{H}om_{\mathcal{H}\text{-}\mathsf{Fra}}(\boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)}, \boldsymbol{S}) \to \mathcal{H}om_{\mathcal{A}\text{-}\mathsf{Mod}}(\mathbb{R}_\delta, \mathbb{S})$ is surjective.

It only remains to prove that it is injective too. We reason towards a contradiction: suppose the contrary. Then there are two different frame morphisms $\boldsymbol{\tau}, \boldsymbol{\rho}\colon \boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)} \to \boldsymbol{S}$ such that $\tau = \rho$. Clearly $\tau' \neq \rho'$. We consider the frame morphism $\mathcal{F}(\boldsymbol{R},\delta)\colon \boldsymbol{R} \to \boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)}$. Since $\mathcal{F}(\boldsymbol{R},\delta)'$ is surjective, it follows that $\boldsymbol{\tau}\mathcal{F}(\boldsymbol{R},\delta), \boldsymbol{\rho}\mathcal{F}(\boldsymbol{R},\delta)\colon \boldsymbol{R} \to \boldsymbol{S}$ are two different frame morphisms such that $\mathcal{S}(\boldsymbol{\tau}\mathcal{F}(\boldsymbol{R},\delta)) = \tau\delta = \rho\delta = \mathcal{S}(\boldsymbol{\rho}\mathcal{F}(\boldsymbol{R},\delta))$ against the assumption that $\boldsymbol{R}$ has the INT.

For the direction (ii)⇒(i) it is enough to observe that, given an evaluational frame $\boldsymbol{T}$, we have that $\mathcal{F}(\boldsymbol{T}, 1_{\mathbb{T}}) = 1_{\boldsymbol{T}}$. Therefore, letting $\delta = 1_{\mathbb{R}}$ and applying the assumption, we are done. ⊠

We turn now to prove that evaluational frames with the INT enjoy an abstract version of the Semantic Isomorphism Theorem. In order to do this, let us state a more general result, i.e., that evaluational frames with the INT are determined up to isomorphism by their syntactic component. The Semantic Isomorphism Theorem will be obtained as a corollary.

Lemma 6.3. *Let $\boldsymbol{R}$ and $\boldsymbol{S}$ have INT. The following conditions are equivalent:*

(i) $\tau\colon \mathbb{R} \to \mathbb{S}$ *is an $\mathcal{A}$-module isomorphism.*
(ii) $\mathcal{I}(\tau)\colon \boldsymbol{R} \to \boldsymbol{S}$ *is a frame isomorphism.*

Proof. (i)⇒(ii): Since $\tau\colon \mathbb{R} \to \mathbb{S}$ is an isomorphism, there is an $\mathcal{A}$-module morphism $\rho\colon \mathbb{S} \to \mathbb{R}$ such that $1_{\mathbb{R}} = \rho\tau$ and $1_{\mathbb{S}} = \tau\rho$. The fact that $\boldsymbol{R}$ and $\boldsymbol{S}$ have the INT guarantees the possibility of interpreting τ and ρ, yielding two frame morphisms $\mathcal{I}(\tau)\colon \boldsymbol{R} \to \boldsymbol{S}$ and $\mathcal{I}(\rho)\colon \boldsymbol{S} \to \boldsymbol{R}$. But observe that $\mathcal{S}(\mathcal{I}(\rho) \circ \mathcal{I}(\tau)) = 1_{\mathbb{R}}$ and $\mathcal{S}(\mathcal{I}(\tau) \circ \mathcal{I}(\rho)) = 1_{\mathbb{S}}$. Now, the fact that $\boldsymbol{R}$ and $\boldsymbol{S}$ have the INT guarantees also the uniqueness of the interpretations, therefore we conclude that $\mathcal{I}(\rho)\circ\mathcal{I}(\tau) = 1_{\boldsymbol{R}}$ and $\mathcal{I}(\tau)\circ\mathcal{I}(\rho) = 1_{\boldsymbol{S}}$. But this is to say that $\mathcal{I}(\tau)\colon \boldsymbol{R} \to \boldsymbol{S}$ is an isomorphism. Direction (ii)⇒(i) is straight-forward. ⊠

Corollary 6.4 (Categorial Semantic Isomorphism)**.** *Let $\boldsymbol{R}$, $\boldsymbol{S}$ have the* INT *and δ, γ be structural closure operators over $\mathbb{R}$ and $\mathbb{S}$ respectively. If*

$\tau\colon \mathbb{R}_\delta \to \mathbb{S}_\gamma$ is an $\mathcal{A}$-module isomorphism, then $\mathcal{I}(\tau)\colon \boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)} \to \boldsymbol{S}_{\mathcal{F}(\boldsymbol{S},\gamma)}$ is a frame isomorphism.

Proof. By Theorem 6.2 we know that $\boldsymbol{R}_{\mathcal{F}(\boldsymbol{R},\delta)}$ and $\boldsymbol{S}_{\mathcal{F}(\boldsymbol{S},\gamma)}$ have the INT. Therefore by Lemma 6.3 we are done. ⊠

Now we turn to characterise the evaluational frames with the INT. A first step in this direction is the observation that the INT is preserved under the formation of coproducts. Notice that this result will be strengthened in Corollary 6.11.

Lemma 6.5. *Let $\{\boldsymbol{R}_i\}_{i\in I}$ be a family of evaluational frames. If $\boldsymbol{R}_i$ has the INT for every $i \in I$, then $\coprod_{i\in I} \boldsymbol{R}_i$ has the INT.*

Proof. Assume that every element of $\{\boldsymbol{R}_i\}_{i\in I}$ has the INT and pick any evaluational frame $\boldsymbol{S}$ equipped with an $\mathcal{A}$-module morphism $\tau\colon \coprod_{i\in I} \mathbb{R}_i \to \mathbb{S}$. Then for every $i \in I$, we consider the $\mathcal{A}$-module morphism $\tau\pi_i\colon \mathbb{R}_i \to \mathbb{S}$. Since $\mathbb{R}_i$ has the INT we can interpret $\tau\pi_i$, obtaining the frame morphism $\mathcal{I}(\tau\pi_i)\colon \boldsymbol{R}_i \to \boldsymbol{S}$.

Now, we can apply the existential condition of the universal property of the coproduct $\coprod_{i\in I} \boldsymbol{R}_i$ yielding a frame morphism $\boldsymbol{\rho}\colon \coprod_{i\in I} \boldsymbol{R}_i \to \boldsymbol{S}$ such that $\boldsymbol{\rho}\boldsymbol{\pi}_i = \mathcal{I}(\tau\pi_i)$. In particular we have that $\rho\pi_i = \tau\pi_i$ for every $i \in I$. Thanks to the unicity condition of the universal property of the coproduct $\coprod_{i\in I} \mathbb{R}_i$, this implies that $\tau = \rho$. Therefore we conclude that the map $\mathcal{S}\colon \mathcal{H}om_{\mathcal{H}\text{-}\mathsf{Fra}}(\coprod_{i\in I} \boldsymbol{R}_i, \boldsymbol{S}) \to \mathcal{H}om_{\mathcal{A}\text{-}\mathsf{Mod}}(\coprod_{i\in I} \mathbb{R}_i, \mathbb{S})$ is surjective.

It only remains to prove its injectivity. Consider two frame morphisms $\boldsymbol{\tau}, \boldsymbol{\rho}\colon \coprod_{i\in I} \boldsymbol{R}_i \to \boldsymbol{S}$ such that $\tau = \rho$. Let $i \in I$. Since $\boldsymbol{R}_i$ has the INT and $\tau\pi_i = \rho\pi_i$, we have that $\boldsymbol{\tau}\boldsymbol{\pi}_i = \boldsymbol{\rho}\boldsymbol{\pi}_i$. Therefore, by the unicity condition of the universal property of the coproduct $\coprod_{i\in I} \boldsymbol{R}_i$, we conclude that $\boldsymbol{\tau} = \boldsymbol{\rho}$. ⊠

Then we observe that the compositional lattice $\mathcal{H}$, seen as an $\mathcal{H}$-evaluational frame itself, has the interpretation property.

Lemma 6.6. *$\boldsymbol{H}$ has the INT.*

Proof. Let $\boldsymbol{R}$ be an evaluational frame. Observe that $\oplus\colon H \times A \to H$ is surjective, since $h \otimes 1 = h$ for every $h \in H$. This yields that the map $\mathcal{S}\colon \mathcal{H}om_{\mathcal{H}\text{-}\mathsf{Fra}}(\boldsymbol{H}, \boldsymbol{R}) \to \mathcal{H}om_{\mathcal{A}\text{-}\mathsf{Mod}}(\mathbb{A}, \mathbb{R})$ is injective.

Therefore it only remains to prove that it is surjective too. Let $\tau\colon \mathbb{A} \to \mathbb{R}$ be a $\mathcal{A}$-module morphism. We let $\tau'\colon H \to R'$ be the function defined as $\tau' h \coloneqq \langle h, \tau 1\rangle_{\boldsymbol{R}}$ for every $h \in H$. First we check that τ' is residuated. In order to do this, let $\{h_i\}_{i\in I} \subseteq H$, we have that

$$\tau' \bigvee_{i\in I}^{\mathbb{H}'} h_i = \langle \bigvee_{i\in I}^{\mathbb{H}'} h_i, \tau 1\rangle_{\boldsymbol{R}} = \bigvee_{i\in I}^{\mathbb{R}'} \langle h_i, \tau 1\rangle_{\boldsymbol{R}} = \bigvee_{i\in I}^{\mathbb{R}'} \tau' h_i.$$

Then we turn to prove that τ' is structural. Let $b \in B$ and $h \in H$, we have that $\tau'(b \otimes h) = \langle b \otimes h, \tau 1\rangle_{\boldsymbol{R}} = b *_{\boldsymbol{R}} \langle h, \tau 1\rangle_{\boldsymbol{R}} = b *_{\boldsymbol{R}} \tau' h$. We conclude that $\tau'\colon \mathbb{H}' \to \mathbb{R}'$ is a $\mathcal{B}$-module morphism. In order to prove that $\boldsymbol{\tau}\colon \boldsymbol{H} \to \boldsymbol{R}$ is a frame morphism, it only remains to check the commutativity condition. Let $h \in H$ and $a \in A$, we have that $\tau'(h \oplus a) = \langle h \oplus a, \tau 1\rangle_{\boldsymbol{R}} = \langle h, a \star_{\boldsymbol{R}} \tau 1\rangle_{\boldsymbol{R}} = \langle h, \tau(a \cdot 1)\rangle_{\boldsymbol{R}} = \langle h, \tau a\rangle_{\boldsymbol{R}}$. ⊠

We are now ready to present our desired characterisation of evaluational frames with the INT. In order to do this recall that, given an arbitrary $\mathcal{A}$-module $\mathbb{R}$, the map $\phi_{\mathbb{R}} \colon \coprod_{x \in R} \mathbb{A}_x \to \mathbb{R}$, defined for every $\overline{a} \in \coprod_{x \in R} A_x$ as

$$\phi_{\mathbb{R}}(\overline{a}) = \bigvee_{x \in R}^{\mathbb{R}} \overline{a}(x) \star_{\boldsymbol{R}} x,$$

is an $\mathcal{A}$-module epimorphism [16, Proposition 4.47]. In particular, together with Lemma 5.5, this implies that the map

$$\phi_{\mathbb{R}} \colon (\coprod_{x \in R} \mathbb{A}_x)_{\phi_{\mathbb{R}}^{+}\phi_{\mathbb{R}}} \to \mathbb{R}$$

is a module isomorphism. Then we let $\mathcal{I}(\mathbb{R})$ be the $\mathcal{H}$-evaluational frame obtained from

$$\Big(\coprod_{x \in R} \boldsymbol{H}_x\Big)_{\mathcal{F}(\coprod\limits_{x \in R} \boldsymbol{H}_x, \phi_{\mathbb{R}}^{+}\phi_{\mathbb{R}})}$$

replacing its syntactic component $(\coprod_{x \in R} \mathbb{A}_x)_{\phi_{\mathbb{R}}^{+}\phi_{\mathbb{R}}}$ by the isomorphic copy $\mathbb{R}$. It is clear that the evaluational frames $\mathcal{I}(\mathbb{R})$ and $(\coprod_{x \in R} \boldsymbol{H}_x)_{\mathcal{F}(\coprod_{x \in R} \boldsymbol{H}_x, \phi_{\mathbb{R}}^{+}\phi_{\mathbb{R}})}$ are isomorphic. However this characterization of $\mathcal{I}(\mathbb{R})$ will simplify some notational issues.

Theorem 6.7. *The following conditions are equivalent:*

(i) $\boldsymbol{R}$ *has the* INT*.*
(ii) $\boldsymbol{R}$ *is isomorphic to* $\mathcal{I}(\mathbb{R})$*.*

Proof. We claim that $\mathcal{I}(\mathbb{R})$ has the INT for every $\mathbb{R}$. In order to prove this consider an arbitrary $\mathcal{A}$-module $\mathbb{R}$. From Lemmas 6.6, 6.5 and Theorem 6.2 it follows that $(\coprod_{x \in R} \boldsymbol{H}_x)_{\mathcal{F}(\coprod_{x \in R} \boldsymbol{H}_x, \phi_{\mathbb{R}}^{+}\phi_{\mathbb{R}})}$ has the INT. Since $\mathcal{I}(\mathbb{R})$ is isomorphic to it, we conclude that $\mathcal{I}(\mathbb{R})$ has the INT too. This concludes the proof of our claim. As a consequence we obtain direction (ii)⇒(i). Then we turn to prove direction (i)⇒(ii). Observe that $\mathcal{S}\mathcal{I}(\mathbb{R}) = \mathbb{R} = \mathcal{S}(\boldsymbol{R})$. From our claim and the assumption, we know that both $\boldsymbol{R}$ and $\mathcal{I}(\mathbb{R})$ have the INT. Therefore from Lemma 6.3 it follows that they are isomorphic. ⊠

From Lemma 6.3 we know that, given an $\mathcal{A}$-module $\mathbb{R}$, there can be (up to isomorphism) at most one evaluational frame $\boldsymbol{R}$ with the INT whose syntactic component is isomorphic to $\mathbb{R}$. We are now able to prove that there is always such an evaluational frame.

Corollary 6.8. *Let* $\mathbb{R}$ *be an* $\mathcal{A}$*-module.* $\mathcal{I}(\mathbb{R})$ *is (up to isomorphism) the unique evaluational frame with the* INT*, whose syntactic component is isomorphic to* $\mathbb{R}$*.*

Proof. This is a direct consequence of Lemma 6.3 and Theorem 6.7. ⊠

As the careful reader could have noticed, we used the notation $\mathcal{I}(\cdot)$ both for $\mathcal{A}$-modules and for $\mathcal{A}$-module morphisms. This was not a coincidence: we would like our map to be a functor $\mathcal{I}\colon \mathcal{A}\text{-}\mathsf{Mod} \to \mathcal{H}\text{-}\mathsf{Fra}$, called the ***interpreting*** functor.

Lemma 6.9. *$\mathcal{I}$ is a functor.*

Proof. Observe that $\mathcal{I}$ is well-defined on arrows since by Theorem 6.7 $\mathcal{I}(\mathbb{R})$ has the INT for every $\mathcal{A}$-module $\mathbb{R}$. The fact that $\mathcal{I}$ respects identity arrows and composition follows from the unicity condition of the INT. ⊠

It turns out that the syntax and the interpreting functors form an adjoint pair between the categories of $\mathcal{A}$-modules and that of evaluational frames. Drawing consequences from this fact we will obtain a general result about the preservation of the INT under the formation of colimits.

Theorem 6.10. *$\mathcal{S}$ and $\mathcal{I}$ form an adjunction $\mathcal{I} \dashv \mathcal{S}$.*

Proof. First observe that, given a $\mathcal{A}$-module $\mathbb{R}$ and an $\mathcal{A}$-module morphism $\tau\colon \mathbb{R} \to \mathbb{S}$, we have $\mathcal{S}\mathcal{I}(\mathbb{R}) = \mathbb{R}$ and $\mathcal{S}\mathcal{I}(\tau) = \tau$. Then we can define a natural transformation $id\colon 1_{\mathcal{A}\text{-}\mathsf{Mod}} \to \mathcal{S}\mathcal{I}$ such that $id_{\mathbb{R}} \coloneqq 1_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ for every $\mathcal{A}$-module $\mathbb{R}$.

Then we turn to define a natural transformation $\eta\colon \mathcal{I}\mathcal{S} \to 1_{\mathcal{H}\text{-}\mathsf{Fra}}$. In order to do this, recall from Theorem 6.7 that $\mathcal{I}\mathcal{S}(\boldsymbol{R})$ has the INT for every evaluational frame $\boldsymbol{R}$. Therefore, given an arbitrary evaluational frame $\boldsymbol{R}$, it makes sense to put $\eta_{\boldsymbol{R}} \coloneqq \mathcal{I}(1_{\mathbb{R}})\colon \mathcal{I}\mathcal{S}(\boldsymbol{R}) \to \boldsymbol{R}$. In order to prove that η is a natural transformation, observe that for every frame morphism $\boldsymbol{\tau}\colon \boldsymbol{R} \to \boldsymbol{S}$ we have that $\mathcal{S}(\eta_{\boldsymbol{S}} \circ \mathcal{I}\mathcal{S}(\boldsymbol{\tau})) = \mathcal{S}(\mathcal{I}(1_{\mathbb{S}}) \circ \mathcal{I}\mathcal{S}(\boldsymbol{\tau})) = 1_{\mathbb{S}}\tau = \tau = \tau 1_{\mathbb{R}} = \mathcal{S}(\boldsymbol{\tau} \circ \mathcal{I}(1_{\mathbb{R}})) = \mathcal{S}(\boldsymbol{\tau} \circ \eta_{\boldsymbol{R}})$. By the uniqueness condition of the INT of $\mathcal{I}\mathcal{S}(\boldsymbol{R})$, we conclude that the following diagram commutes and therefore that η is a natural transformation.

$$\begin{tikzcd}
\mathcal{I}\mathcal{S}(\boldsymbol{R}) \arrow[r, "\eta_{\boldsymbol{R}}"] \arrow[d, "\mathcal{I}\mathcal{S}(\boldsymbol{\tau})"'] & \boldsymbol{R} \arrow[d, "\boldsymbol{\tau}"] \\
\mathcal{I}\mathcal{S}(\boldsymbol{S}) \arrow[r, "\eta_{\boldsymbol{S}}"'] & \boldsymbol{S}
\end{tikzcd}$$

It only remains to prove that id and η satisfy the adjunction conditions. Let $\boldsymbol{R}$ be an evaluational frame. We have that $\mathcal{S}(\eta_{\boldsymbol{R}}) \circ id_{\mathcal{S}(\boldsymbol{R})} = \mathcal{S}\mathcal{I}(1_{\mathbb{R}}) \circ 1_{\mathcal{S}(\boldsymbol{R})} = 1_{\mathbb{R}} \circ 1_{\mathcal{S}(\boldsymbol{R})} = 1_{\mathcal{S}(\boldsymbol{R})}$. Then let $\mathbb{R}$ be an $\mathcal{A}$-module. Observe that $\mathcal{S}(\eta_{\mathcal{I}(\mathbb{R})} \circ \mathcal{I}(id_{\mathbb{R}})) = \mathcal{S}(\mathcal{I}(1_{\mathbb{R}}) \circ \mathcal{I}(1_{\mathbb{R}})) = 1_{\mathbb{R}} = \mathcal{S}(1_{\mathcal{I}(\mathbb{R})})$. Applying the unicity condition of the INT of $\mathcal{I}(\mathbb{R})$, we conclude that $\eta_{\mathcal{I}(\mathbb{R})} \circ \mathcal{I}(id_{\mathbb{R}}) = 1_{\mathcal{I}(\mathbb{R})}$ and therefore we are done. ⊠

Corollary 6.11. *Let $\mathcal{D}$ be a diagram in $\mathcal{H}$-Fra whose objects have the INT. $\mathcal{D}$ have a colimit with the INT.*

Proof. Consider a digram in $\mathcal{H}$-Fra whose objects and arrows are respectively $\{\boldsymbol{R}_i\}_{i \in I}$ and $\{\boldsymbol{\tau}_j\}_{j \in J}$. Suppose then that $\boldsymbol{R}_i$ has the INT for every $i \in I$. By

Theorem 6.7 we can assume that $\boldsymbol{R}_i = \mathcal{I}(\mathbb{R}_i)$ for every $i \in I$. Then consider the diagram made up by $\{\mathbb{R}_i\}_{i\in I}$ and $\{\tau_j\}_{j\in J}$ in $\mathcal{A}$-Mod. From Proposition 4.26 of [16] we know that it has a colimit $\mathbb{S}$. Since $\mathcal{I}$ is a left adjoint functor, it preserves colimits. Therefore we conclude that $\mathcal{I}(\mathbb{S})$ is a coproduct of $\{\boldsymbol{R}_i\}_{i\in I}$ and $\{\mathcal{I}(\tau_j)\}_{j\in J}$. Moreover, from the fact that $\boldsymbol{R}_i$ has the INT for every $i \in I$, it follows that $\{\mathcal{I}(\tau_j)\}_{j\in J} = \{\boldsymbol{\tau}_j\}_{j\in J}$ and therefore that $\mathcal{I}(\mathbb{S})$ is a coproduct of the original diagram. Finally, the fact that $\mathcal{I}(\mathbb{S})$ has the INT follows from Theorem 6.7. ⊠

7. Projective objects

It should be noticed that the categorial analysis of the Syntactic and the Semantic Isomorphism Theorems present an intrinsic asymmetry. This is due to the fact that Galatos and Tsinakis characterised modules for which the Syntactic Isomorphism hold as the projective ones (Theorem 3.4), while in Example 8.11 we will prove that in general the class of evaluational frames for which the Semantic Isomorphism Theorem holds (or, equivalently, with INT) and the class of projective ones do not need to be comparable (see Figure 1)[2]. Nevertheless, the aim of this section is that of proving that projective evaluational frames still play a relevant role in the study of the Semantic Isomorphism Theorem, since they enjoy a weak form of the interpretation property.

In order to do this, we will make a wide use of the following construction. First recall that the unique (up to isomorphism) one-element $\mathcal{A}$-module $\mathbb{1}$ is a terminal object in $\mathcal{A}$-Mod [16, Proposition 4.26]. Then, given any $\mathcal{A}$-module $\mathbb{R}$, we will denote by $\mu_{\mathbb{R}} \colon \mathbb{R} \to \mathbb{1}$ the unique $\mathcal{A}$-module morphism from $\mathbb{R}$ to $\mathbb{1}$. Moreover, we put

$$\mathcal{T}(\mathbb{R}) \coloneqq \langle \mathbb{R}, \mathbb{1}, \langle \cdot, \cdot \rangle_{\mathcal{T}(\mathbb{R})} \rangle$$

where $\langle \cdot, \cdot \rangle_{\mathcal{T}(\mathbb{R})}$ is the unique function from $H \times R$ to 1. It is easy to prove that $\mathcal{T}(\mathbb{R})$ is an evaluational frame. Moreover, given an evaluational frame $\boldsymbol{R}$ and an $\mathcal{A}$-module morphism $\tau \colon \mathbb{R} \to \mathbb{S}$, we put

$$\mathcal{T}(\tau) \coloneqq \langle \tau, \mu_{\mathbb{R}'} \rangle \colon \boldsymbol{R} \to \mathcal{T}(\mathbb{S}).$$

It is easy to check that $\mathcal{T}(\tau)$ is a frame morphism.

Since we are interested in projective objects, it will be useful to understand how epimorphisms look like in $\mathcal{H}$-Mod. It is indeed possible to show, thanks to a duality closer to the one presented by Gil-Férez [16, Theorem 4.18], that frame epimorphisms are frame morphisms which are pairs of surjective functions. However, proving this fact would take us quite a long time and is not necessary for the aim of this paper, therefore we chose to rely on the following weaker result.

Lemma 7.1. *If $\boldsymbol{\tau} \colon \boldsymbol{R} \to \boldsymbol{S}$ is a frame epimorphism, then τ is surjective.*

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[2] But take a look at the commentary right before Theorem 8.4 for a partial reconciliation of the study of the two Isomorphism Theorems within a categorical framework.

Proof. We will reason by contraposition. Suppose that $\tau\colon \mathbb{R} \to \mathbb{S}$ is not surjective. Recall that epis in $\mathcal{A}$-Mod are surjective [16, Proposition 4.21]. Then τ is not an $\mathcal{A}$-module epimorphism. This implies that there are two different $\mathcal{A}$-module morphisms $\rho, \sigma\colon \mathbb{S} \to \mathbb{T}$ such that $\rho\tau = \sigma\tau$. Then consider the frame morphisms $\mathcal{T}(\rho), \mathcal{T}(\sigma)\colon \boldsymbol{S} \to \mathcal{T}(\mathbb{T})$. It is clear that $\mathcal{T}(\rho)\boldsymbol{\tau} = \mathcal{T}(\sigma)\boldsymbol{\tau}$ and that $\mathcal{T}(\rho) \neq \mathcal{T}(\sigma)$. Therefore we conclude that $\boldsymbol{\tau}$ is not a frame epimorphism. ⊠

Corollary 7.2. *The following conditions hold:*

1. *If $\boldsymbol{R}$ is projective, then $\mathbb{R}$ is projective too.*
2. *If $\boldsymbol{R}$ has the* INT*, then $\boldsymbol{R}$ is projective if and only if $\mathbb{R}$ is projective.*

Proof. 1. Let $\boldsymbol{R}$ be projective. Consider two $\mathcal{A}$-module morphisms $\tau\colon \mathbb{S} \to \mathbb{T}$ and $\rho\colon \mathbb{R} \to \mathbb{T}$ such that τ is an epimorphism. Then consider the frame morphisms $\mathcal{T}(\tau)\colon \mathcal{T}(\mathbb{S}) \to \mathcal{T}(\mathbb{T})$ and $\mathcal{T}(\rho)\colon \boldsymbol{R} \to \mathcal{T}(\mathbb{T})$. Since epis in $\mathcal{A}$-Mod are surjective, observe that each component of $\mathcal{T}(\tau)$ is surjective. Therefore $\mathcal{T}(\tau)$ is an epimorphism. By the fact that $\boldsymbol{R}$ is projective, we conclude that there is a frame morphism $\boldsymbol{\sigma}\colon \boldsymbol{R} \to \mathcal{T}(\mathbb{S})$ such that $\mathcal{T}(\tau)\boldsymbol{\sigma} = \mathcal{T}(\rho)$. This yields that $\tau\sigma = \rho$ and therefore we are done.

2. Let $\boldsymbol{R}$ have the INT. By condition 1, we know that if $\boldsymbol{R}$ is projective, then $\mathbb{R}$ is projective too. Therefore it only remains to prove the other direction. Let $\mathbb{R}$ be projective and consider two frame morphisms $\boldsymbol{\tau}\colon \boldsymbol{S} \to \boldsymbol{T}$ and $\boldsymbol{\rho}\colon \boldsymbol{R} \to \boldsymbol{T}$ such that $\boldsymbol{\tau}$ is an epimorphism. From Theorem 7.1 it follows that $\tau\colon \mathbb{S} \to \mathbb{T}$ is surjective and therefore that it is an $\mathcal{A}$-module epimorphism. Since $\mathbb{R}$ is projective, this yields that there is an $\mathcal{A}$-module morphism $\sigma\colon \mathbb{R} \to \mathbb{S}$ such that $\tau\sigma = \rho$. By the assumption $\boldsymbol{R}$ has the INT, therefore we can consider the frame morphism $\mathcal{I}(\sigma)\colon \boldsymbol{R} \to \boldsymbol{S}$. We have that $\mathcal{S}(\boldsymbol{\tau}\mathcal{I}(\sigma)) = \tau\sigma = \rho = \mathcal{S}(\boldsymbol{\rho})$. By the unicity condition of the INT we conclude that $\boldsymbol{\tau}\mathcal{I}(\sigma) = \boldsymbol{\rho}$. ⊠

Now that we learnt enough about projective evaluational frames, let us introduce a weaker property of the interpretation. More precisely we will forget of the unicity condition of the INT.

Definition 7.3. $\boldsymbol{R}$ has the ***weak interpretation*** property (for short WINT), if for every $\boldsymbol{S}$ the map $\mathcal{S}\colon \mathcal{H}om_{\mathcal{H}\text{-}\mathsf{Fra}}(\boldsymbol{R}, \boldsymbol{S}) \to \mathcal{H}om_{\mathcal{A}\text{-}\mathsf{Mod}}(\mathbb{R}, \mathbb{S})$ is surjective.

In other words $\boldsymbol{R}$ has the WINT if for every $\boldsymbol{S}$ and every $\mathcal{A}$-module morphism $\tau\colon \mathbb{R} \to \mathbb{S}$, there is at least one frame morphism $\boldsymbol{\tau}\colon \boldsymbol{R} \to \boldsymbol{S}$ such that $\mathcal{S}(\boldsymbol{\tau}) = \tau$. Clearly if $\boldsymbol{R}$ has the INT, then it has the WINT. The fact that these properties are not equivalent and that the WINT is not a trivial condition will be proved in Example 8.11. We begin our study of evaluational frames with the WINT by observing that they enjoy a nice relation with the ones with the INT.

Theorem 7.4. *$\boldsymbol{R}$ has the* WINT *if and only if $\mathcal{I}(1_{\mathbb{R}})\colon \mathcal{I}(\mathbb{R}) \to \boldsymbol{R}$ is a section.*

Proof. Recall from Theorem 6.7 that $\mathcal{I}(\mathbb{R})$ has the INT. Therefore it makes sense to consider the frame morphism $\mathcal{I}(1_{\mathbb{R}})\colon \mathcal{I}(\mathbb{R}) \to \boldsymbol{R}$. First suppose that $\boldsymbol{R}$ has the WINT. Then there is a frame morphism $\boldsymbol{\tau}\colon \boldsymbol{R} \to \mathcal{I}(\mathbb{R})$ such that $\mathcal{S}(\boldsymbol{\tau}) = 1_{\mathbb{R}}$. Then consider the composition $\boldsymbol{\tau}\mathcal{I}(1_{\mathbb{R}})\colon \mathcal{I}(\mathbb{R}) \to \mathcal{I}(\mathbb{R})$. We have that $\mathcal{S}(\boldsymbol{\tau}\mathcal{I}(1_{\mathbb{R}})) = 1_{\mathbb{R}}$ and therefore, by the unicity condition of the INT of $\mathcal{I}(\mathbb{R})$, that $\boldsymbol{\tau}\mathcal{I}(1_{\mathbb{R}}) = 1_{\mathcal{I}(\mathbb{R})}$. We conclude that $\mathcal{I}(1_{\mathbb{R}})\colon \mathcal{I}(\mathbb{R}) \to \boldsymbol{R}$ is a section.

Now we turn to prove the other direction. Let $\mathcal{I}(1_{\mathbb{R}})\colon \mathcal{I}(\mathbb{R}) \to \boldsymbol{R}$ be a section. Then there is a frame morphism $\boldsymbol{\tau}\colon \boldsymbol{R} \to \mathcal{I}(\mathbb{R})$ such that $\boldsymbol{\tau}\mathcal{I}(1_{\mathbb{R}}) = 1_{\mathcal{I}(\mathbb{R})}$. It is easy to see that $\tau = 1_{\mathbb{R}}$. Then let $\boldsymbol{S}$ be an evaluational frame and $\rho\colon \mathbb{R} \to \mathbb{S}$ an $\mathcal{A}$-module morphism. Since $\mathcal{I}(\mathbb{R})$ has the INT, we can consider the frame morphism $\mathcal{I}(\rho)\colon \mathcal{I}(\mathbb{R}) \to \boldsymbol{S}$ and therefore composition $\mathcal{I}(\rho)\boldsymbol{\tau}\colon \boldsymbol{R} \to \boldsymbol{S}$. Since $\mathcal{S}(\mathcal{I}(\rho)\boldsymbol{\tau}) = \rho 1_{\mathbb{R}} = \rho$, we are done. ⊠

We are now ready to prove the main result of the section, i.e., that projective objects can interpret every syntactic transformer (possibly in more than one way).

Theorem 7.5. *If $\boldsymbol{R}$ is projective, then it has the* WINT.

Proof. Let $\boldsymbol{R}$ be projective. Then consider the (possibly different) frame morphisms $\mathcal{T}(1_{\mathbb{R}})\colon \mathcal{I}(\mathbb{R}) \to \mathcal{T}(\mathbb{R})$ and $\mathcal{T}(1_{\mathbb{R}})\colon \boldsymbol{R} \to \mathcal{T}(\mathbb{R})$. Clearly $\mathcal{T}(1_{\mathbb{R}})\colon \mathcal{I}(\mathbb{R}) \to \mathcal{T}(\mathbb{R})$ is a frame epimorphism, since each one of its components is surjective. From the fact that $\boldsymbol{R}$ is projective it follows that there is a frame morphism $\boldsymbol{\tau}\colon \boldsymbol{R} \to \mathcal{I}(\mathbb{R})$ which makes the following digram commutes.

$$\begin{tikzcd} \boldsymbol{R} \arrow[r, dashed, "\boldsymbol{\tau}"] \arrow[dr, "\mathcal{T}(1_{\mathbb{R}})"'] & \mathcal{I}(\mathbb{R}) \arrow[d, "\mathcal{T}(1_{\mathbb{R}})"] \\ & \mathcal{T}(\mathbb{R}) \end{tikzcd}$$

In particular, this yields that $\tau = 1_{\mathbb{R}}$.

Recall from Theorem 7.4 that, in order to prove that $\boldsymbol{R}$ has the WINT, it is enough to show that the frame morphism $\mathcal{I}(1_{\mathbb{R}})\colon \mathcal{I}(\mathbb{R}) \to \boldsymbol{R}$ is a section. This is what we do now. Consider the composition $\boldsymbol{\tau}\mathcal{I}(1_{\mathbb{R}})\colon \mathcal{I}(\mathbb{R}) \to \mathcal{I}(\mathbb{R})$. We have that $\mathcal{S}(\boldsymbol{\tau}\mathcal{I}(1_{\mathbb{R}})) = \tau 1_{\mathbb{R}} = \tau = 1_{\mathbb{R}} = \mathcal{S}(1_{\mathcal{I}(\mathbb{R})})$. By the unicity condition of the INT of $\mathcal{I}(\mathbb{R})$, we conclude that $\boldsymbol{\tau}\mathcal{I}(1_{\mathbb{R}}) = 1_{\mathcal{I}(\mathbb{R})}$ and therefore we are done. ⊠

The fact that Theorem 7.5 cannot be strengthened, in the sense that its converse does not hold in general and that there are projective evaluational frames without the INT, will be proved in Example 8.11.

8. Full algebraizability

Until now we focused on the study of the conditions under which half part of the Semantic Isomorphism Theorem could be recovered within the context of evaluational frames. In this section we consider the natural problem of characterising evaluational frames for which both the abstract versions of the Syntactic and of the Semantic Isomorphism Theorem hold. In other words, we concentrate on evaluational frames for which the three characterisations of algebraizable logics, i.e., the one in terms of existence of two structural transformers from formulas to equations and vice-versa, the one in terms of the existence of a syntactic isomorphism and the one in terms of a semantic isomorphism, are still equivalent.

Definition 8.1. $\boldsymbol{R}$ has the ***full algebraizability*** property (for short FAL), if $\mathbb{R}$ has the REP and $\boldsymbol{R}$ the INT.

We will see in Corollary 8.10 that the natural examples of evaluational frames of equations, formulas and sequents enjoy the full algebraizability property. But for the moment we rely on the fact that this is the case at least for the compositional lattice $\mathcal{H}$, seen as an evaluational frame itself, as we remark in the following example.

Example 8.2. From Theorem 3.5 we know that $\mathbb{A}$ is projective. Then by Theorem 3.4 we conclude that it has the REP. Moreover in Lemma 6.6 we proved that $\boldsymbol{H}$ has the INT. ⊠

A direct consequence of the definition of the FAL is that it is preserved under the formation of coproducts.

Lemma 8.3. *Let $\{\boldsymbol{R}_i\}_{i\in I}$ be a family of evaluational frames. If $\boldsymbol{R}_i$ has the* FAL *for every $i \in I$, then $\coprod_{i\in I} \boldsymbol{R}_i$ has the* FAL.

Proof. Let $\{\boldsymbol{R}_i\}_{i\in I}$ be a family of evaluational frames such that $\boldsymbol{R}_i$ has the FAL for every $i \in I$. The assumption together with Theorem 3.4, yields that $\mathbb{R}_i$ is projective for every $i \in I$. Since projectivity is preserved under the formation of coproducts, we know that $\coprod_{i\in I} \mathbb{R}_i$ is projective too. Hence, by Theorem 3.4, we conclude that $\coprod_{i\in I} \mathbb{R}_i$ has the REP. Moreover the assumption together with Lemma 6.5 yields that $\coprod_{i\in I} \boldsymbol{R}_i$ has the INT. ⊠

Evaluational frames with the FAL can be described in a nice categorial fashion, as we remark in the next result. This fact can be read as a reconciliation of the study of the abstract versions of the Syntactic and Semantic Isomorphism Theorems. In order to explain this point, recall that in Theorem 3.4 it is claimed that the Syntactic Isomorphism Theorem holds for a module $\mathbb{R}$ if and only if it is projective. Moreover, Gil-Férez proved in [16, Theorem 4.51] that $\mathbb{R}$ is projective if and only if $\phi_{\mathbb{R}} \colon \coprod_{x\in R} \mathbb{A}_x \to \mathbb{R}$ is a retraction. Comparing this condition with point (ii) of the next result, we conclude that the modules for which the Syntactic Isomorphism Theorem holds and the evaluational frames for which the both the Syntactic and the Semantic Isomorphism Theorems hold enjoy an analogous characterization as retracts of coproducts of $\mathbb{A}$ and $\boldsymbol{H}$ respectively.

Theorem 8.4. *The following conditions are equivalent:*

(i) *$\boldsymbol{R}$ has the* FAL.

(ii) $\mathcal{I}(\phi_{\mathbb{R}})\colon \coprod_{x\in R} \boldsymbol{H}_x \to \boldsymbol{R}$ *is a retraction.*
(iii) $\boldsymbol{R}$ *is a retract of a an object with the* FAL*.*

Proof. (i)⇒(ii): Observe that $\mathbb{R}$ is projective. By Theorem 4.51 of [16] this amounts to saying that $\phi_{\mathbb{R}}\colon \coprod_{x\in R} \mathbb{A}_x \to \mathbb{R}$ is a retraction. Therefore there is an $\mathcal{A}$-module morphism $\tau\colon \mathbb{R} \to \coprod_{x\in R} \mathbb{A}_x$ such that $\phi_{\mathbb{R}}\tau = 1_{\mathbb{R}}$. Since $\boldsymbol{R}$ and $\boldsymbol{H}$ have the INT, we can consider the frame morphisms $\mathcal{I}(\tau)\colon \boldsymbol{R} \to \coprod_{x\in R} \boldsymbol{H}_x$ and $\mathcal{I}(\phi_{\mathbb{R}})\colon \coprod_{x\in R} \boldsymbol{H}_x \to \boldsymbol{R}$. Now observe that $\mathcal{S}(\mathcal{I}(\phi_{\mathbb{R}})\mathcal{I}(\tau)) = \phi_{\mathbb{R}}\tau = 1_{\mathbb{R}} = \mathcal{S}(1_{\boldsymbol{R}})$. By the unicity condition of the INT of $\boldsymbol{R}$, we conclude that $\mathcal{I}(\phi_{\mathbb{R}})\mathcal{I}(\tau) = 1_{\boldsymbol{R}}$ and therefore we are done.

(ii)⇒(iii): Recall from Example 8.2 that $\boldsymbol{H}$ has the FAL. Therefore applying the assumption and Lemma 8.3 we are done.

(iii)⇒(i): Let $\boldsymbol{\tau}\colon \boldsymbol{S} \to \boldsymbol{R}$ be a retraction for some $\boldsymbol{S}$ with the FAL. Then there is a frame morphism $\boldsymbol{\sigma}\colon \boldsymbol{R} \to \boldsymbol{S}$ such that $\boldsymbol{\tau\sigma} = 1_{\boldsymbol{R}}$. Since $\mathbb{S}$ is projective and $\boldsymbol{S}$ has the INT, condition 2 of Corollary 7.2 yields that $\boldsymbol{S}$ is projective too. Then $\boldsymbol{R}$ is projective, since retracts of projective objects are projective. From Theorem 7.5 it follows that $\boldsymbol{R}$ has the WINT. Therefore, in order to prove that $\boldsymbol{R}$ has the INT too, it is enough to check the unicity condition.

In order to do this let $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2\colon \boldsymbol{R} \to \boldsymbol{T}$ be two frame morphisms such that $\mathcal{S}(\boldsymbol{\rho}_1) = \mathcal{S}(\boldsymbol{\rho}_2)$. Consider the compositions $\boldsymbol{\rho}_1\boldsymbol{\tau}, \boldsymbol{\rho}_2\boldsymbol{\tau}\colon \boldsymbol{S} \to \boldsymbol{T}$. We have that $\mathcal{S}(\boldsymbol{\rho}_1\boldsymbol{\tau}) = \mathcal{S}(\boldsymbol{\rho}_2\boldsymbol{\tau})$. By the unicity condition of the INT of $\boldsymbol{S}$, this implies that $\boldsymbol{\rho}_1\boldsymbol{\tau} = \boldsymbol{\rho}_2\boldsymbol{\tau}$. Therefore we conclude that $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_1 1_{\boldsymbol{R}} = \boldsymbol{\rho}_1\boldsymbol{\tau\sigma} = \boldsymbol{\rho}_2\boldsymbol{\tau\sigma} = \boldsymbol{\rho}_2 1_{\boldsymbol{R}} = \boldsymbol{\rho}_2$.

Finally recall that $\boldsymbol{R}$ is projective. Therefore, by condition 1 of Corollary 7.2 and Theorem 3.4, we conclude that $\mathbb{R}$ has the REP. ⊠

Now let us take a closer look to a special class of evaluational frames, namely to the ones that enjoy a general kind of variable.

Definition 8.5. $\boldsymbol{R}$ is ***cyclic*** if there is an element $v \in R$ such that $R = \{a \star_{\boldsymbol{R}} v \in R : a \in A\}$ and $R' = \{\langle h, v\rangle_{\boldsymbol{R}} \in R' : h \in H\}$. In this case we say that v is a ***generator*** of $\boldsymbol{R}$.

Observe that if $\boldsymbol{R}$ is cyclic, then $\mathbb{R}$ is cyclic too. Moreover in the case of cyclic evaluational frames, the property of the interpretation coincides with its weak version, as we remark below.

Lemma 8.6. *Let $\boldsymbol{R}$ be cyclic. The following conditions are equivalent:*

(i) $\boldsymbol{R}$ *has the* INT*.*
(ii) $\boldsymbol{R}$ *has the* WINT*.*

Proof. Clearly condition (i) implies condition (ii), then we turn to prove the other direction. Suppose that $\boldsymbol{R}$ has the WINT. In order to prove that $\boldsymbol{R}$ has the INT too, it will be enough to show that for every pair of frame morphisms $\boldsymbol{\tau}, \boldsymbol{\rho}\colon \boldsymbol{R} \to \boldsymbol{S}$ if $\mathcal{S}(\boldsymbol{\tau}) = \mathcal{S}(\boldsymbol{\rho})$, then $\boldsymbol{\tau} = \boldsymbol{\rho}$. But this is a consequence of the fact that the $\langle \cdot, \cdot \rangle_{\boldsymbol{R}}$ is surjective, since $\boldsymbol{R}$ is cyclic. ⊠

Corollary 8.7. *Cyclic and projective evaluational frames have the* FAL*.*

Proof. Let $\boldsymbol{R}$ be cyclic and projective. From Theorem 7.5 and Lemma 8.6, it follows that it has the INT. Moreover from condition 1 of Corollary 7.2 and Theorem 3.4, we know that $\mathbb{R}$ has the REP. ⊠

Condition 2 of Corollary 7.2 implies that, under the assumption of the INT, projectivity transfers from the syntactic component of an evaluational frame to the whole structure. An analogous result can be obtained for what concerns cyclicity.

Lemma 8.8. *If $\boldsymbol{R}$ has the* INT*, then $\boldsymbol{R}$ is cyclic if and only if $\mathbb{R}$ is cyclic.*

Proof. Let $\boldsymbol{R}$ have the INT. Clearly if $\boldsymbol{R}$ is cyclic, so is $\mathbb{R}$. Then we turn to prove the other direction: suppose that $\mathbb{R}$ is cyclic. Let v be a generator of $\mathbb{R}$. It is easy to prove that the function $m_v \colon \mathbb{A} \to \mathbb{R}$, defined as $m_v(a) = a \star_{\boldsymbol{R}} v$ for every $a \in A$, is a surjective $\mathcal{A}$-module morphism. Since $\boldsymbol{H}$ has the INT, we can consider the frame morphism $\boldsymbol{m}_v \coloneqq \mathcal{I}(m_v) \colon \boldsymbol{H} \to \boldsymbol{R}$. By Lemma 4.8 we know that $\boldsymbol{m}_v[\boldsymbol{H}]$ is an evaluational frame.

First we claim that $\boldsymbol{m}_v[\boldsymbol{H}]$ is cyclic with generator v. Since $v = 1 \star_{\boldsymbol{R}} v = m_v 1 \in m_v[A]$, we have that $\{a \star_{\boldsymbol{m}_v[\boldsymbol{H}]} v : a \in A\} = \{a \star_{\boldsymbol{R}} v : a \in A\} = m_v[A]$. Moreover let $y \in m'_v[H]$. We know that there is $h \in H$ such that $m'_v h = y$. Therefore, applying the fact that $\boldsymbol{m}_v$ is a frame morphism and that $v \in m_v[A]$, we have that

$$y = m'_v h = m'_v(h \oplus 1) = \langle h, m_v 1\rangle_{\boldsymbol{R}} = \langle h, v\rangle_{\boldsymbol{R}} = \langle h, v\rangle_{\boldsymbol{m}_v[\boldsymbol{H}]}.$$

Hence we obtain that $\{\langle h, v\rangle_{\boldsymbol{m}_v[\boldsymbol{H}]} : h \in H\} = m'_v[H]$. This concludes the proof of the claim.

Then we turn to prove that $\boldsymbol{m}_v[\boldsymbol{H}]$ has the INT. Let $\boldsymbol{S}$ be an evaluational frame and $\tau \colon m_v[\mathbb{A}] \to \mathbb{S}$ an $\mathcal{A}$-module morphism. Since m_v is surjective, we have that $m_v[\mathbb{A}] = \mathbb{R}$. Therefore we can apply the INT of $\boldsymbol{R}$ and obtain a frame morphism $\mathcal{I}(\tau) \colon \boldsymbol{R} \to \boldsymbol{S}$. Now let $\mathcal{I}(\tau)\boldsymbol{\iota} \colon \boldsymbol{m}_v[\boldsymbol{H}] \to \boldsymbol{S}$ be the composition, where $\boldsymbol{\iota} \colon \boldsymbol{m}_v[\boldsymbol{H}] \to \boldsymbol{R}$ is the inclusion frame morphism. Since $\mathcal{S}(\mathcal{I}(\tau)\boldsymbol{\iota}) = \tau\iota = \tau$, we conclude that $\boldsymbol{R}$ has the WINT. Applying the claim and with Lemma 8.6, this implies that $\boldsymbol{m}_v[\boldsymbol{H}]$ has the INT too.

Finally recall that $m_v[\mathbb{A}] = \mathbb{R}$. Together with Lemma 6.3 and the fact that $\boldsymbol{R}$ and $\boldsymbol{m}_v[\boldsymbol{H}]$ have both the INT, this yields that $\boldsymbol{R}$ is isomorphic to $\boldsymbol{m}_v[\boldsymbol{H}]$. By the claim we conclude that $\boldsymbol{R}$ is cyclic. ⊠

Theorem 3.5 characterises cyclic and projective $\mathcal{A}$-modules. We are now ready to prove that an analogous result holds in the context of evaluational frames too.

Theorem 8.9. *The following conditions are equivalent:*

(i) *$\boldsymbol{R}$ is cyclic and projective.*
(ii) *$\boldsymbol{R}$ is a retract of $\boldsymbol{H}$.*
(iii) *There are $u \in A$ and a generator v of $\boldsymbol{R}$ such that $u \star_{\boldsymbol{R}} v = v$, $((a \star_{\boldsymbol{R}} v) \mathbin{/_{\star_{\boldsymbol{R}}}} v) \cdot u = a \cdot u$ and $(\langle h, v\rangle_{\boldsymbol{R}} \mathbin{/_{\langle\cdot,\cdot\rangle_{\boldsymbol{R}}}} v) \oplus u = h \oplus u$ for every $a \in A$ and $h \in H$.*

Proof. (i)⇒(ii): Suppose that $\boldsymbol{R}$ is cyclic and projective. From the fact that $\boldsymbol{R}$ is projective and Theorem 7.5, we know that $\boldsymbol{R}$ has the WINT. This yields, together with the fact that $\boldsymbol{R}$ is cyclic and Lemma 8.8, that $\boldsymbol{R}$ has the INT too. Moreover, from condition 1 of Corollary 7.2, we know that $\mathbb{R}$ is cyclic and projective. By Theorem 3.5 this yields that $\mathbb{R}$ is a retract of $\mathbb{A}$. We conclude that $\boldsymbol{R}$ is a retract of $\boldsymbol{H}$, since both $\boldsymbol{R}$ and $\boldsymbol{H}$ have the INT.

(ii)⇒(iii): Since $\boldsymbol{R}$ is a retract of $\boldsymbol{H}$, there are two frame morphisms $\boldsymbol{\tau}\colon \boldsymbol{H} \to \boldsymbol{R}$ and $\boldsymbol{\rho}\colon \boldsymbol{R} \to \boldsymbol{H}$ such that $\boldsymbol{\tau}\boldsymbol{\rho} = 1_{\boldsymbol{R}}$. Then we let $u \coloneqq \rho\tau 1$ and $v \coloneqq \tau 1$. First we check that v s a generator of $\boldsymbol{R}$. In order to prove this, observe that the two components of $\boldsymbol{\tau}$ are surjective, since it is a retraction. Then let $x \in R$. Since τ is surjective, there is $c \in A$ such that $\tau c = x$. This yields that $x = \tau c = \tau(c \cdot 1) = c \star_{\boldsymbol{R}} \tau 1 = c \star_{\boldsymbol{R}} v$ and therefore that $\{a \star_{\boldsymbol{R}} v : a \in A\} = R$. From an analogous argument, it follows that $\{\langle h, v\rangle_{\boldsymbol{R}} : h \in H\} = R'$. Therefore we are done.

Then we have that $u \star_{\boldsymbol{R}} v = \rho\tau 1 \star_{\boldsymbol{R}} \tau 1 = \tau(\rho\tau(1) \cdot 1) = \tau\rho\tau 1 = 1_{\mathbb{R}}\tau 1 = \tau 1 = v$. Moreover, observe that

$$(x \,/_{\star_{\boldsymbol{R}}}\, v) \star_{\boldsymbol{R}} v = x \text{ and } \langle y \,/_{\langle \cdot,\cdot\rangle_{\boldsymbol{R}}}\, v, v\rangle_{\boldsymbol{R}} = y \tag{5}$$

for every $x \in R$ and $y \in R'$. We will detail proof of the first condition, since the proof of the other one is analogous. From residuation it easily follows that $(x \,/_{\star_{\boldsymbol{R}}}\, v) \star_{\boldsymbol{R}} v \leqslant x$. Moreover, since v is a generator of $\boldsymbol{R}$, there is $a \in A$ such that $a \star_{\boldsymbol{R}} v = x$. Then in particular $a \star_{\boldsymbol{R}} v \leqslant x$. By residuation this yields that $a \leqslant x \,/_{\star_{\boldsymbol{R}}}\, v$. Since residuated mappings are monotone in both components, we have that $x = a \star_{\boldsymbol{R}} v \leqslant (x \,/_{\star_{\boldsymbol{R}}}\, v) \star_{\boldsymbol{R}} v$ and therefore we are done.

Then let $a \in A$. Applying (5), we have that $((a \star_{\boldsymbol{R}} v) \,/_{\star_{\boldsymbol{R}}}\, v) \cdot u = ((a \star_{\boldsymbol{R}} \tau 1) \,/_{\star_{\boldsymbol{R}}}\, v) \cdot \rho\tau 1 = \rho((\tau(a \cdot 1) \,/_{\star_{\boldsymbol{R}}}\, v) \star_{\boldsymbol{R}} \tau 1) = \rho((\tau a \,/_{\star_{\boldsymbol{R}}}\, v) \star_{\boldsymbol{R}} v) = \rho\tau a = \rho\tau(a \cdot 1) = a \cdot \rho\tau 1 = a \cdot u$. Finally let $h \in H$. Applying our claim, we have that $(\langle h, v\rangle_{\boldsymbol{R}} \,/_{\langle \cdot,\cdot\rangle_{\boldsymbol{R}}}\, v) \oplus u = (\langle h, \tau 1\rangle_{\boldsymbol{R}} \,/_{\langle \cdot,\cdot\rangle_{\boldsymbol{R}}}\, v) \oplus \rho\tau 1 = \rho'\langle \tau'(h \oplus 1) \,/_{\langle \cdot,\cdot\rangle_{\boldsymbol{R}}}\, v, \tau 1\rangle_{\boldsymbol{R}} = \rho'\langle \tau' h \,/_{\langle \cdot,\cdot\rangle_{\boldsymbol{R}}}\, v, v\rangle_{\boldsymbol{R}} = \rho'\tau' h = \rho'\tau'(h \oplus 1) = \rho'\langle h, \tau' 1\rangle_{\boldsymbol{R}} = h \oplus \rho\tau 1 = h \oplus u$.

(iii)⇒(i): From the assumption we know that $\boldsymbol{R}$ is cyclic, then we turn to check that it is projective. Since $\boldsymbol{H}$ is projective and retracts of projective objects are still projective, it will be enough to prove that $\boldsymbol{R}$ is a retract of $\boldsymbol{H}$. Then consider the function $\tau\colon R \to A$ defined as $\tau(a \star_{\boldsymbol{R}} v) = a \cdot u$ for every $a \star_{\boldsymbol{R}} v \in R$. We claim that it is well-defined. In order to prove this, let $a, b \in A$ such that $a \star_{\boldsymbol{R}} v = b \star_{\boldsymbol{R}} v$. By the assumption we have that $\tau(a \star_{\boldsymbol{R}} v) = a \cdot u = ((a \star_{\boldsymbol{R}} v) \,/_{\star_{\boldsymbol{R}}}\, v) \cdot u = ((b \star_{\boldsymbol{R}} v) \,/_{\star_{\boldsymbol{R}}}\, v) \cdot u = b \cdot u$. Then we turn to prove that τ is an $\mathcal{A}$-module morphism from $\mathbb{R}$ to $\mathbb{A}$. First we show that it is residuated. Let $\{a_i \star_{\boldsymbol{R}} v\}_{i \in I} \subseteq R$. We have that

$$\tau \bigvee_{i \in I}^{\mathbb{R}} (a_i \star_{\boldsymbol{R}} v) = \tau(\bigvee_{i \in I}^{\mathbb{A}} a_i \star_{\boldsymbol{R}} v) = \bigvee_{i \in I}^{\mathbb{A}} a_i \cdot u = \bigvee_{i \in I}^{\mathbb{A}} (a_i \cdot u) = \bigvee_{i \in I}^{\mathbb{A}} \tau(a_i \star_{\boldsymbol{R}} v).$$

Then let $b \in A$ and $a \star_{\boldsymbol{R}} v \in R$. We have that $\tau(b \star_{\boldsymbol{R}} (a \star_{\boldsymbol{R}} v)) = \tau((b \cdot a) \star_{\boldsymbol{R}} v) = (b \cdot a) \cdot u = b \cdot (a \cdot u) = b \cdot \tau(a \star_{\boldsymbol{R}} v)$. Hence τ is an $\mathcal{A}$-module morphism.

By an analogous argument it is possible to prove that the function $\tau\colon R' \to H$, defined as $\tau'\langle h, v\rangle_{\boldsymbol{R}} = h \oplus u$ for every $\langle h, v\rangle_{\boldsymbol{R}} \in R'$, is a well-defined residuated function from $\mathbb{R}'$ to $\mathbb{H}$. In order to prove that it is a $\mathcal{B}$-module morphism too, let $b \in B$ and $\langle h, v\rangle_{\boldsymbol{R}} \in R'$. We have that $\tau'(b *_{\boldsymbol{R}} \langle h, v\rangle_{\boldsymbol{R}}) = \tau'\langle b \otimes h, v\rangle_{\boldsymbol{R}} = (b \otimes h) \oplus u = b \otimes (h \oplus u) = b \otimes \tau'\langle h, v\rangle_{\boldsymbol{R}}$. Hence τ is a $\mathcal{B}$-module morphism.

Now, we claim that $\boldsymbol{\tau} \coloneqq \langle \tau, \tau'\rangle$ is a frame morphism from $\boldsymbol{R}$ to $\boldsymbol{H}$. Since τ and τ' are respectively an $\mathcal{A}$ and a $\mathcal{B}$-module morphism, it will be enough to check the commutativity condition. Let $h \in H$ and $a \star_{\boldsymbol{R}} v \in R$. We have that $\tau'\langle h, a \star_{\boldsymbol{R}} v\rangle_{\boldsymbol{R}} = \tau'\langle h \oplus a, v\rangle_{\boldsymbol{R}} = (h \oplus a) \oplus u = h \oplus (a \cdot u) = h \oplus \tau(a \star_{\boldsymbol{R}} v)$. This concludes the proof of our claim.

Then consider the $\mathcal{A}$-module morphism $m_v\colon \mathbb{A} \to \mathbb{R}$, defined as $m_v(a) = a \star_{\boldsymbol{R}} v$ for every $a \in A$. Since $\boldsymbol{H}$ has the INT, we can consider the frame morphism $\boldsymbol{m}_v \coloneqq \mathcal{I}(m_v)\colon \boldsymbol{H} \to \boldsymbol{R}$. By the assumption we have that $m_v \tau v = m_v \tau(u \star_{\boldsymbol{R}} v) = (u \cdot u) \star_{\boldsymbol{R}} v = u \star_{\boldsymbol{R}} (u \star_{\boldsymbol{R}} v) = u \star_{\boldsymbol{R}} v = v$. Since v is a generator of $\boldsymbol{R}$, this yields that $\boldsymbol{m}_v \boldsymbol{\tau} = 1_{\boldsymbol{R}}$. ⊠

Corollary 8.10. *$\boldsymbol{Fm}(\mathbb{A})$ and $\boldsymbol{Eq}(\mathbb{A})$ are cyclic and projective. Moreover $\boldsymbol{Fm}(\mathbb{A})$, $\boldsymbol{Eq}(\mathbb{A})$ and $\boldsymbol{Seq}(\mathbb{A})$ have the* FAL.

Proof. It is easy to prove that $\boldsymbol{Fm}(\mathbb{A})$ and $\boldsymbol{Eq}(\mathbb{A})$ satisfy condition (iii) of Theorem 8.9, by letting u and v as in the commentary right after Theorem 3.5. Therefore, by Theorem 8.9, we conclude that they are cyclic and projective, and from Corollary 8.7 it follows that they enjoy the FAL.

For what concerns $\boldsymbol{Seq}(\mathbb{A})$ we reason as follows. Consider an arbitrary algebra $\boldsymbol{B}$ and let n and m be natural numbers. Then let $\mathrm{B}(n, m)$ be the set of elements of $\mathrm{Seq}(B)$, whose first component has length n and whose second component has length m. Now consider the function

$$\star_{\mathsf{B}(n,m)}\colon \mathcal{E}nd(\boldsymbol{B}) \times \mathrm{B}(n, m) \to \mathrm{B}(n, m)$$

defined as $\sigma \star_{\mathsf{B}(n,m)} \langle \bar{b}, \bar{c}\rangle \coloneqq \langle \sigma(\bar{b}), \sigma(\bar{c})\rangle$ for every $\sigma \in \mathcal{E}nd(\boldsymbol{B})$ and $\langle \bar{b}, \bar{c}\rangle \in \mathrm{B}(n, m)$. It is easy to prove that $\mathsf{B}(n, m) \coloneqq \langle \mathrm{B}(n, m), \star_{\mathsf{B}(n,m)}\rangle$ is an $\mathcal{M}(\boldsymbol{B})$-set.

Then observe that $\mathcal{P}(\mathsf{Fm}(n, m))$ and $\mathcal{P}(\mathsf{A}(n, m))$ are respectively a $\mathcal{P}(\mathcal{M}(\boldsymbol{Fm}))$ and a $\mathcal{P}(\mathcal{M}(\boldsymbol{A}))$-module. Keeping this in mind, it is easy to see that

$$\boldsymbol{R}(n, m) \coloneqq \langle \mathcal{P}(\mathsf{Fm}(n, m)), \mathcal{P}(\mathsf{A}(n, m)), \langle \cdot, \cdot\rangle_{\boldsymbol{R}(n,m)}\rangle,$$

where $\langle h, \langle \overline{x}, \overline{y}\rangle\rangle_{\boldsymbol{R}} \mathcal{E}nd(\boldsymbol{B}) \langle h\overline{x}, h\overline{y}\rangle$ for every $h \in H$ and $\langle \overline{x}, \overline{y}\rangle \in \mathrm{Fm}(n, m)$, is an $\mathcal{H}(\boldsymbol{Fm}, \boldsymbol{A})$-evaluational frame. Moreover, we claim that $\boldsymbol{R}(n, m)$ is cyclic and projective. In order to prove this fact, consider an injective enumeration $x_1, x_2, \ldots, x_i, \ldots$ of the variables of our language. Then we put $v \coloneqq \{\langle \overline{x}, \overline{y}\rangle\}$, where $\overline{x} = \langle x_1, x_2, \ldots, x_n\rangle$ and $\overline{y} = \langle x_{n+1}, x_{n+2}, \ldots, x_{n+m}\rangle$, and $u \coloneqq \{\sigma\}$, where σ is the substitution which is the identity on $\{x_1, x_2, \ldots, x_{n+m}\}$ and

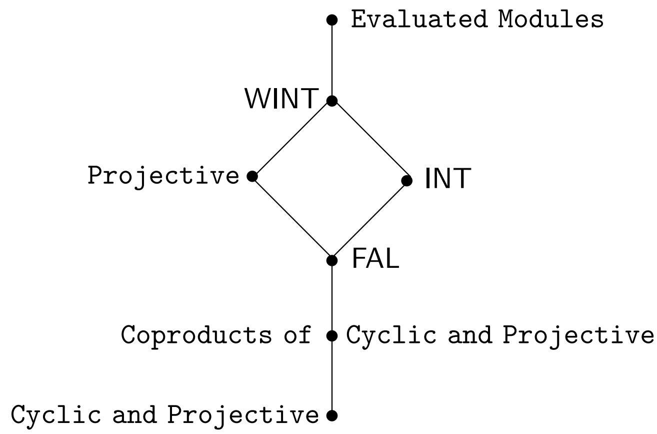

FIGURE 1. Some classes of evaluational frames.

sends other variables to x_1. It is easy to prove that v and u satisfy condition (iii) of Theorem 8.9. Therefore we are done.

Finally observe that the frame morphism $\boldsymbol{\tau}\colon \coprod_{\langle n,m\rangle\in\omega\times\omega} \boldsymbol{R}(n,m) \to \boldsymbol{Seq}(\mathbb{A})$, defined as

$$\tau\overline{x} \coloneqq \bigcup_{\langle n,m\rangle\in\omega\times\omega} \overline{x}(\langle n,m\rangle) \quad \tau'\overline{y} \coloneqq \bigcup_{\langle n,m\rangle\in\omega\times\omega} \overline{y}(\langle n,m\rangle)$$

for every $\overline{x} \in \coprod_{\langle n,m\rangle\in\omega\times\omega} \mathcal{P}(\mathsf{Fm}(n,m))$ and $\overline{y} \in \coprod_{\langle n,m\rangle\in\omega\times\omega} \mathcal{P}(\mathsf{A}(n,m))$, is indeed an isomorphism. The fact that $\boldsymbol{R}(n,m)$ is cyclic and projective for every $\langle n,m\rangle \in \omega\times\omega$, together with Corollary 8.7 and Lemma 8.3, yields that $\boldsymbol{Seq}(\mathbb{A})$ has the FAL. ⊠

We conclude this section by presenting a quite long example, which establishes that the various classes of evaluational frames we have considered so far are indeed different one from the other and that the inclusions that we proved until now are the only ones that hold in general. More precisely, we will prove that the classes of evaluational frames, evaluational frames with the WINT, evaluational frames with the INT, projective evaluational frames, evaluational frames with the FAL, coproducts of cyclic and projective evaluational frames and the one of cyclic and projective evaluational frames are different and form, when ordered under the inclusion relation, the poset depicted in Figure 1 (which shall not be read as a lattice diagram).

Example 8.11. Let $\mathcal{B}$ be the (complete) residuated lattice with universe $\{0,1\}$ such that $0 < 1$ and whose residuated operation coincides with the infimum. Clearly the neutral element of $\mathcal{B}$ is 1. Moreover, let $\langle G,\wedge,\vee\rangle$ be the two element (complete) lattice with universe $\{\bot,\top\}$ such that $\bot < \top$. Then let $\oplus\colon G\times B \to G$ be the function defined for every $g \in G$ and $a \in B$

as

$$g \oplus a = \begin{cases} \top & \text{if } g = \top \text{ and } a = 1 \\ \bot & \text{otherwise.} \end{cases}$$

Moreover, let $\otimes\colon B \times G \to G$ be the function defined as $a \otimes g = g \oplus a$ for every $g \in G$ and $a \in B$. It is easy to prove that $\mathcal{G} = \langle G, \wedge, \vee, \oplus, \otimes\rangle$ is a $\langle \mathcal{B}, \mathcal{B}\rangle$-compositional lattice.

We will make use of several properties of the categories $\mathcal{B}$-Mod and $\mathcal{G}$-Fra. More precisely the following facts hold in $\mathcal{B}$-Mod:

Fact 1. *Let $\langle R, \wedge, \vee\rangle$ be a complete lattice and $\star$ a function from $B \times R$ to R. $\langle R, \wedge, \vee, \star\rangle$ is a $\mathcal{B}$-module if and only if $1 \star x = x$ and $0 \star x = \bot^R$ for every $x \in R$.*

Fact 2. *Let $\mathbb{R}$ and $\mathbb{S}$ be $\mathcal{B}$-modules and τ a function from R to S. τ is an $\mathcal{B}$-module morphism if and only if it is residuated.*

Fact 3. *$\mathbb{1}$ and $\mathbb{B}$ are the unique cyclic and projective $\mathcal{B}$-modules.*

Fact 4. *For every pair of $\mathcal{B}$-modules $\mathbb{R}$ and $\mathbb{S}$ the constant function $\kappa\colon R \to S$, which sends every element of R to the bottom $\bot^{\mathbb{S}}$, is a $\mathcal{B}$-module morphism.*

Fact 5. *$\mathbb{1}$ is an initial object: For every $\mathcal{B}$-module $\mathbb{R}$, the constant function $\kappa\colon \mathbb{1} \to \mathbb{R}$ is the unique $\mathcal{B}$-module morphism from 1 to R.*

The proof of Facts 1, 2 and 3 is an easy exercise and depends on the simple structure of the complete residuated lattice $\mathcal{B}$, whereas Facts 4 and 5 hold in every category of modules over a complete residuated lattice. Moreover, we will use the following properties of $\mathcal{G}$-Fra:

Fact 6. *Let $\mathbb{R}$ and $\mathbb{S}$ be $\mathcal{B}$-modules and $\langle\cdot,\cdot\rangle$ a function from $H \times R$ to S. $\langle \mathbb{R}, \mathbb{S}, \langle\cdot,\cdot\rangle\rangle$ is a $\mathcal{G}$-evaluational frame if and only if $\langle\cdot,\cdot\rangle$ is residuated.*

Fact 7. *Let $\boldsymbol{R}$ and $\boldsymbol{S}$ be $\mathcal{G}$-evaluational frames, τ a function from R to S and τ' a function from R' to S'. $\langle \tau, \tau'\rangle$ is a frame morphism if and only if τ and τ' are residuated and $\tau'\langle \top, x\rangle_{\boldsymbol{R}} = \langle \top, \tau x\rangle_{\boldsymbol{S}}$ for every $x \in R$.*

Facts 6 and 7 depend on the simple structure of $\mathcal{G}$ and on Facts 1 and 2. Now observe that from Theorem 7.5, condition 1 of Corollary 7.2, Theorem 3.4, Corollary 8.3 and Corollary 8.7 it follows that the inclusion relations depicted in Fig. 1 hold. Therefore it only remains to prove that these classes are different.

In order to do this, let us state some more involved claim about $\mathcal{G}$-Fra. Given a $\mathcal{B}$-module $\mathbb{R}$, we put $\mathcal{D}(\mathbb{R}) \coloneqq \langle \mathbb{R}, \mathbb{R}, \langle\cdot,\cdot\rangle_{\mathcal{D}(\mathbb{R})}\rangle$, where $\langle\cdot,\cdot\rangle_{\mathcal{D}(\mathbb{R})}$ is the function from $H \times R$ to R such that $\langle \bot, x\rangle_{\mathcal{D}(\mathbb{R})} = \bot^{\mathbb{R}}$ and $\langle \top, x\rangle_{\mathcal{D}(\mathbb{R})} = x$ for every $x \in R$. It is easy to prove that $\mathcal{D}(\mathbb{R})$ is a $\mathcal{G}$-evaluational frame.

Claim 1. *Let $\mathbb{R}$ be a $\mathcal{B}$-module. $\mathcal{I}(\mathbb{R})$ is isomorphic to $\mathcal{D}(\mathbb{R})$.*

Proof. We begin by proving that $\mathcal{D}(\mathbb{R})$ has the INT. Let $\tau\colon \mathbb{R} \to \mathbb{S}$ be a $\mathcal{B}$-module morphism and $\boldsymbol{S}$ a $\mathcal{G}$-evaluational frame. We consider the function $\tau'\colon \mathbb{R}' \to \mathbb{S}'$ defined as $\tau'(x) \coloneqq \langle \top, \tau x\rangle_{\boldsymbol{S}}$ for every $x \in R$. By Facts 2 and 7 we conclude that $\langle \tau, \tau' \rangle \colon \mathcal{D}(\mathbb{R}) \to \boldsymbol{S}$ is a frame morphism, since τ' is residuated and $\tau' \langle \top, x \rangle_{\mathcal{D}(\mathbb{R})} = \tau' x = \langle \top, \tau x \rangle_{\boldsymbol{S}}$ for every $x \in R$. Hence $\mathcal{D}(\mathbb{R})$ has the WINT. Since $\langle \cdot, \cdot \rangle_{\mathcal{D}(\mathbb{R})}$ is surjective, this implies that $\mathcal{D}(\mathbb{R})$ has the INT too. From Lemma 6.3, it follows that $\mathcal{I}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ are isomorphic. ⊠

Claim 2. *Frame epimorphisms are frame morphisms which are pairs of surjective mappings.*

Proof. It is clear that if a frame morphism is a pair of surjective mappings, then it is a frame epimorphism. Then we turn to prove the other direction. Let $\boldsymbol{\tau} \colon \boldsymbol{R} \to \boldsymbol{S}$ be a frame epimorphism. From Lemma 7.1, it follows that τ is surjective.

It only remains to prove that τ' is surjective too. We reason towards a contradiction: suppose that this is not the case. Since $\mathcal{B}$-module epimorphisms are surjective, this is to say that τ' is not a $\mathcal{B}$-module epimorphism. Then there are two different $\mathcal{B}$-module morphisms $\rho_1, \rho_2 \colon \mathbb{S}' \to \mathbb{T}$ such that $\rho_1 \tau' = \rho_2 \tau'$. Put $\boldsymbol{T} \coloneqq \langle \tau'[\mathbb{R}], \mathbb{T}, \langle \cdot, \cdot \rangle_{\boldsymbol{T}} \rangle$, where $\langle \cdot, \cdot \rangle_{\boldsymbol{T}} \colon H \times \tau'[R'] \to T$ is the function defined as $\langle \top, x \rangle_{\boldsymbol{T}} = \rho_1(x)$ and $\langle \bot, x \rangle_{\boldsymbol{T}} = \bot^{\mathbb{T}}$ for every $x \in \tau'[R']$. By Fact 6, we know that $\boldsymbol{T}$ is a $\mathcal{G}$-evaluational frame.

We will prove that $\langle \langle \top, \cdot \rangle_{\boldsymbol{S}}, \rho_i \rangle \colon \boldsymbol{S} \to \boldsymbol{T}$ is a frame morphism, for every $i \in \{1, 2\}$. Let $i \in I$. Then observe that the function $\langle \top, \cdot \rangle_{\boldsymbol{S}} \colon \mathbb{S} \to \tau'[\mathbb{R}]$ is a well-defined, since $\boldsymbol{\tau}$ is a $\mathcal{B}$-module morphism and τ is surjective. Moreover it is residuated, therefore by Fact 2 we conclude that it is a $\mathcal{B}$-module morphism. Then observe that for every $x \in S$, we have $\rho_i \langle \top, x \rangle_{\boldsymbol{S}} = \rho_1 \langle \top, x \rangle_{\boldsymbol{S}} = \langle \top, \langle \top, x \rangle_{\boldsymbol{S}} \rangle_{\boldsymbol{T}}$, since $\langle \top, x \rangle_{\boldsymbol{S}} \in \tau'[R']$ and $\rho_1 \tau' = \rho_2 \tau'$. By Fact 7, this yields that $\langle \langle \top, \cdot \rangle_{\boldsymbol{S}}, \rho_i \rangle$ is a frame morphism. Since $\langle \langle \top, \cdot \rangle_{\boldsymbol{S}}, \rho_1 \rangle \boldsymbol{\tau} = \langle \langle \top, \cdot \rangle_{\boldsymbol{S}}, \rho_2 \rangle \boldsymbol{\tau}$ and clearly $\langle \langle \top, \cdot \rangle_{\boldsymbol{S}}, \rho_1 \rangle \neq \langle \langle \top, \cdot \rangle_{\boldsymbol{S}}, \rho_2 \rangle$, we conclude that $\boldsymbol{\tau}$ is not a frame epimorphism against the assumption. ⊠

Claim 3. *Let $\mathbb{R}$ be a $\mathcal{B}$-module. $\mathbb{R}$ is projective if and only if $\langle R, \wedge, \vee \rangle$ is completely distributive.*

Proof. Let $\mathcal{C}$ be the category of complete lattices with residuated mappings as arrows. Given any $\mathcal{B}$-module $\mathbb{R}$, we put $\mathcal{F}(\mathbb{R}) = \langle R, \wedge, \vee \rangle$ and, given a $\mathcal{B}$-module morphism $\tau \colon \mathbb{R} \to \mathbb{S}$, we let $\mathcal{F}(\tau) \colon \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{S})$ be the function from R to S defined as $\mathcal{F}(\tau)(x) = \tau(x)$ for every $x \in R$. Clearly $\mathcal{F} \colon \mathcal{B}$-$\mathsf{Mod} \to \mathcal{C}$ is a functor. From Facts 1 and 2 it follows that the categories $\mathcal{B}$-Mod and $\mathcal{C}$ are isomorphic via $\mathcal{F}$. In [7] the projective objects of $\mathcal{C}$ are characterised as the completely distributive lattices, therefore we are done. ⊠

We begin by showing that there is a $\mathcal{G}$-evaluational frame without the WINT. Suppose towards a contradiction that this is not the case. Then consider the $\mathcal{G}$-evaluational frame $\mathcal{T}(\mathbb{B})$, defined at the beginning of Section 7. Observe that $\mathcal{S}\mathcal{T}(\mathbb{B}) = \mathbb{B} = \mathcal{S}(\boldsymbol{G})$. From the fact that $\mathcal{T}(\mathbb{B})$ has the WINT, it follows that there is a frame morphism $\boldsymbol{\tau} \colon \mathcal{T}(\mathbb{B}) \to \boldsymbol{G}$ such that $\mathcal{S}(\boldsymbol{\tau}) = 1_{\mathbb{B}}$. By Fact 5, we know that $\boldsymbol{\tau} = \langle 1_{\mathbb{B}}, k \rangle$. But this yields that

$\perp = \kappa\langle\top, 1\rangle_{\mathcal{T}(\mathbb{B})} = \tau'\langle\top, 1\rangle_{\mathcal{T}(\mathbb{B})} = \langle\top, \tau 1\rangle_{\boldsymbol{G}} = \langle\top, 1_{\mathbb{B}}1\rangle_{\boldsymbol{G}} = \langle\top, 1\rangle_{\boldsymbol{G}} = \top$, which is a contradiction.

Now we construct a $\mathcal{G}$-evaluational frame with the INT, which is not projective. Consider any non-distributive complete lattice $\langle R, \wedge, \vee\rangle$. By Fact 1, we can consider the unique $\mathcal{B}$-module $\mathbb{R}$ whose lattice component is $\langle R, \wedge, \vee\rangle$. We know that $\mathcal{I}(\mathbb{R})$ has the INT. Moreover from Claim 3, it follows that $\mathbb{R}$ is not projective. By condition 1 of Corollary 7.2, we conclude that $\mathcal{I}(\mathbb{R})$ is not projective.

Then we turn to construct a projective $\mathcal{G}$-evaluational frame which does not have the INT. Put $\boldsymbol{R} := \langle \mathbb{1}, \mathbb{B}, \langle\cdot,\cdot\rangle_{\boldsymbol{R}}\rangle$, where $\langle\cdot,\cdot\rangle_{\boldsymbol{R}}$ is the constant function which sends every element to 0. From Claim 1, it follows that $\boldsymbol{R}$ does not have the INT. Then we turn to prove that $\boldsymbol{R}$ is projective. Let $\boldsymbol{\tau}\colon \boldsymbol{S} \to \boldsymbol{T}$ and $\boldsymbol{\rho}\colon \boldsymbol{R} \to \boldsymbol{T}$ be two frame morphisms such that $\boldsymbol{\tau}$ is epi. By Claim 2, we know that τ' is surjective. Then there is $y \in S'$ such that $\tau' y = \rho' 1$. We put $\boldsymbol{\sigma} := \langle \kappa, \sigma'\rangle$, where $\kappa\colon \mathbb{1} \to \mathbb{S}$ is defined in Fact 4 and $\sigma'\colon \mathbb{B} \to \mathbb{S}'$ is the function such that $\sigma' 1 = y$ and $\sigma' 0 = \perp^{\mathbb{S}'}$. Since σ' is residuated and $\sigma'\langle\top, x\rangle_{\boldsymbol{R}} = \sigma'\perp^{\mathbb{R}'} = \perp^{\mathbb{S}'} = \langle\top, \perp^{\mathbb{S}}\rangle_{\boldsymbol{S}} = \langle\top, \kappa x\rangle_{\boldsymbol{S}}$ for every $x \in \{1\}$, from Facts 2 and 7 it follows that $\boldsymbol{\sigma}$ is a frame morphism. We conclude that $\boldsymbol{R}$ is projective, since $\boldsymbol{\tau}\boldsymbol{\sigma} = \boldsymbol{\rho}$.

Now we construct an example of $\mathcal{G}$-evaluational frame with the FAL, which is not the coproduct of a family of cyclic and projective objects. By Fact 1, we can consider the $\mathcal{B}$-module $\mathbb{R}$ whose lattice component is the three element chain. We know that $\mathcal{I}(\mathbb{R})$ has the INT. Moreover, from Claim 3 it follows that $\mathbb{R}$ is projective. By Theorem 3.4, we conclude that $\mathbb{R}$ has the REP. Hence $\mathcal{I}(\mathbb{R})$ has the FAL. Moreover, from Fact 3, it follows that $\mathbb{R}$ is not the coproduct of cyclic and projective $\mathcal{B}$-modules. Together with condition 1 of Corollary 7.2, this implies that $\mathcal{I}(\mathbb{R})$ is not the coproduct of a family of cyclic and projective $\mathcal{G}$-evaluational frames.

It only remains to construct a coproduct of cyclic and projective $\mathcal{G}$-evaluational frames, which is not cyclic and projective. This is very easy. From condition (ii) of Theorem 8.9, it follows that $\boldsymbol{G}$ is cyclic and projective. But the coproduct $\coprod_{i\in\{0,1\}} \boldsymbol{G}_i$ is not cyclic and projective, by Fact 3. ⊠

9. Evaluational sets

In Example 8.11 we showed that in general is not true that every evaluational frame with the FAL is the coproduct of a family of cyclic and projective objects. Nevertheless, we would like to conclude this paper by proving that this is the case if we restrict to a more concrete class of evaluational frames, namely the one of evaluational frames arising from structures which play a role analogous the one of $\mathcal{M}$-sets in the case of the Syntactic Isomorphism Theorem. In order to do this, let us introduce some new concept.

Definition 9.1. Let $\mathcal{M}$ and $\mathcal{N}$ be two monoids. Then $\mathcal{K} = \langle K, \oplus, \otimes\rangle$ is a $\langle\mathcal{M}, \mathcal{N}\rangle$-***compositional monoid*** if the following conditions hold:

(i) $\langle K, \oplus\rangle$ is a right $\mathcal{M}$-set;

(ii) $\langle K, \otimes \rangle$ is a left $\mathcal{N}$-set; and
(iii) $n \otimes (k \oplus m) = (n \otimes k) \oplus m$ for every $n \in N$, $k \in K$ and $m \in M$.

Given a compositional monoid $\mathcal{K}$, we put

$$\mathcal{P}(\mathcal{K}) := \langle \mathcal{P}(K), \cap, \cup, \mathcal{P}(\oplus), \mathcal{P}(\otimes) \rangle$$

where $\mathcal{P}(\oplus)$ and $\mathcal{P}(\otimes)$ are just the functions $\oplus$ and $\otimes$, respectively, lifted to the power sets.

Lemma 9.2. *If $\mathcal{K}$ is a $\langle \mathcal{M}, \mathcal{N} \rangle$-compositional monoid, then $\mathcal{P}(\mathcal{K})$ is a $\langle \mathcal{P}(\mathcal{M}), \mathcal{P}(\mathcal{N}) \rangle$-compositional lattice.*

It is worth remarking that the example which motivated the introduction of compositional lattices can be seen as particular cases of this construction, as we record below.

Example 9.3. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two algebras. Then consider the composition functions

$$\begin{aligned} \oplus &\colon \mathcal{H}om(\boldsymbol{A}, \boldsymbol{B}) \times \mathcal{E}nd(\boldsymbol{A}) \to \mathcal{H}om(\boldsymbol{A}, \boldsymbol{B}) \\ \otimes &\colon \mathcal{E}nd(\boldsymbol{B}) \times \mathcal{H}om(\boldsymbol{A}, \boldsymbol{B}) \to \mathcal{H}om(\boldsymbol{A}, \boldsymbol{B}). \end{aligned}$$

It is easy to prove that $\mathcal{K}(\boldsymbol{A}, \boldsymbol{B}) := \langle \mathcal{H}om(\boldsymbol{A}, \boldsymbol{B}), \oplus, \otimes \rangle$ is a $\langle \mathcal{M}(\boldsymbol{A}), \mathcal{M}(\boldsymbol{B}) \rangle$-compositional monoid. Moreover $\mathcal{P}(\mathcal{K}(\boldsymbol{A}, \boldsymbol{B}))$ coincides with the $\langle \mathcal{P}(\mathcal{M}(\boldsymbol{A})), \mathcal{P}(\mathcal{M}(\boldsymbol{B})) \rangle$-compositional lattice $\mathcal{H}(\boldsymbol{A}, \boldsymbol{B})$. ⊠

Let us now introduce a new kind of structures that, when lifted to the power set, give rise to $\mathcal{P}(\mathcal{K})$-evaluational frames.

Definition 9.4. Let $\mathcal{K}$ be a $\langle \mathcal{M}, \mathcal{N} \rangle$-compositional monoid. $\mathbb{R} = \langle \mathsf{R}, \mathsf{R}', \langle \cdot, \cdot \rangle_{\mathbb{R}} \rangle$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}} \colon K \times R \to R'$, is a $\mathcal{K}$-***evaluational set*** if the following conditions hold:

(i) R is a left $\mathcal{M}$-set;
(ii) R' is a left $\mathcal{N}$-set;
(iii) $\langle k \oplus m, x \rangle_{\mathbb{R}} = \langle k, m \star_{\mathsf{R}} x \rangle_{\mathbb{R}}$ for every $k \in K$, $m \in M$ and $x \in R$; and
(iv) $\langle n \otimes k, x \rangle_{\mathbb{R}} = n \star_{\mathsf{R}'} \langle k, x \rangle_{\mathbb{R}}$ for every $n \in N$, $k \in K$ and $x \in R$.

Given a $\mathcal{K}$-evaluated set $\mathbb{R}$, we put

$$\mathcal{P}(\mathbb{R}) := \langle \mathcal{P}(\mathsf{R}), \mathcal{P}(\mathsf{R}'), \mathcal{P}(\langle \cdot, \cdot \rangle_{\mathbb{R}}) \rangle$$

where $\mathcal{P}(\langle \cdot, \cdot \rangle_{\mathbb{R}})$ is obtained lifting to the power sets the function $\langle \cdot, \cdot \rangle_{\mathbb{R}}$.

Lemma 9.5. *If $\mathbb{R}$ is a $\mathcal{K}$-evaluational set, then $\mathcal{P}(\mathbb{R})$ is a $\mathcal{P}(\mathcal{K})$-evaluational frame.*

It is easy to prove that our motivating examples, coming from the field of logic, are in fact particular cases of this construction.

Example 9.6. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two algebras. Then consider the functions

$$\begin{aligned}\langle\cdot,\cdot\rangle_{\mathbb{R}(\mathsf{A},\mathsf{B})} &\colon \mathcal{H}om(\boldsymbol{A},\boldsymbol{B}) \times A \to B\\ \langle\cdot,\cdot\rangle_{\mathbb{E}\mathrm{q}(\mathsf{A},\mathsf{B})} &\colon \mathcal{H}om(\boldsymbol{A},\boldsymbol{B}) \times \mathrm{Eq}(A) \to \mathrm{Eq}(B)\\ \langle\cdot,\cdot\rangle_{\mathbb{S}\mathrm{eq}(\mathsf{A},\mathsf{B})} &\colon \mathcal{H}om(\boldsymbol{A},\boldsymbol{B}) \times \mathrm{Seq}(A) \to \mathrm{Seq}(B)\end{aligned}$$

defined as $\langle h,a\rangle_{\mathbb{R}(\mathsf{A},\mathsf{B})} \coloneqq ha$, $\langle h,\langle a,c\rangle\rangle_{\mathbb{E}\mathrm{q}(\mathsf{A},\mathsf{B})} \coloneqq \langle ha,hc\rangle$ and $\langle h,\langle \overline{a},\overline{c}\rangle\rangle_{\mathbb{S}\mathrm{eq}(\mathsf{A},\mathsf{B})} \coloneqq \langle h\overline{a},h\overline{c}\rangle$ for every $h\in\mathcal{H}om(\boldsymbol{A},\boldsymbol{B})$, $a\in A$, $\langle a,c\rangle\in\mathrm{Eq}(A)$ and $\langle\overline{a},\overline{c}\rangle\in\mathrm{Seq}(A)$. It is easy to prove that the structures

$$\begin{aligned}\mathbb{R}(\mathsf{A},\mathsf{B}) &\coloneqq \langle \mathsf{A},\mathsf{B},\langle\cdot,\cdot\rangle_{\mathbb{R}(\mathsf{A},\mathsf{B})}\rangle\\ \mathbb{E}\mathrm{q}(\mathsf{A},\mathsf{B}) &\coloneqq \langle \mathrm{Eq}(\mathsf{A}),\mathrm{Eq}(\mathsf{B}),\langle\cdot,\cdot\rangle_{\mathbb{E}\mathrm{q}(\mathsf{A},\mathsf{B})}\rangle\\ \mathbb{S}\mathrm{eq}(\mathsf{A},\mathsf{B}) &\coloneqq \langle \mathrm{Seq}(\mathsf{A}),\mathrm{Seq}(\mathsf{B}),\langle\cdot,\cdot\rangle_{\mathbb{S}\mathrm{eq}(\mathsf{A},\mathsf{B})}\rangle\end{aligned}$$

are $\mathcal{K}(\boldsymbol{A},\boldsymbol{B})$-evaluational sets. Moreover $\mathcal{P}(\mathbb{R}(\mathsf{A},\mathsf{B}))$, $\mathcal{P}(\mathbb{E}\mathrm{q}(\mathsf{A},\mathsf{B}))$ and $\mathcal{P}(\mathbb{S}\mathrm{eq}(\mathsf{A},\mathsf{B}))$ coincide respectively with the $\mathcal{H}(\boldsymbol{A},\boldsymbol{B})$-evaluational frames $\boldsymbol{R}(\mathbb{A},\mathbb{B})$, $\boldsymbol{Eq}(\mathbb{A},\mathbb{B})$ and $\boldsymbol{Seq}(\mathbb{A},\mathbb{B})$. ⊠

We are now ready to prove our desired characterization of evaluational frames arising from evaluational sets, which enjoy the full algebraizability property. This result can be read as stating that these evaluational frames preserve a feature typical of the behaviour of sets of formulas equipped with algebraic models, in the sense that they can be cut into slices each of which enjoys a general kind of variable (compare with Theorem 3.6). We believe that this is a nice way to come full circle in the analysis of the generalisation of the Syntactic and the Semantic Isomorphism Theorems, since it allows us to recover the logical flavour of Blok and Pigozzi's theory of algebraizability within the context of evaluational frames.

Theorem 9.7. *Let $\mathbb{R}$ be an $\mathcal{K}$-evaluational set. $\mathcal{P}(\mathbb{R})$ has the* FAL *if and only if it is a coproduct of cyclic and projective $\mathcal{P}(\mathcal{K})$-evaluational frames.*

Proof. Let $\mathbb{R}$ be an $\mathcal{K}$-evaluational set. From Fig 1, we know that if $\mathcal{P}(\mathbb{R})$ is a coproduct of cyclic and projective $\mathcal{P}(\mathcal{K})$-evaluational frames, then it has the FAL.

Then we turn to prove the other direction: suppose that $\mathcal{P}(\mathbb{R})$ has the FAL. From Lemma 6.3, it follows that $\mathcal{P}(\mathbb{R})$ is isomorphic to $\mathcal{I}(\mathcal{P}(\mathsf{R}))$. Therefore it will be enough to prove that $\mathcal{I}(\mathcal{P}(\mathsf{R}))$ is a coproduct of cyclic and projective $\mathcal{P}(\mathcal{K})$-evaluational frames. Now, from the assumption we know that $\mathcal{P}(\mathsf{R})$ has the REP. From Theorem 3.6, it follows that $\mathcal{P}(\mathsf{R})$ is the coproduct of a family $\{\mathbb{R}_i\}_{i\in I}$ of cyclic and projective $\mathcal{P}(\mathcal{M})$-modules. Moreover, from Theorem 6.10 we know that $\mathcal{I}$ is a left adjoint functor and therefore that it preserves colimits. This yields, in particular, that $\mathcal{I}(\mathcal{P}(\mathsf{R}))$ is a coproduct of the family of $\mathcal{P}(\mathcal{K})$-evaluational frames $\{\mathcal{I}(\mathbb{R}_i)\}_{i\in I}$. Moreover, by condition 2 of Corollary 7.2 and Lemma 8.8, we conclude that $\mathcal{I}(\mathbb{R}_i)$ is cyclic and projective for every $i\in I$. ⊠

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