M-SETS AND THE REPRESENTATION PROBLEM

JOSEP MARIA FONT AND TOMMASO MORASCHINI

Abstract. The "representation problem" in abstract algebraic logic is that of finding necessary and sufficient conditions for a structure, on a well defined abstract framework, to have the following property: that for every structural closure operator on it, every structural embedding of the expanded lattice of its closed sets into that of the closed sets of another structural closure operator on another similar structure is induced by a structural transformer between the base structures. This question arose from Blok and Jónsson abstract analysis of one of Blok and Pigozzis's characterizations of algebraizable logics. The problem, which was later on reformulated independently by Gil-Férez and by Galatos and Tsinakis, was solved by Galatos and Tsinakis in the more abstract framework of the category of modules over a complete residuated lattice, and by Galatos and Gil-Férez in the even more abstract setting of modules over a quantaloid. We solve the representation problem in Blok and Jónsson's original context of M-sets, where M is a monoid, and characterise the corresponding M-sets both in categorical terms and in terms of their inner structure, using the notions of a graded M-set and a generalized variable introduced by Gil-Férez.

Introduction

Blok and Pigozzi's celebrated notion of algebraizability of a deductive system [3] was given its first truly abstract treatment in Blok and Jónsson's [2], in the framework of $\mathcal{M}$-sets, where $\mathcal{M}$ is a monoid (corresponding to the substitutions) acting on a set (corresponding to the language on which the deductive system is defined, such as sentential formulas, equations, etc.). Two structural closure operators on two different $\mathcal{M}$-sets are then defined to be equivalent when there is an isomorphism between the respective lattices of closed sets expanded by the operations induced by the actions of the monoid. In the second part of the paper, Blok and Jónsson find a sufficient condition for $\mathcal{M}$-sets to have the property that all such equivalences are induced by a pair of mutually inverse structural transformers between the $\mathcal{M}$-sets themselves. This is what happens in the case of algebraizability of ordinary deductive systems. However, this condition does not encompass the case of the algebraization of Gentzen systems introduced in [15], neither that of more complicated formalisms such as hypersequents.

A finer analysis was then performed, independently, by Gil-Férez [12] and by Galatos and Tsinakis [11], and finally by Galatos and Gil-Férez [9].

Date: December 14, 2013.

These papers show that the bulk of technical work can be done by restricting the problem to one of its two symmetrical halves. Thus, even if for instance [9] uses the expression "isomorphism problem", we prefer to speak of the "representation problem", and accordingly say that an $\mathcal{M}$-set "has the representation property" (the REP) when every representation (i.e., structural embedding) of the expanded lattice of the closed sets of a structural closure operator over it into those of another one over another $\mathcal{M}$-set is induced by a structural transformer from the first $\mathcal{M}$-set into the second one. At the level of $\mathcal{M}$-sets, a more sophisticated sufficient condition, encompassing not just Gentzen systems but all the known examples coming from logic, but again not a necessary one, was found by Gil-Férez [13, Theorem 34].

The problem was reformulated and solved in [11, 9] in the considerably more abstract frameworks of modules over complete residuated lattices (quantales) and of modules over quantaloids, by adopting a categorical approach. Theorem 5.1 of [11] proves that, in the category of modules over a fixed complete residuated lattice with structural residuated maps as arrows, the modules that have the REP are exactly the projective ones. This may help in solving the problem in its original formulation, because if $\mathbb{R}$ is an $\mathcal{M}$-set, then lifting everything to power sets produces a complete residuated lattice $\mathcal{P}(\mathcal{M})$ and a $\mathcal{P}(\mathcal{M})$-module $\mathcal{P}(\mathbb{R})$; however, it is not immediately obvious that $\mathbb{R}$ has the REP as an $\mathcal{M}$-set if and only if $\mathcal{P}(\mathbb{R})$ has the REP as a $\mathcal{P}(\mathcal{M})$-module.

The aim of this paper is to prove the just mentioned equivalence and to obtain characterizations of the $\mathcal{M}$-sets having the REP, especially characterizations intrinsic to the framework of $\mathcal{M}$-sets rather than coming from the one of $\mathcal{P}(\mathcal{M})$-modules. The structure of the paper is as follows.

After a section containig a few points of the residuation theory and category theory we will make use of during our analysis, Section 1 reviews the work done on the problem, from its logical formulation to the latest developments using category-theoretic tools[1]. The technical contents of this section allows to introduce almost all the already existing notions used in the paper, in particular those of a "graded" $\mathcal{M}$-set and a "generalised variable", due to Gil-Férez [12, 13].

In Section 2 we introduce a suitable category of $\mathcal{M}$-sets, where the arrows between two $\mathcal{M}$-sets are functions between their power sets that commute with "substitutions" (the maps induced by the actions of the monoid). We call these arrows "structural transformers" because they are the natural abstraction of the transformers between formulas and equations that appear in the theory of algebraizable logics. In one of the main results of the paper (Theorem 18) we show that the $\mathcal{M}$-sets with the REP are those that are onto-projective in this category and that these coincide with those such that

[1] We do not need the extremely general framework of modules over quantaloids of [9]; we stick to some results on modules over quantales appearing in the unpublished [12], quoted here with permission (and which have homologue results in [9]).

$\mathcal{P}(\mathbb{R})$ has the REP as a $\mathcal{P}(\mathcal{M})$-module (hence, with those such that $\mathcal{P}(\mathbb{R})$ is projective in the category of $\mathcal{P}(\mathcal{M})$-modules).

Therefore the task of characterising $\mathcal{M}$-sets with the REP reduces to that of characterising onto-projective $\mathcal{M}$-sets. This is what we do in Section 3: we first develop a decomposition technique and prove (Theorem 25) that an $\mathcal{M}$-set is onto-projective if and only if it can be built by "pasting together" (technically: is isomorphic to a coproduct of) retracts of the monoid $\mathcal{M}$ considered as an $\mathcal{M}$-set. Then we characterise such retracts as the $\mathcal{M}$-sets that have a generalised variable. This fact leads naturally to the introduction of the notion of a generalised graded variable and to the characterisation of $\mathcal{M}$-sets with the REP as the ones which can be equipped with a graduation which enjoys a generalised graded variable (Theorem 28). These two results are arguably the main and most innovative in the paper.

The paper ends by closing two additional issues. First, we show that our solution to the representation problem is intrinsic to the context of $\mathcal{M}$-sets. To this end, we provide a counterexample in the category of $\mathcal{P}(\mathcal{N})$-modules, with $\mathcal{N}$ the trivial monoid, where the expected natural extension of the result does not hold. And second, we prove (Theorem 30) that in the original setting of [2] (with an isomorphism rather than a representation), the fact that the two $\mathcal{M}$-sets involved have both the REP guarantees that the two structural transformers inducing the isomorphism and its inverse are not only mutually inverse modulo the closure operators, but they are such that their residuals coincide with the isomorphism and its inverse on the closed sets. In this way, the original situation of Blok and Jónsson's abstraction of algebraizability is fully restored.

Preliminaries

In this section we fix some terminology and notation in residuation theory and category theory we will make use of; for standard background on these topics we refer the reader respectively to [10] and [1]. For information on abstract algebraic logic we refer the reader to [7, 8]. The symbol "$\mathcal{P}(\cdot)$" always denotes the power set construction. We write $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$ ... for posets, in particular for lattices. In order to reduce the notation burden, in general we use the same symbol for a structure and for its universe, unless we consider several structures over the same universe; if necessary, universes of structures are denoted by the same letter in italic typeface (i.e., Q, R, S in the previous examples). A **closure operator** $\gamma\colon \mathcal{R} \to \mathcal{R}$ over a poset is a monotone function such that $x \leqslant \gamma x = \gamma\gamma x$ for all $x \in R$. When γ is a closure operator over a complete lattice $\mathcal{R}$ we denote by $\mathcal{R}_\gamma = \langle \gamma[R], \bigwedge, \gamma\bigvee \rangle$ the complete lattice of its fixed points.

A map $f\colon \mathcal{Q} \to \mathcal{R}$ is **residuated** when there is another map $f^+\colon \mathcal{R} \to \mathcal{Q}$ such that for all $x \in Q$, $y \in R$ the following holds:

$$fx \leqslant y \iff x \leqslant f^+ y.$$

In this case we say that $\langle f, f^+\rangle$ forms a **residuated pair** and that f^+ is the **residuum** of f. It can be easily proved that if $f\colon \mathcal{Q} \to \mathcal{R}$ is residuated, then its residuum f^+ is uniquely determined and is defined as follows: for every $y \in R$,

$$f^+y = \max\{x \in Q : fx \leqslant y\}.$$

Since we will deal only with complete lattices we will make use of a nice characterisation result, namely that residuated maps between complete lattices coincide with the functions that preserve arbitrary suprema. This means that in the particular case of power set lattices, a map $f\colon \mathcal{P}(Q) \to \mathcal{P}(R)$ is residuated if and only if it is determined by its restriction to unitary subsets, i.e., $fX = \bigcup\big\{f\{x\} : x \in X\big\}$ for all $X \subseteq Q$; then the residuum can be defined as $f^+Y = \big\{x \in R : f\{x\} \subseteq Y\big\}$ for all $Y \subseteq R$. In the (very common) case where f has been actually defined from a point function from Q to R, then the residuum f^+ is just the ordinary "inverse image" function.

The notion of residuation can be applied to binary functions by fixing one argument; the resulting notions will provide a fundamental tool in our analysis of structures abstracting languages of a certain type. More precisely, we say that a map $\cdot\colon \mathcal{Q}\times\mathcal{R} \to \mathcal{S}$ is **residuated** when there are two functions $\backslash.\colon \mathcal{Q}\times\mathcal{S} \to \mathcal{R}$ and $/.\colon \mathcal{S}\times\mathcal{R} \to \mathcal{Q}$ such that for every $x \in Q, y \in R$ and $z \in S$, the following holds:

$$x \cdot y \leqslant z \iff y \leqslant x \backslash.\, z \iff x \leqslant z \,/.\, y.$$

In this case we say that $\backslash.$ and $/.$ are respectively the **left** and **right residuum** of $\cdot$. The residua are uniquely determined and can be defined as follows: for every $x \in Q, y \in R$ and $z \in S$

$$x \backslash.\, z = \max\{r \in R : x \cdot r \leqslant z\} \qquad z \,/.\, y = \max\{q \in Q : q \cdot y \leqslant z\}.$$

Also in this case we get a nicer characterization for complete structures: if $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ are complete lattices, then residuated maps $\cdot\colon \mathcal{Q}\times\mathcal{R} \to \mathcal{S}$ coincide with functions that preserve arbitrary suprema in both coordinates.

As to category theory, we need just to review three basic notions. Given a category $\mathcal{C}$, an object A is **projective** in $\mathcal{C}$ when for every pair of arrows $f\colon B \to C$ and $g\colon A \to C$ such that f is an epimorphism, there is an arrow $h\colon A \to B$ which makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, dashed, "\exists h"] \arrow[dr, "g"'] & B \arrow[d, two heads, "f"] \\ & C \end{tikzcd}$$

When arrows in a category are real set functions we can talk about **onto-projective** objects by requiring the f in the definition to be surjective.

The second notion we recall is that of a coproduct. Given a category $\mathcal{C}$, an object $\coprod_{i\in I} A_i$ is the **coproduct** of a family of objects $\{A_i\}_{i\in I}$ when there is a family of arrows $\pi_i\colon A_i \to \coprod_{i\in I} A_i$ for every $i \in I$ such that for every other family of arrows $f_i\colon A_i \to B$ for every $i \in I$ there is a unique arrow $g\colon \coprod_{i\in I} A_i \to B$ which makes the following diagram commute for every $i \in I$:

$$\begin{tikzcd}
& A_i \arrow[dl, "\pi_i"'] \arrow[dr, "f_i"] & \\
\coprod_{i\in I} A_i \arrow[rr, dashed, "\exists! g"] & & B
\end{tikzcd}$$

Finally we have retractions. Given a category $\mathcal{C}$, we say that an arrow $f\colon A \to B$ is a **retraction** if there is an arrow $g\colon B \to A$ such that $f \circ g = id_B$. If there is a retraction $f\colon A \to B$ we say that B is a **retract** of A.

1. The Representation Problem

Blok and Pigozzi introduced in their monograph [3] the theory of algebraizable logics. Its key point is the usage of what are called, in the present terminology, structural transformers to establish a deductive equivalence between the consequence of the logic and the relative equational consequence of a class of algebras, called the equivalent algebraic semantics of the logic.

More precisely, let Fm be the algebra of formulas over some fixed but arbitrary algebraic similarity type, and denote by Eq the set of equations over this language; equations are just pairs of formulas but are written in the more suggestive notation $\alpha \approx \beta$. A map $\tau\colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ is a **structural transformer** (from formulas to equations) when there is a set $E(x)$ of equations in a single variable x such that for all $\Gamma \subseteq Fm$,

$$\tau(\Gamma) = \left\{\sigma_\varphi \alpha \approx \sigma_\varphi \beta : \alpha \approx \beta \in E(x)\, , \varphi \in \Gamma\right\}$$

where $\sigma_\varphi\colon Fm \to Fm$ is any substitution sending the variable x to φ. It is easy to see that this is equivalent to requiring that $\tau\colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ commutes with unions and with substitutions, that is, it is residuated and commutes with substitutions. Accordingly, we say that a map $\rho\colon \mathcal{P}(Eq) \to \mathcal{P}(Fm)$ is a structural transformer (from equations into formulas) when it is residuated and commutes with substitutions.

According to Blok and Pigozzi's original definition [3], a logic $\mathscr{L}$ is **algebraizable** with equivalent algebraic semantics the generalised quasi-variety[2] K when there are two structural transformers $\tau\colon \mathcal{P}(Fm) \longleftrightarrow \mathcal{P}(Eq)\colon \rho$ satisfying the following conditions:

A1. $\Gamma \vdash_{\mathscr{L}} \varphi$ if and only if $\tau\Gamma \vDash_{\mathsf{K}} \tau\varphi$;
A2. $x \approx y \mathrel{=\!\mid\!\vDash_{\mathsf{K}}} \tau\rho(x \approx y)$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$ and $x \approx y \in Eq$. Here $\vDash_{\mathsf{K}}$ is the consequence relation between equations relative to the class K, and corresponds to the validity of

[2] By a *generalised quasi-variety* we mean a class of algebras axiomatised by generalised quasi-equations, i.e., quasi-equations where the antecedent is the conjunction of a possibly infinite set of equations.

generalised quasi-equations in K. The closure operators associated with the consequence relations $\vdash_{\mathscr{L}}$ and $\vDash_{\mathsf{K}}$ are denoted by $C_{\mathscr{L}}$ and C_{K} respectively.

One of their most important results is the characterisation of algebraizability as the existence of an isomorphism $\Phi\colon \mathcal{T}h(\mathscr{L}) \to \mathcal{T}h(\vDash_{\mathsf{K}})$ between the expanded lattices of theories $\mathcal{T}h(\mathscr{L})$ and $\mathcal{T}h(\vDash_{\mathsf{K}})$ of both consequences, where the expansion is given by the endomorphisms of term algebra $\boldsymbol{End}(Fm)$, which induce unary operations on these lattices after closing under the corresponding closure operator. To be more precise: they prove the following.

Theorem 1 (Blok-Pigozzi)**.** *Let $\mathscr{L}$ be a logic and K a generalised quasi-variety. $\mathscr{L}$ is algebraizable with equivalent algebraic semantics K if and only if there is an isomorphism $\Phi\colon \mathcal{T}h(\mathscr{L}) \to \mathcal{T}h(\vDash_{\mathsf{K}})$ such that $\Phi C_{\mathscr{L}}\sigma = C_{\mathsf{K}}\sigma\Phi$ for every $\sigma \in \boldsymbol{End}(Fm)$.*

It was soon understood that the idea of algebraizability is not intrinsic of the connection between two structural consequence one of which is defined over formulas and the other on equations. This lead to the extension of the theory to k-dimensional systems by Blok and Pigozzi themselves [4] and independently to the case of Gentzen systems, studied first by Rebagliato and Verdú [15] and then by Raftery [14]. These investigations suggested that what lies behind the idea of algebraizability is simply a notion of equivalence between two structural closure operators, but it was only with the introduction of the notion of $\mathcal{M}$-set by Blok and Jónsson [2] that an appropriate mathematical framework to formulate such intuitions was found.

In fact $\mathcal{M}$-sets abstract the notion of a collection of syntactic objects (formulas, equations, sequents, hypersequents, etc.) built up from a fixed algebraic language, equipped with the natural action of ordinary substitutions on them. More precisely, let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be a monoid; then $\mathbb{R} = \langle R, \star_{\mathbb{R}}\rangle$ is an **$\mathcal{M}$-set** when R is non-empty and $\star_{\mathbb{R}}\colon \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is a map, called the **action** of the monoid on $\mathbb{R}$, satisfying the following conditions:

M1. $(\sigma \cdot \sigma') \star_{\mathbb{R}} x = \sigma \star_{\mathbb{R}} (\sigma' \star_{\mathbb{R}} x)$; and
M2. $1 \star_{\mathbb{R}} x = x$,

for every $\sigma, \sigma' \in M$ and $x \in R$. From now on we will work with a fixed monoid $\mathcal{M}$ and use letters $\mathbb{R}$, $\mathbb{S}$, $\mathbb{T}$ ... to denote $\mathcal{M}$-sets. It is important to observe that $\mathbb{M} \coloneqq \langle M, \cdot\rangle$ is an $\mathcal{M}$-set too; it will play an important rôle in this paper.

Example 2. In order to build $\mathcal{M}$-sets out of typical objects from logic, consider the monoid of substitutions $\boldsymbol{End}(Fm)$ and let Seq be the set of sequents over Fm. It is easy to prove that $\langle Fm, \star\rangle$, $\langle Eq, *\rangle$, $\langle Seq, \circ\rangle$ are all $\boldsymbol{End}(Fm)$-sets, where $\sigma \star \varphi \coloneqq \sigma\varphi$, $\sigma * (\alpha \approx \beta) \coloneqq \sigma\alpha \approx \sigma\beta$ and $\sigma \circ (\Gamma \rhd \Delta) \coloneqq \sigma\Gamma \rhd \sigma\Delta$ for every $\varphi \in Fm$, $\alpha \approx \beta \in Eq$, $\Gamma \rhd \Delta \in Seq$ and $\sigma \in End(Fm)$. The notation $\Gamma \rhd \Delta$ for sequents is just an alternative notation for the pair $\langle \Gamma, \Delta\rangle$, in order to avoid using symbols with other connotations such as $\vdash$ or $\Rightarrow$ as sequent separator.

In order to extend the notion of algebraizability to $\mathcal{M}$-sets we need to introduce the notions of structural closure operator and of structural transformer in this context. Beginning with the first one: we say that a function $\gamma\colon \mathcal{P}(R) \to \mathcal{P}(R)$ is a **structural closure operator** on $\mathbb{R}$ when it is a closure operator (with respect to the set-theoretical inclusion $\subseteq$) such that for every $\sigma \in M$ and $X \subseteq R$ the following condition holds:

$$\sigma \star_{\mathbb{R}} \gamma X \leqslant \gamma(\sigma \star_{\mathbb{R}} X).$$

As before we denote by $\mathcal{P}(R)_\gamma$ the corresponding lattice of closed sets. Moreover we say that a function $\tau\colon \mathcal{P}(R) \to \mathcal{P}(S)$ is a **structural transformer from $\mathbb{R}$ to $\mathbb{S}$** when it is residuated and

$$\tau(\sigma \star_{\mathbb{R}} X) = \sigma \star_{\mathbb{S}} \tau X$$

for every $\sigma \in M$ and $X \subseteq R$. We are now ready to extend the notion of algebraizability: let γ and δ be structural closure operators on $\mathbb{R}$ and $\mathbb{S}$ respectively, and let $\tau\colon \mathcal{P}(R) \longleftrightarrow \mathcal{P}(S)\colon \rho$ be two structural transformers from $\mathbb{R}$ to $\mathbb{S}$ and vice-versa. The following conditions are the natural translations of A1 and A2 into the context of $\mathcal{M}$-sets:

A1$'$. $\gamma = \tau^+\delta\tau$;
A2$'$. $\delta = \delta\tau\rho$.

Note that the two dual conditions ($\delta = \rho^+\gamma\rho$ and $\gamma = \gamma\rho\tau$) hold as well. Blok and Jónsson [2] focused their attention on the abstract version of the isomorphism that appears in Theorem 1, and proved that these conditions ensure its validity.

Theorem 3 (Blok-Jónsson)**.** *Let γ and δ be structural closure operators on the $\mathcal{M}$-sets $\mathbb{R}$ and $\mathbb{S}$ respectively. If there are two structural transformers $\tau\colon \mathcal{P}(R) \longleftrightarrow \mathcal{P}(S) : \rho$ from $\mathbb{R}$ to $\mathbb{S}$ and vice-versa satisfying* A1$'$ *and* A2$'$*, then there is an isomorphism $\Phi\colon \mathcal{P}(R)_\gamma \to \mathcal{P}(S)_\delta$ such that $\Phi\gamma(\sigma \star_{\mathbb{R}} X) = \delta(\sigma \star_{\mathbb{S}} \Phi X)$ for every $\sigma \in M$ and $X \subseteq R$.*

The isomorphism Φ is actually the restriction of the residual function ρ^+ to the closed sets, and by condition A2$'$ and its dual it coincides with the inverse of τ^+.

Unfortunately it is not possible to obtain an equivalence like that in Theorem 1, and actually a counterexample to the converse was found in [13]. The subsequent investigation of conditions under which such a converse might hold shifted its attention from the whole isomorphism to one of its symmetrical halves: given two structural closure operators γ and δ on $\mathbb{R}$ and $\mathbb{S}$ respectively, we say that $\Phi\colon \mathcal{P}(R)_\gamma \to \mathcal{P}(S)_\delta$ is a **structural representation** of γ into δ when the following conditions hold: It is an order embedding, i.e.,

$$X \subseteq Y \Longleftrightarrow \Phi X \subseteq \Phi Y$$

for every $X, Y \in \gamma\big[\mathcal{P}(R)\big]$, and the following diagram commutes for every $\sigma \in M$.

$$\begin{tikzcd}
\mathcal{P}(R)_\gamma \arrow[r, "\Phi"] \arrow[d, "\gamma\sigma\star_{\mathbb{R}}"'] & \mathcal{P}(S)_\delta \arrow[d, "\delta\sigma\star_{\mathbb{S}}"] \\
\mathcal{P}(R)_\gamma \arrow[r, "\Phi"'] & \mathcal{P}(S)_\delta
\end{tikzcd} \tag{1}$$

Moreover we say that a structural representation $\Phi\colon \mathcal{P}(R)_\gamma \to \mathcal{P}(S)_\delta$ of γ into δ is **induced** by a structural transformer $\tau\colon \mathcal{P}(R) \to \mathcal{P}(S)$ from $\mathbb{R}$ to $\mathbb{S}$ when the following diagram commutes.

$$\begin{tikzcd}
\mathcal{P}(R) \arrow[r, "\tau"] \arrow[d, two heads, "\gamma"'] & \mathcal{P}(S) \arrow[d, two heads, "\delta"] \\
\mathcal{P}(R)_\gamma \arrow[r, "\Phi"'] & \mathcal{P}(S)_\delta
\end{tikzcd}$$

Keeping in mind that in the situation of Theorem 3 Φ is actually $(\tau^+)^{-1}$, condition A1$'$ can be rewritten as $\Phi\gamma = \delta\tau$, that is, it says exactly that this Φ is "induced" by τ in the above sense. So this notion is really a generalisation of the original situation of algebraizability, when focusing on one representation only. Condition A2$'$ says that in the stronger case of having an isomorphism, the transformers inducing it and its inverse are mutually inverse modulo the closure operators. We will return to this situation at the very end of the paper (Theorem 30).

What makes the converse of Theorem 3 fail is the fact that in the context of $\mathcal{M}$-sets it is not true that every structural representation is induced by a structural transformer. This gives rise to *the representation problem*, which consists in understanding when that is the case. To be more precise, we say that "$\mathbb{R}$ **has the REP**", i.e., the **property of the representation**, when for every $\mathbb{S}$ and every γ and δ structural closure operators on $\mathbb{R}$ and $\mathbb{S}$ respectively, every structural representation of γ into δ is induced by a structural transformer from $\mathbb{R}$ to $\mathbb{S}$. Thus, the representation problem consists in finding a necessary and sufficient condition for an $\mathcal{M}$-set $\mathbb{R}$ to have the REP.

A sufficient condition was already found in [2], but it had a very limited application, e.g., it could not be applied to the algebraizability of Gentzen systems developed in [15], nor to more complicated formalisms such as hypersequents. The best (up to the present) approximation to the solution of the problem is probably the one presented by Gil-Férez in [13], which provides a sufficient condition embracing almost every $\mathcal{M}$-set arising from logic. In order to present it we need to introduce some new concepts, which will be also used in our solution. The first one describes a partition of an $\mathcal{M}$-set into a family of subsets stable with respect to the action of the monoid. More precisely, $\langle R, \star_{\mathbb{R}}, \pi\rangle$ is a **graded $\mathcal{M}$-set** when $\mathbb{R} = \langle R, \star_{\mathbb{R}}\rangle$ is an $\mathcal{M}$-set and $\pi\colon R \to I$ is a surjective map such that $\pi x = \pi(\sigma \star_{\mathbb{R}} x)$ for every $\sigma \in M$

and $x \in R$. In this case we will say that π is a **graduation** on $\mathbb{R}$, and for each $i \in I$ the subset $R_{\restriction i} \coloneqq \{x \in R : \pi x = i\}$ is a **layer**. It is easy to see that each layer is the universe of an $\mathcal{M}$-set, which we denote by $\mathbb{R}_{\restriction i}$. The second one provides a notion of variable for this kind of $\mathcal{M}$-sets. We say that a map $\tau\colon R \to M$ is a **coherent family of substitutions** when $\tau(\sigma \star_{\mathbb{R}} x) = \sigma \cdot \tau x$ for every $\sigma \in M$ and $x \in R$. Then, if $\langle R, \star_{\mathbb{R}}, \pi\rangle$ is a graded $\mathcal{M}$-set, we say that a map $\alpha\colon I \to R$ is a **graded variable** for it when $\pi\alpha(i) = i$ for every $i \in I$ (i.e., α selects a point in each layer) and there exists a coherent family of substitutions τ such that

$$\tau(x) \star_{\mathbb{R}} \alpha(\pi x) = x$$

for every $x \in R$. The main result from this point of view is Theorem 34 of [13]:

Theorem 4 (Gil-Férez)**.** *If an $\mathcal{M}$-set $\mathbb{R}$ can be equipped with a graduation that enjoys a graded variable, then it has the REP.*

All the $\boldsymbol{End}(Fm)$-sets of Example 2 can be equipped in a natural way with a graduation that enjoys a graded variable; this means that in all these cases the isomorphisms appearing in Theorem 3 due to algebraizability are all the possible ones. So, this result really accounts for all these natural examples. However, it leaves the general representation problem unsolved, since it is possible to construct a counterexample which shows that this is not a necessary condition (and that not every graded $\mathcal{M}$-set has a graded variable), as we will do in Example 10. At the end of the paper we will prove that $\mathcal{M}$-sets having the REP can be characterised in terms of what we call generalised graded variable, which are obtained by lifting the original concept to power sets (Theorem 28).

The quest for a sufficient and necessary condition led Galatos and Tsinakis [11] to move the problem to the yet more abstract setting of categories of modules over complete residuated lattices. In order to do this they "lift" to power sets all the constructions done until now; observe that closure operators and structural transformers are by definition maps on the power sets, so only the notion of $\mathcal{M}$-set has to be appropriatedly lifted. Then, the abstract notion corresponding to that of the power set is here that of a complete lattice. More precisely, given a complete residuated lattice $\mathcal{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ we say that $\boldsymbol{R} = \langle \mathcal{R}, *_{\boldsymbol{R}}\rangle$ is a **module over** $\mathcal{A}$, or an $\mathcal{A}$**-module**, when $\mathcal{R} = \langle R, \wedge, \vee\rangle$ is a complete lattice and $*_{\boldsymbol{R}}\colon \mathcal{A} \times \boldsymbol{R} \to \boldsymbol{R}$ a residuated mapping satisfying the following conditions:

R1. $(\sigma \cdot \sigma') *_{\boldsymbol{R}} x = \sigma *_{\boldsymbol{R}} (\sigma' *_{\boldsymbol{R}} x)$; and
R2. $1 *_{\boldsymbol{R}} x = x$,

for all $\sigma, \sigma' \in A$ and $x \in R$. From now on we will assume we are working with a fixed complete residuated lattice $\mathcal{A}$ and will use letters $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$... to denote $\mathcal{A}$-modules. A **module morphism** $\tau\colon \boldsymbol{R} \to \boldsymbol{S}$ from $\boldsymbol{R}$ to $\boldsymbol{S}$ is a residuated mapping $\tau\colon \mathcal{R} \to \mathcal{S}$ such that $\tau(\sigma *_{\boldsymbol{R}} x) = \sigma *_{\boldsymbol{S}} \tau x$ for every

$\sigma \in A$ and $x \in R$. $\mathcal{A}$-modules and module morphisms between them form a category which we denote by $\mathcal{A}$-`Mod`.

It is easy to see that given a monoid $\mathcal{M}$ the power set monoid, where $X \cdot Y \coloneqq \{x \cdot y : x \in X\,, y \in Y\}$ for all $X, Y \subseteq M$, equipped with intersection and union operations is a complete residuated lattice, which we will denote by $\mathcal{P}(\mathcal{M})$. Moreover, given any $\mathcal{M}$-set $\mathbb{R}$, if we jump to its power set and lift the operation $\star_{\mathbb{R}}$ to the power set in a similar way we obtain a $\mathcal{P}(\mathcal{M})$-module, which we will denote by $\mathcal{P}(\mathbb{R})$.

In this new framework some of the notions presented before for $\mathcal{M}$-sets are introduced in a very natural way. We say that a function $\gamma \colon \mathcal{R} \to \mathcal{R}$ is a **structural closure operator** on $\boldsymbol{R}$ when it is a closure operator such that for every $\sigma \in A$ and $x \in R$

$$\sigma *_{\boldsymbol{R}} \gamma x \leqslant \gamma(\sigma *_{\boldsymbol{R}} x).$$

Given a structural closure operator γ on $\boldsymbol{R}$ we put $\boldsymbol{R}_\gamma \coloneqq \langle \mathcal{R}_\gamma, \gamma *_{\boldsymbol{R}} \rangle$. It is easy to prove that $\boldsymbol{R}_\gamma$ is still a module over $\mathcal{A}$ and that $\gamma \colon \boldsymbol{R} \to \boldsymbol{R}_\gamma$ is a surjective module morphism. Given two structural closure operators γ and δ on $\boldsymbol{R}$ and $\boldsymbol{S}$ respectively, we say that $\Phi \colon \boldsymbol{R}_\gamma \to \boldsymbol{S}_\delta$ is a **structural representation** (of γ into δ) when it is a module morphism such that

$$x \leqslant y \Longleftrightarrow \Phi x \leqslant \Phi y$$

for every $x, y \in R$. It is worth to remark that this condition abstracts the situation considered in diagram (1), since the commutation of that diagram was just requiring Φ to be a module morphism in $\mathcal{P}(\mathcal{M})$-`Mod`. As before we say that a structural representation $\Phi \colon \boldsymbol{R}_\gamma \to \boldsymbol{S}_\delta$ is **induced** by a module morphism $\tau \colon \boldsymbol{R} \to \boldsymbol{S}$ when the following diagram commutes.

$$\begin{tikzcd}
\boldsymbol{R} \arrow[r, "\tau"] \arrow[d, two heads, "\gamma"'] & \boldsymbol{S} \arrow[d, two heads, "\delta"] \\
\boldsymbol{R}_\gamma \arrow[r, "\Phi"'] & \boldsymbol{S}_\delta
\end{tikzcd}$$

By analogy with the case of $\mathcal{M}$-sets, we say that an $\mathcal{A}$-module $\boldsymbol{R}$ **has the REP** when for every $\mathcal{A}$-module $\boldsymbol{S}$ and every pair of structural closure operators γ and δ on $\boldsymbol{R}$ and $\boldsymbol{S}$ respectively, every structural representation $\Phi \colon \boldsymbol{R}_\gamma \to \boldsymbol{S}_\delta$ is induced by a module morphism $\tau \colon \boldsymbol{R} \to \boldsymbol{S}$.

Gil-Férez proved in [12, Proposition 4.21] that epis in the category $\mathcal{A}$-`Mod` coincide with surjective module morphisms. Therefore we can express the solution of the representation problem in the context of $\mathcal{A}$-modules given in [11, Theorem 5.1] as follows:

Theorem 5 (Galatos-Tsinakis)**.** *An $\mathcal{A}$-module has the REP if and only if it is projective.*

Therefore the quest for the understanding of the representation problem in this new framework turned out to coincide with the study of projective objects in $\mathcal{A}$-`Mod`. In [11] it is proved that the complete residuated lattice $\mathcal{A}$

is itself an $\mathcal{A}$-module, which we will denote by $\boldsymbol{A}$, and as an easy consequence of [11, Theorem 5.7] we obtain a first example of a projective object in $\mathcal{A}$-$\mathtt{Mod}$, namely the residuated lattice $\mathcal{A}$ viewed as an $\mathcal{A}$-module:

Lemma 6 (Galatos-Tsinakis)**.** *Let $\mathcal{A}$ be a complete residuated lattice. The associated $\mathcal{A}$-module $\boldsymbol{A}$ is projective in $\mathcal{A}$-$\mathtt{Mod}$.*

But the more general results on projective $\mathcal{A}$-modules are due to Gil-Férez [12]. In order to present the main one we need to recall some facts about $\mathcal{A}$-$\mathtt{Mod}$. In [11] it is proved that $\mathcal{A}$-$\mathtt{Mod}$ is closed under coproducts and that they coincide with products. Therefore when we are given a coproduct $\coprod_{i\in I}\boldsymbol{R}_i$ of a family of $\mathcal{A}$-modules $\{\boldsymbol{R}_i\}_{i\in I}$, we denote by $p_i\colon \coprod_{i\in I}\boldsymbol{R}_i \to \boldsymbol{R}_i$ the projection arrow on the i-th component associated with it as a product. Therefore, given an element $x\in R$, one can prove that $m_x\colon \boldsymbol{A}\to\boldsymbol{R}$, defined as $m_x\sigma \coloneqq \sigma *_{\mathcal{R}} x$ for every $\sigma\in A$, is a module morphism. Thank to these facts in Theorem 4.51 of [12] the following is proved:

Theorem 7 (Gil-Férez)**.** *For every $\mathcal{A}$-module $\boldsymbol{R}$ the following conditions are equivalent:*

(i) *$\boldsymbol{R}$ is projective.*
(ii) *Every epi $\tau\colon \boldsymbol{S}\to\boldsymbol{R}$ is a retraction.*
(iii) *The map $\bigvee_{x\in R} m_x p_x\colon \coprod_{x\in R}\boldsymbol{A}\to\boldsymbol{R}$ is a retraction.*
(iv) *$\boldsymbol{R}$ is a retract of some projective $\boldsymbol{S}$.*

However, Theorem 7 tells us very little about the inner structure of projective objects in $\mathcal{A}$-$\mathtt{Mod}$. In order to get more information in this direction we shall restrict to a particular class of well-behaved $\mathcal{A}$-modules: $\mathcal{A}$-modules which have a special element which plays a rôle somewhat similar to the one of variables in the formula algebra. More precisely we say that $\boldsymbol{R}$ is **cyclic** when there is an element $x\in R$ such that $A *_{\boldsymbol{R}} \{x\} \coloneqq \{\sigma *_{\boldsymbol{R}} x : \sigma\in A\} = R$. In this case we say that x is a **generator** of $\boldsymbol{R}$. Enhancing a result of Galatos and Tsinakis [11] we can get the following characterization.

Theorem 8. *Let $\boldsymbol{R}$ be an $\mathcal{A}$-module. The following conditions are equivalent:*

(i) *$\boldsymbol{R}$ is cyclic and projective.*
(ii) *There are $\sigma\in A$, $x\in R$ such that $\sigma *_{\boldsymbol{R}} x = x$, $A *_{\boldsymbol{R}} \{x\} = R$ and for every $\sigma'\in A$, $[(\sigma' *_{\boldsymbol{R}} x)/_{*_{\boldsymbol{R}}} x]\cdot\sigma = \sigma'\cdot\sigma$.*
(iii) *There is $\sigma\in A$ such that $\boldsymbol{R}\cong\langle A\cdot\{\sigma\},\cdot\rangle$ and $\sigma^2=\sigma$.*
(iv) *$\boldsymbol{R}$ is a retract of $\boldsymbol{A}$.*

Proof. For the equivalence between (i), (ii) and (iii) see [11, Theorem 5.7]. We prove the remaining implications. For (iii)⇒(iv) observe that it is enough to prove that $\langle A\cdot\sigma,\cdot\rangle$ is a retract of $\boldsymbol{A}$. To do this observe that $m_\sigma\colon \boldsymbol{A}\to\langle A\cdot\sigma,\cdot\rangle$ is an onto module morphism since $\sigma^2=\sigma$. Now consider the identity embedding $id\colon\langle A\cdot\sigma,\cdot\rangle\to\boldsymbol{A}$. Since $\sigma^2=\sigma$ we have that $m_\sigma id = id_{\langle A\cdot\sigma,\cdot\rangle}$.

(iv)⇒ (i): Observe that by Lemma 6 and condition (iv) of Theorem 7 we know that $\boldsymbol{R}$ is projective, therefore we need only prove it is cyclic. Since $\boldsymbol{R}$ is a retract of $\boldsymbol{A}$, there is an onto module morphism $\tau\colon \boldsymbol{A} \to \boldsymbol{R}$. We check that $\tau(1)$ is a generator of $\boldsymbol{R}$. For any element $x \in R$ we have that there is $\sigma \in A$ such that $\tau\sigma = x$ and therefore we get that $x = \tau\sigma = \tau(\sigma \cdot 1) = \sigma *_{\boldsymbol{R}} \tau(1)$. ☐

It is natural to ask now whether it is possible to use Theorem 5 in order to solve the representation problem in the original context, namely that of $\mathcal{M}$-sets, and whether this will yield a characterization of objects enjoying the REP intrinsically linked with the context of $\mathcal{M}$-sets. The crucial fact that all the maps involved in the definition of an $\mathcal{M}$-set having the REP are actually maps between the corresponding power sets allows to easily prove:

Lemma 9. *If $\mathbb{R}$ is an $\mathcal{M}$-set such that $\mathcal{P}(\mathbb{R})$ has the REP as a $\mathcal{P}(\mathcal{M})$-module, then $\mathbb{R}$ has the REP as an $\mathcal{M}$-set.* ☐

The converse is not at all obvious, because not all $\mathcal{P}(\mathcal{M})$-modules are of the form $\mathcal{P}(\mathbb{S})$ for some $\mathcal{M}$-set $\mathbb{S}$, but we will show that it holds as well (Theorem 15). However this result allows us to provide the counterexample to the converse of Theorem 4 we promised above.

Example 10. Consider the two-element monoid $\mathcal{M} \coloneqq \big\langle \{0,1\}, \cdot, 1 \big\rangle$ defined by the following table:

$\cdot$	0	1
0	1	0
1	0	1

and consider the trivial $\mathcal{M}$-set $\mathbb{T} \coloneqq \big\langle \{a\}, \star_{\mathbb{T}} \big\rangle$. We begin by proving that $\mathcal{P}(\mathbb{T})$ is projective in $\mathcal{P}(\mathcal{M})$-`Mod`. To do this consider the function $\tau\colon \mathcal{P}(M) \to \mathcal{P}(\{a\})$ defined as

$$\tau(\sigma) \coloneqq \begin{cases} \{a\} & \text{if } \sigma \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

for every $\sigma \in \mathcal{P}(M)$. It is easy to prove that $\tau\colon \mathcal{P}(\mathbb{M}) \to \mathcal{P}(\mathbb{T})$ is a $\mathcal{P}(\mathcal{M})$-module morphism. Now we show it is a retraction. Let $\rho\colon \mathcal{P}(\{a\}) \to \mathcal{P}(M)$ be defined as

$$\rho(x) \coloneqq \begin{cases} \{0,1\} & \text{if } x = \{a\} \\ \emptyset & \text{otherwise} \end{cases}$$

for every $x \in \mathcal{P}(\{a\})$. Then $\rho\colon \mathcal{P}(\mathbb{T}) \to \mathcal{P}(\mathbb{M})$ is a $\mathcal{P}(\mathcal{M})$-module morphism, because it is clearly residuated, and to check that it commutes with substitutions it is enough (the other cases are trivial) to observe that

$$\rho(\{0\} *_{\mathcal{P}(\mathbb{T})} \{a\}) = \rho\{a\} = \{0,1\} = \{0 \cdot 1, 0 \cdot 0\} = \{0\} \cdot \{0,1\} = \{0\} \cdot \rho\{a\}.$$

Now observe that $\tau\rho = id_{\mathcal{P}(\mathbb{T})}$, therefore $\mathcal{P}(\mathbb{T})$ is a retract of $\mathcal{P}(\mathbb{M})$. By Lemma 6 we know that $\mathcal{P}(\mathbb{M})$ is projective and therefore applying condition (iv) of Theorem 7 we conclude that $\mathcal{P}(\mathbb{T})$ is projective in $\mathcal{P}(\mathcal{M})$-`Mod`. Therefore by Theorem 5 $\mathcal{P}(\mathbb{T})$ has the REP, and by Lemma 9 we conclude that $\mathbb{T}$ has the REP. It only remains to prove that $\mathbb{T}$ cannot be equipped with a graduation which enjoys a graded variable. Actually, $\mathbb{T}$ can be graded (possibly in several ways, but at least by a trivial graduation with only one layer), but it can never have a graded variable, simply because there is no coherent family of substitutions $\tau\colon \{a\} \to M$.

One interesting result, which we will make use of, is given by Gil-Férez in [12] and provides a characterisation of $\mathcal{M}$-sets giving rise to cyclic and projective $\mathcal{P}(\mathcal{M})$-modules. If $\mathbb{R}$ is an $\mathcal{M}$-set, then we say that $x \in R$ is a **generalised variable** when there is $u \subseteq M$ satisfying the following conditions:

G1. $u \star_{\mathbb{R}} \{x\} = \{x\}$;
G2. $\{\sigma \star_{\mathbb{R}} x : \sigma \in M\} = R$;
G3. for every $\sigma, \sigma' \in M$ if $\sigma \star_{\mathbb{R}} x = \sigma' \star_{\mathbb{R}} x$, then $\{\sigma'\} \cdot u = \{\sigma\} \cdot u$.

Thanks to this new concept one obtains the following result [12, Theorem 4.12], which will play a fundamental rôle in our characterization of $\mathcal{M}$-sets enjoying the REP.

Theorem 11 (Gil-Férez)**.** *Let $\mathbb{R}$ be an $\mathcal{M}$-set. The following conditions are equivalent:*

(i) $\mathcal{P}(\mathbb{R})$ *is cyclic and projective in* $\mathcal{P}(\mathcal{M})$-$\mathtt{Mod}$*;*
(ii) $\mathbb{R}$ *has a generalised variable.*

From this and Theorem 5 it follows that $\mathbb{R}$ has the REP whenever it has a generalized variable. The converse is however not true, as shown by the example of the $\boldsymbol{End}(Fm)$-set of sequents, which has the REP by Theorem 4 but does not have a generalized variable, because of condition G2.

2. The category of $\mathcal{M}$-Sets

In order to bring the characterisation of Theorem 5 from the land of $\mathcal{A}$-modules to the land of $\mathcal{M}$-sets we shall equip the class of $\mathcal{M}$-sets with a categorical structure.

Lemma 12. *$\mathcal{M}$-sets with structural transformers between them form a category, which we denote by* $\mathcal{M}$-$\mathtt{Set}$*.*

Proof. Take the usual composition of functions as composition, and the usual identity function as identity arrow. □

Note that arrows in this category, i.e., structural transformers τ from $\mathbb{R}$ to $\mathbb{S}$, are actually set functions from the power set of R to the power set of S; however, most often we prefer to denote them by $\tau\colon \mathbb{R} \to \mathbb{S}$ according to the standard categorical notation, in order to emphasize that we are regarding τ as an arrow from the object $\mathbb{R}$ to the object $\mathbb{S}$ in the category $\mathcal{M}$-$\mathtt{Set}$. Often as well, such maps are first defined on the base universe as a map from R to $\mathcal{P}(S)$ and then lifted to the power set of R in the natural way, that is, by taking unions. In such situations it is often the case that in order to check a property of these maps it is enough to check it for points, and the extension to sets is straightforward. In particular this is the case when checking that a transformer defined in this way is structural; we will do this without notice.

In the previous section we recalled how to construct, given an $\mathcal{M}$-set $\mathbb{R}$, the corresponding $\mathcal{P}(\mathcal{M})$-module $\mathcal{P}(\mathbb{R})$. It is easy to extend this construction to $\mathcal{M}$-arrows as follows: recall that an $\mathcal{M}$-arrow $\tau\colon \mathbb{R} \to \mathbb{S}$ is a function from $\mathcal{P}(R)$ to $\mathcal{P}(S)$; therefore we can have $\mathcal{P}(\tau)\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{S})$ by simply defining $\mathcal{P}(\tau)X \coloneqq \tau X$ for every $X \in \mathcal{P}(\mathbb{R})$. Thus, we can say that arrows in $\mathcal{M}$-$\mathtt{Set}$ are also arrows in $\mathcal{P}(\mathcal{M})$-$\mathtt{Mod}$. This fact will have the effect of making some homologous properties in the two categories equivalent. As we expect, it turns out that $\mathcal{P}\colon \mathcal{M}\text{-}\mathtt{Set} \to \mathcal{P}(\mathcal{M})\text{-}\mathtt{Mod}$ is indeed a functor.

Our first goal is to find a categorical characterization of graded $\mathcal{M}$-sets. To do this we need to prove that $\mathcal{M}$-$\mathtt{Set}$ has coproducts. Given a family of $\mathcal{M}$-sets $\{\mathbb{R}_i\}_{i\in I}$, we denote by $\coprod_{i\in I}\mathbb{R}_i$ its disjoint union $\{\langle x,i\rangle : x \in R_i\}$ equipped with the action $\star\colon \mathcal{M}\times\coprod_{i\in I}\mathbb{R}_i \to \coprod_{i\in I}\mathbb{R}_i$ defined for every $\sigma \in M$ and $\langle x,i\rangle$ as $\sigma\star\langle x,i\rangle \coloneqq \langle \sigma\star_i x, i\rangle$.

Theorem 13. $\mathcal{M}$-$\mathtt{Set}$ *has coproducts: If* $\{\mathbb{R}_i\}_{i\in I}$ *is a family of* $\mathcal{M}$*-sets, then* $\coprod_{i\in I}\mathbb{R}_i$ *is an* $\mathcal{M}$*-set and it coincides up to isomorphism with the coproduct of the family* $\{\mathbb{R}_i\}_{i\in I}$*. Moreover the functor* $\mathcal{P}\colon \mathcal{M}\text{-}\mathtt{Set} \to \mathcal{P}(\mathcal{M})\text{-}\mathtt{Mod}$ *preserves coproducts, in the sense that for every family* $\{\mathbb{R}_i\}_{i\in I}$ *of* $\mathcal{M}$*-sets,*

$$\mathcal{P}\big(\textstyle\coprod_{i\in I}\mathbb{R}_i\big) \cong \coprod_{i\in I}\mathcal{P}(\mathbb{R}_i).$$

Proof. We first prove that $\coprod_{i\in I}\mathbb{R}_i$ is an $\mathcal{M}$-set: for every $\sigma,\sigma' \in M$ and $\langle x,i\rangle \in \coprod_{i\in I}R_i$, we have that $(\sigma\cdot\sigma')\star\langle x,i\rangle = \big\langle(\sigma\cdot\sigma')\star_i x, i\big\rangle = \big\langle\sigma\star_i(\sigma'\star_i x), i\big\rangle = \sigma\star\langle\sigma'\star_i x, i\rangle = \sigma\star\big(\sigma'\star\langle x,i\rangle\big)$, and $1\star\langle x,i\rangle = \langle 1\star_i x, i\rangle = \langle x,i\rangle$. Moreover observe that for every $i \in I$ the function $\pi_i\colon R_i \to \{\langle x,i\rangle : x\in R_i\}$ defined for every $x \in R_i$ as $\pi_i x \coloneqq \langle x,i\rangle$ and extended to the power set in the natural way is a structural transformer, because for every $\sigma \in M$ and $x \in R_i$ we have that $\pi_i(\sigma\star_i x) = \langle\sigma\star_i x, i\rangle = \sigma\star\langle x,i\rangle = \sigma\star\pi_i x$.

Now recall from [11] that $\mathcal{P}(\mathcal{M})$-$\mathtt{Mod}$ has coproducts, then the fact that $\mathcal{P}\big(\coprod_{i\in I}\mathbb{R}_i\big) \cong \coprod_{i\in I}\mathcal{P}(\mathbb{R}_i)$ would imply that $\coprod_{i\in I}\mathbb{R}_i$ is a coproduct of $\{\mathbb{R}_i\}_{i\in I}$ in $\mathcal{M}$-$\mathtt{Set}$. Therefore it only remains to prove that $\mathcal{P}$ preserves $\coprod$. In order to do this recall from [11] how coproducts are constructed in $\mathcal{P}(\mathcal{M})$-$\mathtt{Mod}$. Then consider the function $\tau\colon \coprod_{i\in I}\mathcal{P}(R_i) \longrightarrow \mathcal{P}(\{\langle x,i\rangle : x \in R_i\})$ defined for every $\overline{x} \in \coprod_{i\in I}\mathcal{P}(R_i)$ as $\tau\overline{x} = \bigcup_{i\in I}\big(\overline{x}(i)\times\{i\}\big)$. First we check that τ is an arrow in the category. In order to do this we denote by $\overline{\star}$ the action of $\coprod_{i\in I}\mathcal{P}(\mathbb{R}_i)$. We begin by the commutativity condition: if $\sigma \in P(M)$ and $\overline{x} \in \coprod_{i\in I}\mathcal{P}(R_i)$, then

$$\begin{aligned}\tau(\sigma\overline{\star}\,\overline{x}) &= \bigcup_{i\in I}\Big((\sigma\overline{\star}\,\overline{x})(i)\times\{i\}\Big) = \bigcup_{i\in I}\Big(\big(\sigma\star_i\overline{x}(i)\big)\times\{i\}\Big)\\ &= \sigma\star\bigcup_{i\in I}(\overline{x}(i)\times\{i\}) = \sigma\star\tau\overline{x}.\end{aligned}$$

Now we prove that it is residuated: to do this let $\overline{X} \subseteq \coprod_{i\in I} \mathcal{P}(R_i)$; then

$$\begin{aligned}\tau \bigvee^{\coprod \mathcal{P}} \overline{X} &= \bigcup_{i\in I} \Big((\bigvee^{\coprod \mathcal{P}} \overline{X})(i) \times \{i\} \Big) = \bigcup_{i\in I} \Big((\bigvee^{\mathcal{P}}_{\overline{x}\in\overline{X}} \overline{x}(i)) \times \{i\} \Big) \\ &= \bigcup_{i\in I} \Big((\bigcup_{\overline{x}\in\overline{X}} \overline{x}(i)) \times \{i\} \Big) = \bigcup_{\overline{x}\in\overline{X}} \Big((\bigcup_{i\in I} \overline{x}(i)) \times \{i\} \Big) \\ &= \bigcup_{\overline{x}\in\overline{X}} \tau\overline{x} = \bigvee^{\mathcal{P}\coprod}_{\overline{x}\in\overline{X}} \tau\overline{x}.\end{aligned}$$

This ends the proof that τ is an arrow, and it only remains to prove that it is a bijection. We first prove it is an injection: let $\overline{x}, \overline{y} \in \coprod_{i\in I} \mathcal{P}(R_i)$ be such that $\overline{x} \neq \overline{y}$; then there is $i \in I$ such that $\overline{x}(i) \neq \overline{y}(i)$. We can assume without loss of generality that there is $a \in \overline{x}(i)$ such that $a \notin \overline{y}(i)$; therefore we conclude that $\langle a, i\rangle \in \tau\overline{x}$ and $\langle a, i\rangle \notin \tau\overline{y}$, which yields $\tau\overline{x} \neq \tau\overline{y}$. In order to prove that τ is surjective take any $x \in \mathcal{P}(\{\langle x, i\rangle : x \in R_i\})$ and define $\overline{x} \in \coprod_{i\in I} \mathcal{P}(R_i)$ as $\overline{x}(i) \coloneqq \{a : \langle a, i\rangle \in x\}$ for every $i \in I$. It is clear that $\tau\overline{x} = x$, therefore we are done. □

There is a strong connection between coproducts and *graded* $\mathcal{M}$-sets: on the one hand each graded $\mathcal{M}$-set $\langle R, \star_{\mathbb{R}}, \pi\rangle$ is isomorphic (as an $\mathcal{M}$-set) to the coproduct of its layers $\coprod_{i\in I} \mathbb{R}_{\restriction i} \cong \mathbb{R}$; on the other hand each coproduct $\coprod_{i\in I} \mathbb{R}_i$ of $\mathcal{M}$-sets determines a natural graduation on the coproduct, namely the map $\delta\colon \{\langle x, i\rangle : x \in R_i\} \to I$ given by $\delta\langle x, i\rangle \coloneqq i$ for every $\langle x, i\rangle \in \{\langle x, i\rangle : x \in R_i\}$. This idea will be fundamental in the proof of Theorem 28.

Another fact it is worth to remark is that this result allows us to gain a characterisation of $\mathcal{P}(\mathcal{M})$-modules, where $\mathcal{M}$ is a monoid, which is indeed a particular version of Theorem 12.12 of [9]. More precisely it turns out that they coincide with quotients of $\mathcal{P}(\mathcal{M})$-modules coming from $\mathcal{M}$-sets. As a consequence we will get the converse of Lemma 9, as we claimed before.

Lemma 14. *Let $\mathcal{M}$ be a monoid. For every $\mathcal{P}(\mathcal{M})$-module $\boldsymbol{R}$, there is an $\mathcal{M}$-set $\mathbb{R}$ and a structural closure operator γ on $\mathcal{P}(\mathbb{R})$ such that $\boldsymbol{R} \cong \mathcal{P}(\mathbb{R})_\gamma$.*

Proof. Let $\mathcal{M}$ be a monoid and $\boldsymbol{R}$ a $\mathcal{P}(\mathcal{M})$-module. Then we consider the surjective module morphism $\tau \coloneqq \bigvee_{x\in R} m_x p_x \colon \coprod_{x\in R} \mathcal{P}(\mathbb{M}) \to \boldsymbol{R}$, which appeared before in Theorem 7. Now, recall from Lemma 4.1 of [11] that $\tau^+\tau$ is a structural closure operator. Then obviously τ is injective on the fixed points of $\tau^+\tau$ and therefore $\tau\colon (\coprod_{x\in R} \mathcal{P}(\mathbb{M}))_{\tau^+\tau} \to \boldsymbol{R}$ is a module isomorphism. Finally by Theorem 13 we conclude that $\boldsymbol{R} \cong (\coprod_{x\in R} \mathcal{P}(\mathbb{M}))_{\tau^+\tau} \cong \mathcal{P}(\coprod_{x\in R} \mathbb{M})_{\tau^+\tau}$. □

The direct proof of the following result is inspired in a suggestion of José Gil-Férez. The result itself was originally obtained as a consequence of Theorem 18 (which had a more complicated proof), taking Theorem 5 into account.

Theorem 15. *Let $\mathbb{R}$ be an $\mathcal{M}$-set. $\mathcal{P}(\mathbb{R})$ has the REP as a $\mathcal{P}(\mathcal{M})$-module if and only if $\mathbb{R}$ has the REP as an $\mathcal{M}$-set.*

Proof. The "only if" direction is exactly Lemma 9. For the "if" direction we reason as follows. Let $\mathbb{R}$ be an $\mathcal{M}$-set with the REP and consider the $\mathcal{P}(\mathcal{M})$-module $\mathcal{P}(\mathbb{R})$. Then let $\boldsymbol{S}$ be a $\mathcal{P}(\mathcal{M})$-module and γ and δ two structural closure operators on $\mathcal{P}(\mathbb{R})$ and $\boldsymbol{S}$ respectively such that there is a structural representation $\Phi\colon \mathcal{P}(\mathbb{R})_\gamma \to \boldsymbol{S}_\delta$. By Lemma 14 we know that there is an $\mathcal{M}$-set $\mathbb{S}$ and a module isomorphism $\rho\colon \mathcal{P}(\mathbb{S})_\varepsilon \to \boldsymbol{S}$ for some structural closure operator ε on $\mathcal{P}(\mathbb{S})$. Therefore there is a structural closure operator δ' on $\mathcal{P}(\mathbb{S})_\varepsilon$ such that $\boldsymbol{S}_\delta \cong (\mathcal{P}(\mathbb{S})_\varepsilon)_{\delta'}$. Now observe that γ and $\zeta \coloneqq \delta'\varepsilon$ are structural closure operators respectively on $\mathbb{R}$ and $\mathbb{S}$ and that there is structural representation $\Phi'\colon \mathcal{P}(\mathbb{R})_\gamma \to \mathcal{P}(\mathbb{S})_\zeta$. Since $\mathbb{R}$ has the REP, there is a structural transformer $\tau\colon \mathbb{R} \to \mathbb{S}$ which induces Φ'. From this it is easy to prove that Φ is induced by $\rho\varepsilon\tau$. $\square$

Since the $\mathcal{A}$-modules that enjoy the REP are the projective objects of $\mathcal{A}$-`Mod`, in our search for a characterization of $\mathcal{M}$-sets with the REP we want to "translate" Theorem 7 into the context of $\mathcal{M}$-`Set`. As we do not know whether epis are surjective in this category (while this is the case in $\mathcal{A}$-`Mod`), we will work with onto-projective objects. The first step in this direction will be to prove that our monoid $\mathcal{M}$ can be seen as an onto-projective $\mathcal{M}$-set; recall that $\mathbb{M}$ denotes the monoid $\mathcal{M}$ viewed as an $\mathcal{M}$-set.

Lemma 16.

1. *$\mathcal{P}(\mathbb{M})$ is projective in $\mathcal{P}(\mathcal{M})$-`Mod`.*
2. *If $\mathcal{P}(\mathbb{R})$ is projective in $\mathcal{P}(\mathcal{M})$-`Mod`, then $\mathbb{R}$ is onto-projective in $\mathcal{M}$-`Set`.*
3. *$\mathbb{M}$ is onto-projective in $\mathcal{M}$-`Set`.*

Proof. 1 is a particular case of Lemma 6. 2 is a straightforward consequence of the fact that the arrows in $\mathcal{M}$-`Set` are actually arrows in $\mathcal{P}(\mathcal{M})$-`Mod`. Finally 3 follows from 1 and 2. $\square$

It is a general fact of category theory that projectiveness with respect to any family of arrows is preserved under coproducts (see for example Proposition 10.40 of [1], which provides the dual result); below we record the particular case we are going to use.

Lemma 17. *The coproduct of any family of onto-projective $\mathcal{M}$-sets is onto-projective.*

We are finally ready to provide our desired characterization result for $\mathcal{M}$-sets that are onto-projective in $\mathcal{M}$-`Set`. It can be read in two ways: It provides a necessary and sufficient condition for an $\mathcal{M}$-set to have the REP; and moreover it shows that the problem of the representation in the context of $\mathcal{M}$-sets coincides with that of finding a necessary and sufficient condition for an $\mathcal{M}$-set to give rise to a projective $\mathcal{P}(\mathcal{M})$-module.

The following construction will be used in the proof of the next result and also outside it. Given an $\mathcal{M}$-set $\mathbb{R}$, consider the coproduct of R copies of $\mathbb{M}$. Then define a function

$$\begin{aligned} \phi\colon \{\langle\sigma, x\rangle : \sigma \in M, x \in R\} &\longrightarrow R \\ \langle\sigma, x\rangle &\longmapsto \sigma \star_{\mathbb{R}} x \end{aligned}$$

and extend it to the power sets in the natural way so that it can become an arrow $\phi\colon \coprod_{x\in R} \mathbb{M} \to \mathbb{R}$ in $\mathcal{M}$-$\texttt{Set}$. It is easy to prove that ϕ is indeed a surjective structural transformer.

Theorem 18. *Let $\mathbb{R}$ be an $\mathcal{M}$-set. The following conditions are equivalent:*

(i) $\mathbb{R}$ *is onto-projective in* $\mathcal{M}$-$\texttt{Set}$*.*
(ii) *Every surjective structural transformer* $\tau\colon \mathbb{S} \to \mathbb{R}$ *is a retraction.*
(iii) *The map* $\phi\colon \coprod_{x\in R} \mathbb{M} \to \mathbb{R}$ *defined above is a retraction.*
(iv) $\mathbb{R}$ *is a retract of some* $\mathbb{S}$ *that is onto-projective in* $\mathcal{M}$-$\texttt{Set}$*.*
(v) $\mathcal{P}(\mathbb{R})$ *is projective in* $\mathcal{P}(\mathcal{M})$-$\texttt{Mod}$*.*
(vi) $\mathbb{R}$ *has the REP as an* $\mathcal{M}$*-set.*

Proof. For (i)⇒ (ii) consider the diagram made up by τ and the identity $id_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$. Applying onto-projectiveness one proves that τ is a retraction. (ii)⇒(iii) follows directly from the fact that ϕ is a surjective structural transformer. For (iii)⇒(iv) observe that by Lemmas 16 and 17 we know that $\coprod_{x\in R} \mathbb{M}$ is onto-projective in $\mathcal{M}$-$\texttt{Set}$. Therefore applying the assumption we are done.

(iv)⇒(i): Let $\tau\colon \mathbb{K} \to \mathbb{T}$ and $\rho\colon \mathbb{R} \to \mathbb{T}$ be two structural transformers such that τ is surjective. Now by assumption there is a retraction $\lambda\colon \mathbb{S} \to \mathbb{R}$, i.e., there is a structural transformer $\pi\colon \mathbb{R} \to \mathbb{S}$ such that $\lambda\pi = id_{\mathbb{R}}$. If we now consider the composition $\rho\lambda\colon \mathbb{S} \to \mathbb{T}$, by onto-projectiveness of $\mathbb{S}$ there is a structural transformer $\varepsilon\colon \mathbb{S} \to \mathbb{K}$ such that $\tau\varepsilon = \rho\lambda$. Thus, the situation is as follows.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\pi"] & \mathbb{S} \arrow[r, "\lambda"] \arrow[d, dashed, "\varepsilon"'] & \mathbb{R} \arrow[d, "\rho"] \\ & \mathbb{K} \arrow[r, two heads, "\tau"'] & \mathbb{T} \end{tikzcd}$$

Then the composition $\varepsilon\pi\colon \mathbb{R} \to \mathbb{K}$ is the structural transformer we are looking for, because $\tau(\varepsilon\pi) = (\tau\varepsilon)\pi = (\rho\lambda)\pi = \rho(\lambda\pi) = \rho\, id_{\mathbb{R}} = \rho$. This shows $\mathbb{R}$ is onto-projective.

(iii)⇒(v): It is easy to see that if $\mathbb{R}$ is a retract of $\coprod_{x\in R} \mathbb{M}$, then $\mathcal{P}(\mathbb{R})$ is a retract of $\mathcal{P}\big(\coprod_{x\in R} \mathbb{M}\big)$ in the category $\mathcal{P}(\mathcal{M})$-$\texttt{Mod}$. Now by Theorem 13 $\mathcal{P}\big(\coprod_{x\in R} \mathbb{M}\big) \cong \coprod_{x\in R} \mathcal{P}(\mathbb{M})$, therefore by Lemma 16 and Lemma 5.12 of [11], $\mathcal{P}(\mathbb{R})$ is a retract of a projective $\mathcal{P}(\mathcal{M})$-module. Now Theorem 7 implies that $\mathcal{P}(\mathbb{R})$ is projective as well.

Finally the equivalence (v)⇔(vi) follows from Theorem 5 and Theorem 15, and direction (v)⇒ (i) follows from condition 2 of Lemma 16. □

3. Slices Of Cake

Theorem 18 extends a Gil-Férez-style characterization of $\mathcal{M}$-sets that are onto-projective in $\mathcal{M}$-$\texttt{Set}$, connecting them with relevant logical issues, but it does not give information about the inner structure of such objects. From this point of view the most informative condition is (iii), which tells us that they can be seen as special parts of coproducts of the monoid which lies behind the category. The aim of this section is to use this intuition to get more information about their inner structure in order to obtain further characterisations of onto-projective $\mathcal{M}$-sets of either a categorical or a set-theoretic flavour and more intrinsic to the context of $\mathcal{M}$-sets. As a consequence we will obtain that onto-projective $\mathcal{M}$-sets are built by pasting together some special fundamental bricks which can be nicely characterised. In order to start this journey through the decomposition of onto-projective $\mathcal{M}$-sets, we need to introduce a new concept.

Definition 19. Let $\mathbb{R}$ be an $\mathcal{M}$-set. A subset $S \subseteq R$ is a **slice** of $\mathbb{R}$ when either $S \in \{\emptyset, R\}$ or the structure $\langle R, \star_{\mathbb{R}}, \pi_S\rangle$, where $\pi_S \colon R \to \{0,1\}$ is defined for every $x \in R$ as

$$\pi_S(x) \coloneqq \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise,} \end{cases}$$

is a graded $\mathcal{M}$-set.

In other words, a non-trivial slice of an $\mathcal{M}$-set $\mathbb{R}$ is a subset of its universe R determining a graduation with just two layers, it and its complement; note that we had to include separately the trivial cases $\{\emptyset, R\}$ because by definition each layer in a graded $\mathcal{M}$-set should be non-empty. Thus, not every $\mathcal{M}$-set will have non-trivial slices; only those with non-trivial graduations can. Observe also that every single layer of a graded $\mathcal{M}$-set, as well as every union of layers, constitutes a slice.

Given an $\mathcal{M}$-set we want the set of all its slices to be a closure system, in order to be able to define the *slice generation* closure operator. Given an $\mathcal{M}$-set $\mathbb{R}$, we put

$$\mathcal{S}(\mathbb{R}) \coloneqq \{S \subseteq M : S \text{ is a slice of } \mathbb{R}\}.$$

Lemma 20. *Let $\mathbb{R}$ be an $\mathcal{M}$-set. Then $\mathcal{S}(\mathbb{R})$ is a closure system.*

Proof. First observe that $\mathcal{S}(\mathbb{R})$ has a maximum R and a minimum $\emptyset$. Now we turn to check that $\mathcal{S}(\mathbb{R})$ is closed under intersections. Let $\{S_i\}_{\in I} \subseteq \mathcal{S}(\mathbb{R})$. If $\bigcap_{i\in I} S_i = \emptyset$ or $\bigcap_{i\in I} S_i = R$ we are done. Thus, suppose that $\bigcap_{i\in I} S_i \neq \emptyset$ and $R \setminus \bigcap_{i\in I} S_i \neq \emptyset$. We check that $\langle R, \star_{\mathbb{R}}, \pi_{\bigcap_{i\in I} S_i}\rangle$ is a graded $\mathcal{M}$-set. In order to do this let $\sigma \in M$ and $x \in \bigcap_{i\in I} S_i$; we have that $x \in S_i$ for every $i \in I$. Then take any $i \in I$. If $S_i = R$, then $\sigma \star_{\mathbb{R}_i} x \in S_i$. Then suppose $S_i \neq R$. Since $S_i \neq \emptyset$ and S_i is a slice we know that $\langle R, \star_{\mathbb{R}}, \pi_{S_i}\rangle$ is a graded $\mathcal{M}$-set, which yields that $\sigma \star_{\mathbb{R}} x \in S_i$. We conclude that $\sigma \star_{\mathbb{R}} x \in \bigcap_{i\in I} S_i$. Now let $y \in R \setminus \bigcap_{i\in I} S_i = \bigcup_{i\in I}(R \setminus S_i)$; there is $i \in I$ such that $y \in R \setminus S_i$.

Then we have that $S_i \neq R$ and, since we assumed that $S_i \neq \emptyset$ and that S_i is a slice of $\mathbb{R}$, we know that $\langle R, \star_{\mathbb{R}}, \pi_{S_i}\rangle$ is a graded $\mathcal{M}$-set. We conclude that $\sigma \star_{\mathbb{R}} y \in R \setminus S_i$ and therefore that $\sigma \star_{\mathbb{R}} y \in R \setminus \bigcap_{i\in I} S_i$. This ends the proof that $\langle R, \star_{\mathbb{R}}, \pi_{\bigcap_{i\in I} S_i}\rangle$ is a graded $\mathcal{M}$-set. □

We denote by $\mathcal{S}_{\mathbb{R}} \colon \mathcal{P}(R) \to \mathcal{P}(R)$ the associated closure operator, and will omit the subindex when it will be clear from the context. With this notation one can get some extra information on the structure of $\mathcal{S}(\mathbb{R})$.

Theorem 21. *$\mathbf{S}(\mathbb{R}) \coloneqq \left\langle \mathcal{S}(\mathbb{R}), \bigcap, \bigcup, \sim, \emptyset, R\right\rangle$ is a complete and atomic Boolean algebra, with atoms $\mathcal{A}(\mathbb{R}) \coloneqq \{\mathcal{S}\{x\} : x \in R\}$.*

Proof. By Lemma 20, $\mathcal{S}(\mathbb{R})$ is closed under arbitrary intersections. It is clear from the definition of slice that the set-theoretical complement $\sim$ of a slice is a slice as well. Therefore $\mathcal{S}(\mathbb{R})$ is closed under complement and hence under arbitrary unions, and we conclude that it is a complete Boolean algebra.

It only remains to prove that $\mathbf{S}(\mathbb{R})$ is atomic with atoms $\mathcal{A}(\mathbb{R})$. We first show that $\mathcal{A}(\mathbb{R})$ is the set of atoms of $\mathbf{S}(\mathbb{R})$. Let $S \in \mathcal{A}(\mathbb{R})$, then $S = \mathcal{S}\{x\}$ for some $x \in R$. Then $x \in S$ and therefore $\emptyset \subsetneq S$. Moreover suppose towards a contradiction that there is $K \in \mathcal{S}(\mathbb{R})$ such that $\emptyset \subsetneq K \subsetneq S$. Since $\emptyset \subsetneq K$, there is $y \in R$ such that $y \in K$. Moreover since $K \subsetneq S$, we know that $x \notin K$. Since $\mathcal{S}(\mathbb{R})$ is closed under the set-theoretical complement $\sim$, we have that $R \setminus K = {\sim}K \in \mathcal{S}(\mathbb{R})$. Clearly $x \in R \setminus K$. Therefore, since $\mathcal{S}\{x\} = \bigcap\{T \in \mathcal{S}(\mathbb{R}) : x \in T\}$, we have that $\mathcal{S}\{x\} \subseteq R \setminus K$ and therefore $K \subseteq {\sim}K$ which contradicts the fact that $K \neq \emptyset$. Now assume that S is an atom of $\mathbf{S}(\mathbb{R})$. We know that $\emptyset \subsetneq S$, therefore there is $x \in R$ such that $x \in S$. Clearly $\emptyset \subsetneq \mathcal{S}\{x\} \subseteq S$. Therefore we conclude that $\mathcal{S}\{x\} = S$. We conclude that $\mathcal{A}(\mathbb{R})$ is indeed the set of atoms of $\mathbf{S}(\mathbb{R})$.

Finally we check that $\mathbf{S}(\mathbb{R})$ is atomic. Let $S \in \mathcal{S}(\mathbb{R})$ such that $S \neq \emptyset$; then there is $x \in S$. This yields that $\mathcal{S}\{x\} \subseteq S$. Since we proved that $\mathcal{S}\{x\}$ is an atom we are done. □

Example 22. Consider the set of sequents

$$Seq \coloneqq \left\{\mathit{\Gamma} \rhd \mathit{\Delta} \,|\, \text{ there are } n, m \in \mathbb{N} \text{ such that } \mathit{\Gamma} \in Fm^n \text{ and } \mathit{\Delta} \in Fm^m\right\}$$

and let $\langle Seq, \circ\rangle$ be the corresponding $\mathcal{M}$-set described in Example 2. We say that $\mathit{\Gamma} \rhd \mathit{\Delta} \in Seq$ has *trace* $\langle n, m\rangle$ if $\mathit{\Gamma} \in Fm^n$ and $\mathit{\Delta} \in Fm^m$. It is easy to prove that $S \subseteq Seq$ is an atomic slice if and only if $S = \left\{\mathit{\Gamma} \rhd \mathit{\Delta} \in Set \,|\, \mathit{\Gamma} \rhd \mathit{\Delta} \text{ has trace } \langle n, m\rangle\right\}$ for some $n, m \in \mathbb{N}$.

Now, observe that every non-empty slice $S \in \mathcal{S}(\mathbb{R})$ determines in a natural way an $\mathcal{M}$-set, which we denote by $\mathbb{S}$. Therefore it seems natural to think that pasting together the slices in $\mathcal{A}(\mathbb{R})$, i.e., the smallest non-empty slices of $\mathbb{R}$, one should recover the original cake $\mathbb{R}$. Next result tells us that up to isomorphism this is the case.

Lemma 23. *Let $\mathbb{R}$ be an $\mathcal{M}$-set. Then $\mathbb{R} \cong \coprod_{S\in\mathcal{A}(\mathbb{R})} \mathbb{S}$.*

Proof. Let $\tau\colon \mathbb{R} \to \coprod_{S\in\mathcal{A}(\mathbb{R})} \mathbb{S}$ be the map defined as $\tau x := \big\langle x, \mathcal{S}\{x\}\big\rangle$ for every $x \in R$. That τ is well defined is clear because $x \in \mathcal{S}\{x\} \in \mathcal{A}(\mathbb{R})$. It is also clear that it is an injection. We check that it is surjective. Let $\langle y, \mathcal{S}\{x\}\rangle \in \coprod_{S\in\mathcal{A}(\mathbb{R})} \mathbb{S}$. This means that $y \in \mathcal{S}\{x\}$, and by Theorem 21 this implies that $\mathcal{S}\{x\} = \mathcal{S}\{y\}$, and therefore that $\big\langle y, \mathcal{S}\{x\}\big\rangle = \big\langle y, \mathcal{S}\{y\}\big\rangle = \tau y$. Thus, τ is a bijection, and it is also structural: let $\sigma \in M$ and $x \in R$. We have that

$$\sigma \star_{\coprod} \tau x = \sigma \star_{\coprod} \big\langle x, \mathcal{S}\{x\}\big\rangle = \big\langle \sigma \star_{\mathbb{R}} x, \mathcal{S}\{x\}\big\rangle = \big\langle \sigma \star_{\mathbb{R}} x, \mathcal{S}(\sigma \star_{\mathbb{R}} x)\big\rangle = \tau(\sigma \star_{\mathbb{R}} x).$$

Thus, τ is a bijection and a structural map at the level of points. It is straightforward to see that this implies its power set extension is also bijective and structural, that is, an $\mathcal{M}$-isomorphism. □

This is all for what concerns the idea of pasting together different slices of $\mathbb{R}$ in order to recover the original $\mathbb{R}$. On the other hand, in general slices may overlap, so that there may be redundant information, and it is natural to think that in order to characterise $\mathbb{R}$ it is enough to have enough slices to gain information about the whole $\mathbb{R}$, even if they cannot be pasted together in a natural way. The next definition formalises this idea.

Definition 24. Let $\mathbb{R}$ be an $\mathcal{M}$-set. A **covering** of $\mathbb{R}$ is a family $\{S_i\}_{i\in I} \subseteq \mathcal{S}(\mathbb{R})$ such that $\bigcup_{i\in I} S_i = R$.

We are now ready to give our insight on the inner structure of $\mathcal{M}$-sets that are onto-projective in $\mathcal{M}$-`Set`. Drawing consequences from our theorem we will obtain also a characterization of the fundamental bricks from which every onto-projective object can be built. In order to do this, given an $\mathcal{M}$-set $\mathbb{R}$ and an element $x \in R$, we denote by $m_x\colon M \to R$ the function defined for every $\sigma \in M$ as $m_x\sigma := \sigma \star_{\mathbb{R}} x$ and extended to the power set in the natural way. It is easy to prove that $m_x\colon \mathbb{M} \to \mathbb{R}$ is a structural transformer.

Theorem 25. *Let $\mathbb{R}$ be an $\mathcal{M}$-set. The following conditions are equivalent:*

(i) *$\mathbb{R}$ is onto-projective in $\mathcal{M}$-`Set`.*
(ii) *There is a covering of $\mathbb{R}$ whose elements are retracts of $\mathbb{M}$.*
(iii) *$\mathcal{A}(\mathbb{R})$ is a set of retracts of $\mathbb{M}$.*
(iv) *$\mathbb{R}$ is (isomorphic to) the coproduct of a family of retracts of $\mathbb{M}$.*

Proof. (i)⇒(ii): Since $\mathbb{R}$ is onto-projective we know by Theorem 18 that $\phi\colon \coprod_{x\in R} \mathbb{M} \to \mathbb{R}$ is a retraction, i.e., there is a structural transformer $\tau\colon \mathbb{R} \to \coprod_{x\in R} \mathbb{M}$ such that $\phi\tau = id_{\mathbb{R}}$. Then for every $x \in R$ we define a function $\tau_x\colon R \to \mathcal{P}(M)$ as $\tau_x y := \big\{\sigma \in M : \langle \sigma, x\rangle \in \tau\{y\}\big\}$ for every $y \in R$. It is easy to check that τ_x is a structural transformer, when extended to the power set of R: if $\sigma \in M$ and $y \in R$, we have that

$$\begin{aligned}\tau_x\{\sigma \star_{\mathbb{R}} y\} &= \big\{\sigma' \in M : \langle \sigma', x\rangle \in \tau\{\sigma \star_{\mathbb{R}} y\}\big\}\\ &= \big\{\sigma' \in M : \langle \sigma', x\rangle \in \sigma \star_{\coprod} \tau\{y\}\big\}\\ &= \sigma \cdot \big\{\sigma'' \in M : \langle \sigma'', x\rangle \in \tau\{y\}\big\}\\ &= \sigma \cdot \tau_x\{y\}.\end{aligned}$$

Moreover $\bigcup_{x\in R} m_x\tau_x = id_{\mathbb{R}}$, because for each $y \in R$ we have that set of R: if $\sigma \in M$ and $y \in R$, we have that

$$\begin{aligned}\bigcup_{x\in R} m_x\tau_x\{y\} &= \bigcup_{x\in R} m_x\big\{\sigma \in M : \langle \sigma, x\rangle \in \tau\{y\}\big\}\\ &= \bigcup_{x\in R}\big\{\sigma \star_{\mathbb{R}} x : \langle \sigma, x\rangle \in \tau\{y\}\big\}\\ &= \{y\}.\end{aligned}$$

As a consequence we get that $m_x\tau_x \subseteq id_{\mathbb{R}}$ for every $x \in R$. This means in particular that $m_x\tau_x\{y\} \in \big\{\emptyset, \{y\}\big\}$ for every $y \in R$.

Now for every $x \in R$, we put $S_x \coloneqq \{y \in R : m_x\tau_x\{y\} = \{y\}\}$ and check that S_x is a slice of $\mathbb{R}$ for every $x \in R$. If $S_x \in \{\emptyset, R\}$ we are done. Suppose the contrary. Let $\sigma \in M$ and $y \in S$. We have that $\emptyset \neq \sigma \star_{\mathbb{R}} \{y\} = \sigma \star_{\mathbb{R}} m_x\tau_x\{y\} = m_x\tau_x\{\sigma \star_{\mathbb{R}} y\}$. Since $m_x\tau_x\{\sigma \star_{\mathbb{R}} y\} \in \big\{\emptyset, \{\sigma \star_{\mathbb{R}} y\}\big\}$, this yields that $m_x\tau_x\{\sigma \star_{\mathbb{R}} y\} = \{\sigma \star_{\mathbb{R}} y\}$ and therefore that $\sigma \star_{\mathbb{R}} y \in S_x$. Now let $y \in R \setminus S_x$. We have that $m_x\tau_x\{\sigma \star_{\mathbb{R}} y\} = \sigma \star_{\mathbb{R}} m_x\tau_x\{y\} = \sigma \star_{\mathbb{R}} \emptyset = \emptyset$ and therefore $\sigma \star_{\mathbb{R}} y \in R \setminus S_x$. We conclude that π_S is a graduation on $\mathbb{R}$ and therefore that S_x is a slice of $\mathbb{R}$ for every $x \in R$.

We turn now to check that $\{S_x\}_{x\in R}$ is a covering of $\mathbb{R}$. For every $y \in R$ we have $\bigcup_{x\in R} m_x\tau_x\{y\} = \{y\}$. Hence there is $x \in R$ such that $m_x\tau_x\{y\} = \{y\}$, and therefore such that $y \in S_x$. We conclude that $\bigcup_{x\in R} S_x = R$.

It only remains to prove that every non-empty element of $\{S_x\}_{x\in R}$ is a retract of $\mathbb{M}$. Let $S_x \neq \emptyset$, we consider two transformers: the first one is given by the function $\tau_x \restriction_{S_x} : \mathbb{S}_x \to \mathbb{M}$ and is clearly a structural transformer.

We construct the second transformer $m_x^{\perp} : \mathbb{M} \to \mathbb{S}_x$ starting with the function $m_x^{\perp} : M \to \mathcal{P}(S_x)$ defined as

$$m_x^{\perp}\sigma \coloneqq \begin{cases}\{\sigma \star_{\mathbb{R}} x\} & \text{if } \sigma \star_{\mathbb{R}} x \in S_x\\ \emptyset & \text{otherwise}\end{cases}$$

for every $\sigma \in M$ and extending it to power sets as usual. Now we prove that it is a structural transformer. Let $\sigma, \sigma' \in M$. We have two cases:

1) $m_x\{\sigma' \cdot \sigma\} \subseteq S_x$. Then $\sigma' \star_{\mathbb{R}} (\sigma \star_{\mathbb{R}} x) \in S_x$. Since S_x is a slice this yields that $m_x\{\sigma\} = \{\sigma \star_{\mathbb{R}} x\} \subseteq S_x$. This yields that $m_x^{\perp}\{\sigma' \cdot \sigma\} = \{(\sigma' \cdot \sigma) \star_{\mathbb{R}} x\} = \{\sigma' \star_{\mathbb{R}} (\sigma \star_{\mathbb{R}} x)\} = \sigma' \star_{\mathbb{R}} m_x^{\perp}\{\sigma\} = \sigma' \star_{\mathbb{S}} m_x^{\perp}\{\sigma\}$;
2) $m_x\{\sigma' \cdot \sigma\} \nsubseteq S_x$. Then $\sigma' \star_{\mathbb{R}} (\sigma \star_{\mathbb{R}} x) \notin S_x$. Since S_x is a slice this yields that $m_x\{\sigma\} = \{\sigma \star_{\mathbb{R}} x\} \nsubseteq S_x$. This yields that $m_x^{\perp}\{\sigma' \cdot \sigma\} = \emptyset = \sigma' \star_{\mathbb{R}} \emptyset = \sigma' \star_{\mathbb{R}} m_x^{\perp}\{\sigma\} = \sigma' \star_{\mathbb{S}} m_x^{\perp}\{\sigma\}$.

Finally we prove that $m_x^{\perp}$ is a retraction: Observe that for every $y \in S_x$ we have that $m_x(\tau_x\{y\}) = \{y\} \subseteq S_x$ which yields $m_x^{\perp}\tau_x\{y\} = m_x\tau_x\{y\}$. Therefore we conclude that $m_x^{\perp}\tau_x \upharpoonright_{S_x} \{y\} = m_x^{\perp}\tau_x\{y\} = m_x\tau_x\{y\} = \{y\}$, that is $m_x^{\perp}\tau_x \upharpoonright_{S_x} = id_{\mathbb{S}_x}$. We conclude that the family $\{S_x : x \in R \text{ and } S_x \neq \emptyset\}$ is a covering of $\mathbb{R}$ whose elements are retracts of $\mathbb{M}$.

(ii)⇒(iii): Assume that there is a covering $\{S_i\}_{i\in I}$ of $\mathbb{R}$ whose elements are retracts of $\mathbb{M}$. Then let $\mathcal{S}\{x\} \in \mathcal{A}(\mathbb{R})$. Since $\{S_i\}_{i\in I}$ is a covering of $\mathbb{R}$, there is an $i \in I$ such that $x \in S_i$ and therefore such that $\mathcal{S}\{x\} \subseteq S_i$. Since $\mathbb{S}_i$ is a retract of $\mathbb{M}$, there are two structural transformers $\tau\colon \mathbb{S}_i \longleftrightarrow \mathbb{M} \colon \rho$ such that $\rho\tau = id_{\mathbb{S}_i}$. Therefore we define a transformer starting with the function $\pi\colon S_i \to \mathcal{P}(\mathcal{S}\{x\})$ defined as

$$\pi y \coloneqq \begin{cases} \{y\} & \text{if } y \in \mathcal{S}\{x\} \\ \emptyset & \text{otherwise} \end{cases}$$

for every $y \in S_i$. We check that π is a structural transformer, when lifter to the power set. Let $\sigma \in M$ and $y \in S_i$. We have two cases:

1) $\sigma \star_{\mathbb{S}_i} y \in \mathcal{S}\{x\}$. Since $\mathcal{S}\{x\}$ is a slice this yields that $y \in \mathcal{S}\{x\}$. Hence we have that $\pi\{\sigma \star_{\mathbb{S}_i} y\} = \{\sigma \star_{\mathbb{S}_i} y\} = \{\sigma \star_{\mathcal{S}\{x\}} y\} = \sigma \star_{\mathcal{S}\{x\}} \pi\{y\}$;
2) $\sigma \star_{\mathbb{S}_i} y \notin \mathcal{S}\{x\}$. Since $\mathcal{S}\{x\}$ is a slice this yields that $y \notin \mathcal{S}\{x\}$, therefore we have that $\pi\{\sigma \star_{\mathbb{S}_i} y\} = \emptyset = \sigma \star_{\mathcal{S}\{x\}} \emptyset = \sigma \star_{\mathcal{S}\{x\}} \pi\{y\}$.

We conclude that π is a structural transformer. It is also clear that $\pi\iota = id_{\mathcal{S}\{x\}}$, where ι is the inclusion structural transformer from $\mathcal{S}\{x\}$ into $\mathbb{S}_i$. Therefore the following diagram commutes and $\mathcal{S}\{x\}$ is a retract of $\mathbb{M}$.

$$\begin{tikzcd}
\mathcal{S}\{x\} \arrow[r, "\iota"] \arrow[d, "id_{\mathcal{S}\{x\}}"'] & \mathbb{S}_i \arrow[r, "\tau"] \arrow[d, "id_{\mathbb{S}_i}"'] & \mathbb{M} \arrow[dl, "\rho"] \\
\mathcal{S}\{x\} & \mathbb{S}_i \arrow[l, "\pi"] &
\end{tikzcd}$$

(iii)⇒(iv) follows from Lemma 23. For (iv)⇒(i), observe that by Lemma 16 and Theorem 18 retracts of $\mathbb{M}$ are onto-projective, and apply Lemma 17. $\square$

We believe that this result gives more information about the inner structure of onto-projective $\mathcal{M}$-sets, i.e., it tells us that they are built up pasting together retracts of $\mathbb{M}$. However the flavour of this characterization is still a categorical one, as it makes use of the notion of retraction. This is only an outward obstacle, as we will show in the next result. In order to do this José Gil-Férez suggested us the following new notion of variable.

Definition 26. Let $\langle R, \star_{\mathbb{R}}, \pi\rangle$ be a graded $\mathcal{M}$-set. A function $\nu\colon I \to R$ is a **generalised graded variable** if $\pi\nu = id_I$ and there is a structural transformer $\tau\colon \mathbb{R} \to \mathbb{M}$ such that $m_{\nu(i)}\tau = id_{\mathbb{R}_{\upharpoonright i}}$ for every $i \in I$.

We are going to see that this is the right notion of variable, in the sense that it characterises $\mathcal{M}$-sets with the REP. We believe this fact yields a deeper undestanding of the representation problem in the context of $\mathcal{M}$-sets, since it points out that in $\mathcal{M}$-$\texttt{Set}$ the objects with the REP are very close to the usual notion of a logic (which is defined on some kind of language) since they enjoy some sort of variable, while in the context of $\mathcal{P}(\mathcal{M})$-$\texttt{Mod}$ it is not the case as we will show in Example 29. Note that every graded variable in the sense of [13] induces a generalised graded variable, but the converse does not hold since graded variables send singletons to singletons.

Example 27. In Example 10 we considered a monoid $\mathcal{M}$ and and the trivial $\mathcal{M}$-set $\mathbb{T}$. Now let $\coprod_{n\in\{0,1\}}\mathbb{T}_n$ be the coproduct of two copies of $\mathbb{T}$. In Example 10 we proved that $\mathbb{T}$ has the REP, therefore by Lemma 17 and Theorem 18 we conclude that $\coprod_{n\in\{0,1\}}\mathbb{T}_n$ has the REP too. We now turn to prove that $\coprod_{n\in\{0,1\}}\mathbb{T}_n$ has a generalised graded variable. We first consider the graduation π naturally associated with the coprodut. It is easy to check that $\nu\colon\{0,1\}\to\coprod_{n\in\{0,1\}}\mathbb{T}_n$ defined as $\nu(n)\coloneqq\langle a,n\rangle$ for every $n\in\{0,1\}$, with the structural transformer $\tau\colon\coprod_{n\in\{0,1\}}\mathbb{T}_n\to\mathbb{M}$ defined as $\tau\{x\}\coloneqq\{0,1\}$ for every $x\in\big\{\langle a,n\rangle : n\in\{0,1\}\big\}$, is a generalised graded variable for $\langle\big\{\langle a,n\rangle : n\in\{0,1\}\big\},\star_{\coprod},\pi\rangle$.

But $\coprod_{n\in\{0,1\}}\mathbb{T}_n$ has neither a generalised variable, nor a graded one. It is clear that $\coprod_{n\in\{0,1\}}\mathbb{T}_n$ has no generalised variable, since there is no element $x\in\{\langle a,n\rangle : n\in\{0,1\}\}$ which satisfies condition G1. Moreover it is easy to prove that it cannot be equipped with a graded variable since there is no coherent family of substitutions $\tau\colon\big\{\langle a,n\rangle : n\in\{0,1\}\big\}\to M$.

The following result provides a characterization of $\mathcal{M}$-sets only in terms of their inner structure and establishes a nice connection with the work of [12, 13].

Theorem 28. *Let $\mathbb{R}$ be an $\mathcal{M}$-set. The following conditions are equivalent:*

(i) $\mathbb{R}$ *has the REP;*
(ii) $\mathbb{R}$ *can be given the structure of a graded $\mathcal{M}$-set with a generalised graded variable;*
(iii) $\mathbb{R}$ *can be given the structure of a graded $\mathcal{M}$-set such that each layer has a generalised variable;*
(iv) $\mathcal{P}(\mathbb{R})\cong\coprod_{i\in I}\mathcal{P}(\mathbb{S}_i)$ *for a family $\{\mathcal{P}(\mathbb{S}_i)\}_{i\in I}$ of cyclic and projective $\mathcal{P}(\mathcal{M})$-modules.*

Proof. (i)⇒(ii): Recall from Lemma 23 that $\mathbb{R}\cong\coprod_{S\in\mathcal{A}(\mathbb{R})}\mathbb{S}$, therefore it will be enough to prove that $\coprod_{S\in\mathcal{A}(\mathbb{R})}\mathbb{S}$ can be equipped with the structure of a graded $\mathcal{M}$-set with a generalised graded variable. As we remarked after Theorem 13, the fact that $\coprod_{S\in\mathcal{A}(\mathbb{R})}\mathbb{S}$ is a coproduct implies that $\langle\{\langle x,S\rangle : x\in S\},\star_{\coprod},\pi\rangle$ is a graded $\mathcal{M}$-set, where $\pi\colon\{\langle x,S\rangle : x\in S\}\to\mathcal{A}(\mathbb{R})$ is defined as $\pi\langle y,\mathcal{S}\{x\}\rangle\coloneqq\mathcal{S}\{x\}$.

Now, by Theorem 25 we know that $\mathcal{A}(\mathbb{R})$ is a set of retracts of $\mathbb{M}$. This means that for every $S\in\mathcal{A}(\mathbb{R})$ there are two structural transformers $\tau_{\mathbb{S}}\colon\mathbb{S}\to\mathbb{M}$ and $\rho_{\mathbb{S}}\colon\mathbb{M}\to\mathbb{S}$ such that $\rho_{\mathbb{S}}\tau_{\mathbb{S}}=id_{\mathbb{S}}$. Then consider the map $\nu\colon\mathcal{A}(\mathbb{R})\to$

$\{\langle x, S\rangle : x \in S\}$ defined as $\nu(S) = \langle \rho_{\mathbb{S}}(1), S\rangle$ for every $S \in \mathcal{A}(\mathbb{R})$. It is well-defined since $\rho_{\mathbb{S}} \colon \mathbb{M} \to \mathbb{S}$ for every $S \in \mathcal{A}(\mathbb{R})$.

We claim that ν is a generalised graded variable. First observe that $\pi\nu = id_{\mathcal{A}(\mathbb{R})}$. Moreover by the universal property of the coproduct there is a structural transformer $\tau \colon \coprod_{S \in \mathcal{A}(\mathbb{R})} \mathbb{S} \to \mathbb{M}$ such that $\tau\pi_{\mathbb{S}} = \tau_{\mathbb{S}}$ for every $S \in \mathcal{A}(\mathbb{R})$, where $\pi_{\mathbb{S}} \colon \mathbb{S} \to \coprod_{S \in \mathcal{A}(\mathbb{R})} \mathbb{S}$ is defined as $\pi\{x\} = \big\{\langle x, \mathcal{S}\{x\}\rangle\big\}$ for every $x \in S$. If $\langle x, S\rangle \in \coprod_{S \in \mathcal{A}(\mathbb{R})} \mathbb{S}$, then $m_{\nu(S)}\tau\{\langle x, S\rangle\} = m_{\nu(S)}\tau\pi_{\mathbb{S}}\{x\} = m_{\nu(S)}\tau_{\mathbb{S}}\{x\} = \tau_{\mathbb{S}}\{x\} \star_{\coprod} \nu(S) = \tau_{\mathbb{S}}\{x\} \star_{\coprod} \langle \rho_{\mathbb{S}}(1), S\rangle = \langle \tau_{\mathbb{S}}\{x\} \star_{\mathbb{S}} \rho_{\mathbb{S}}(1), S\rangle = \langle \rho_{\mathbb{S}}(\tau_{\mathbb{S}}\{x\} \cdot 1), S\rangle = \langle \rho_{\mathbb{S}}\tau_{\mathbb{S}}\{x\}, S\rangle = \{\langle x, S\rangle\}$.

(ii)$\Rightarrow$(iii) Let $\pi \colon R \to I$ be a graduation and $\nu \colon I \to R$ a generalised graded variable with corresponding structural transformer $\tau \colon \mathbb{R} \to \mathbb{M}$. We have to prove that each layer $\mathbb{R}_{\restriction i}$ has a generalised variable. Then let $i \in I$, we claim that $\nu(i)$ is a generalised variable for the layer $\mathbb{R}_{\restriction i}$ with corresponding set of substitutions $\tau\{\nu(i)\}$. The proof is very easy. First observe that $\tau\{\nu(i)\} \star_{\mathbb{R}} \nu(i) = m_{\nu(i)}\tau\{\nu(i)\} = \{\nu(i)\}$. Moreover we have that $R_{\restriction i} = m_{\nu(i)}\tau[R] \subseteq m_{\nu(i)}[M] = \{\sigma \star_{\mathbb{R}} \nu(i) : \sigma \in M\}$. Since $\nu(i) \in R_{\restriction i}$ and $\mathbb{R}_{\restriction i}$ is a layer, we conclude that $R_{\restriction i} = \{\sigma \star_{\mathbb{R}} \nu(i) : \sigma \in M\}$. Finally let $\sigma, \sigma' \in M$ such that $\sigma \star_{\mathbb{R}_{\restriction i}} \{\nu(i)\} = \sigma' \star_{\mathbb{R}_{\restriction i}} \{\nu(i)\}$. Then clearly $\{\sigma\} \cdot \tau\{\nu(i)\} = \tau\big(\{\sigma\} \star_{\mathbb{R}_{\restriction i}} \{\nu(i)\}\big) = \tau\big(\{\sigma'\} \star_{\mathbb{R}_{\restriction i}} \{\nu(i)\}\big) = \{\sigma'\} \cdot \tau\{\nu(i)\}$.

(iii)$\Rightarrow$(iv). Let $\pi \colon R \to I$ be the stated graduation. Since $\mathbb{R} \cong \coprod_{i \in I} \mathbb{R}_{\restriction i}$, as we remarked after Theorem 13, by Theorem 13 we have that

$$\mathcal{P}(\mathbb{R}) \cong \mathcal{P}\big(\textstyle\coprod_{i \in I} \mathbb{R}_{\restriction i}\big) \cong \textstyle\coprod_{i \in I} \mathcal{P}\big(\mathbb{R}_{\restriction i}\big).$$

Therefore it only remains to prove that $\mathcal{P}\big(\mathbb{R}_{\restriction i}\big)$ is cyclic and projective for every $i \in I$. This follows from Theorem 11 because by assumption $\mathbb{R}_{\restriction i}$ has a generalised variable, for each $i \in I$.

(iv)$\Rightarrow$(i): By [11, Lemma 5.12], projectiveness in $\mathcal{P}(\mathcal{M})$-`Mod` is preserved under coproducts, therefore we conclude that $\mathcal{P}(\mathbb{R})$ is projective in $\mathcal{P}(\mathcal{M})$-`Mod`. Then we apply Theorem 18 and are done. □

Before concluding let us remark a curious fact. It turned out that $\mathcal{M}$-sets which are retracts of $\mathbb{M}$ (or equivalently $\mathcal{M}$-sets with a generalised variable) are the fundamental bricks from which $\mathcal{M}$-sets with the REP are built. This allows us to infer that the number (up to isomorphism) of such fundamental bricks can be bounded trivially by $2^{2^{|\mathcal{M}|}}$. This means for example that when $\mathcal{M}$ is finite the number of these bricks is finite.

We consider that Theorem 28 provides a solution to the representation problem in the context of $\mathcal{M}$-sets; therefore a natural question is whether this solution can be translated back to the context of $\mathcal{A}$-modules, where $\mathcal{A}$ is an arbitrary complete residuated lattice. More precisely, Theorem 28 tells us that $\mathcal{P}(\mathbb{R})$ is projective in $\mathcal{P}(\mathcal{M})$-`Mod` if and only if it is (isomorphic to) a coproduct of cyclic and projective $\mathcal{P}(\mathcal{M})$-modules coming from $\mathcal{M}$-sets. The best approximation to this result in $\mathcal{A}$-`Mod` would be the following: $\boldsymbol{R}$ is a projective $\mathcal{A}$-module if and only if $\boldsymbol{R}$ is (isomorphic to) the coproduct of cyclic and projective $\mathcal{A}$-modules. However, the next counterexample shows that this is not the case, and therefore that our characterisation is intrinsic to the context of $\mathcal{M}$-sets.

Example 29. Let $\mathcal{N} \coloneqq \langle\{1\}, \cdot, 1\rangle$ be the trivial monoid and consider the category $\mathcal{P}(\mathcal{N})$-`Mod`. Using condition (iii) of Theorem 8, it is easy to see that the only cyclic and projective $\mathcal{P}(\mathcal{N})$-modules are, up to isomorphism, $\mathcal{P}(\mathbb{N}) \coloneqq \big\langle\big\{\emptyset, \{1\}\big\}, \cdot\big\rangle$ and the trivial module $\boldsymbol{R} \coloneqq \langle\{a\}, *_{\boldsymbol{R}}\rangle$. Therefore non-trivial coproducts of cyclic and projective $\mathcal{P}(\mathcal{N})$-modules have the form $\coprod_{i\in I} \mathcal{P}(\mathbb{N})$ for some I. By Theorem 13, this implies that they coincide (up to isomorphism) with power set lattices.

Due to the really simple structure of $\mathcal{P}(\mathcal{N})$, it is easy to prove that, given two $\mathcal{P}(\mathcal{N})$-modules $\mathbb{R}$ and $\mathbb{S}$, a module morphism $\tau\colon \mathbb{R} \to \mathbb{S}$ is just a residuated map from $\mathcal{P}(\mathbb{R})$ to $\mathcal{P}(\mathbb{S})$. Moreover each complete lattice can be easily equipped with an action that converts it into a $\mathcal{P}(\mathcal{N})$-module. Therefore the category $\mathcal{P}(\mathcal{N})$-`Mod` turns out to coincide with the category of complete lattices with residuated mappings between them. Now, in [6] it is proved that in this last category projective objects coincide with completely distributive complete lattices, therefore we conclude that projective $\mathcal{P}(\mathcal{N})$-modules are exactly $\mathcal{P}(\mathcal{N})$-modules whose lattice reduct is completely distributive. As a consequence we get that every finite distributive $\mathcal{P}(\mathcal{N})$-module is projective in $\mathcal{P}(\mathcal{N})$-`Mod`. Since clearly there are finite $\mathcal{P}(\mathcal{N})$-modules which are not (isomorphic to) power set lattices (for example finite chains of more than 2 elements) we conclude that there are projective $\mathcal{P}(\mathcal{N})$-modules which are not coproducts of cyclic and projective $\mathcal{P}(\mathcal{N})$-modules.

Finally let us return to the original setting where the "isomorphism problem", as a forerunner of the "representation problem", appeared; namely, let us see that, for $\mathcal{M}$-sets with the REP, the converse of Theorem 3 actually holds. In order to do this, given a pair of structural closure operators γ and δ on two $\mathcal{M}$-sets $\mathbb{R}$ and $\mathbb{S}$ we say a function $\Phi\colon \mathcal{P}(R)_\gamma \to \mathcal{P}(S)_\delta$ is an **equivalence between** γ and δ if it is a surjective structural representation of γ into δ; then, Φ is a structural isomorphism between the complete lattices $\mathcal{P}(\mathbb{R})_\gamma$ and $\mathcal{P}(\mathbb{S})_\delta$.

Theorem 30. *Let γ and δ be structural closure operators respectively on two $\mathcal{M}$-sets $\mathbb{R}$ and $\mathbb{S}$ with the REP. If there is an equivalence $\Phi\colon \mathcal{P}(R)_\gamma \to \mathcal{P}(S)_\delta$ between γ and δ, then there exist structural transformers $\tau\colon \mathbb{R} \longleftrightarrow \mathbb{S} : \rho$ satisfying* A1′ *and* A2′ *and such that $\tau^+ = \Phi^{-1}$ on $\mathcal{P}(S)_\delta$ and $\rho^+ = \Phi$ on $\mathcal{P}(R)_\gamma$.*

Proof. If Φ is an equivalence, then in particular both Φ and Φ^{-1} are structural representations, of γ into δ and conversely. Since the two $\mathcal{M}$-sets have the REP, Φ and Φ^{-1} are induced respectively by two structural transformers $\tau\colon \mathbb{R} \longleftrightarrow \mathbb{S} : \rho$; this means that the following diagram commutes in both senses

$$\begin{tikzcd}
\mathcal{P}(\mathbb{R}) \arrow[r, shift right, "\tau"'] \arrow[d, twoheadrightarrow, "\gamma"'] & \mathcal{P}(\mathbb{S}) \arrow[l, shift right, "\rho"'] \arrow[d, twoheadrightarrow, "\delta"] \\
\mathcal{P}(\mathbb{R})_\gamma \arrow[r, shift right, "\Phi"'] & \mathcal{P}(\mathbb{S})_\delta \arrow[l, shift right, "\Phi^{-1}"']
\end{tikzcd}$$

that is, that, $\Phi\gamma = \delta\tau$ and $\Phi^{-1}\delta = \gamma\rho$. Now, the fact that ρ induces the inverse of the representation induced by τ implies that $\delta\tau\rho = \Phi\gamma\rho = \Phi\Phi^{-1}\delta = \delta$, which is A2$'$, and the dual $\gamma\rho\tau = \gamma$ is proved similarly. The commutativity conditions imply that $\gamma = \Phi^{-1}\delta\tau$ and $\delta = \Phi\gamma\rho$, so in order to prove A1$'$ and its dual it will be enough to show that $\tau^+ = \Phi^{-1}$ on $\mathcal{P}(S)_\delta$ and $\rho^+ = \Phi$ on $\mathcal{P}(R)_\gamma$. Let us show the second equality, that is, that for each $X \subseteq R$, $\Phi\gamma X = \rho^+\gamma X$, or equivalently that $\delta\tau X = \rho^+\gamma X$. By the definition of residual mapping, we have to show that

$$\delta\tau X = \bigcup\big\{Y \subseteq S : \rho Y \subseteq \gamma X\big\}. \tag{2}$$

On the one hand, we have, by properties of the closures and those already shown, that

$$\rho\,\delta\tau X \subseteq \gamma\rho\delta\tau X = \Phi^{-1}\delta\delta\tau X = \Phi^{-1}\delta\tau X = \gamma X.$$

On the other hand, assume $\rho Y \subseteq \gamma X$ for some $Y \subseteq S$. Then $\Phi^{-1}\delta Y = \gamma\rho Y \subseteq \gamma\gamma X = \gamma X$, and since Φ is order-preserving, we obtain that $Y \subseteq \delta Y = \Phi\Phi^{-1}\delta Y \subseteq \Phi\gamma X = \delta\tau X$. This completes the proof of (2), and hence of the equality $\rho^+ = \Phi$ on $\mathcal{P}(R)_\gamma$. The proof of the other equality is similar. □

Acknowledgements. We want to thank José Gil-Férez for the inspiration we gained from his early work on the topic, for his care in reading a first version of the paper and for his many suggestions to improve it. The first author was partially funded by the research project MTM2011-25747 from the government of Spain, which includes FEDER funds from the European Union, and the research grant 2009SGR-1433 from the government of Catalonia. The second author wants to thank his parents for funding his studies and Amanda Vidal for everything else.

References

[1] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and concrete categories: the joy of cats*, Number 17 of Reprints in Theory and Applications of Categories, 2006. Reprint of the 1990 original [Wiley, New York].

[2] W. J. Blok and B. Jónsson. *Equivalence of consequence operations*, Studia Logica, 83(1-3):91–110, 2006.

[3] W. J. Blok and D. Pigozzi. *Algebrizable logics*, Volume 396 of Mem. Amer. Math. Soc., Providence, January 1989.

[4] W. J. Blok and D. Pigozzi. *Local deduction theorems in algebraic logic*, In Algebraic Logic, H. Andreka, J. D. Monk, and I. Nemeti, Eds., vol. 54 of Colloq. Math. Soc. Janos Bolyai. North-Holland, Amsterdam, 1991, pp. 75–109.

[5] S. Burris and H. P. Sankappanavar. *A course in universal algebra*, Springer-Verlag Graduate Texts in Mathematics, 1981.

[6] G. D. Crown. *Projectives and injectives in the category of complete lattices with residuated mappings*, Math. Ann., 187:295–299, 1970.

[7] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, Second revised edition, vol. 7 of Lecture Notes in Logic. Association for Symbolic Logic, 2009. Electronic version freely available through Project Euclid at http://projecteuclid.org/euclid.lnl/1235416965.

[8] J. M. Font, R. Jansana, and D. Pigozzi. *A survey of abstract algebraic logic*, Studia Logica (Special issue on Abstract Algebraic Logic, Part II), 74:13–97, 2003. With an update in vol. 91:125–130, 2009.

[9] N. Galatos and J. Gil-Férez. *Modules over quantaloids: Applications to the isomorphism problem in algebraic logic and π-institutions*, Journal of Pure and Applied Algebra, to appear.

[10] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: an algebraic glimpse at substructural logics*, Volume 151 of Studies in Logic and the Foundations of Mathematics. Elsevier, Amsterdam, 2007.

[11] N. Galatos and C. Tsinakis. *Equivalence of closure operators: an order-theoretic and categorical perspective*, Journal of Symbolic Logic, 74(3):780–810, 2009.

[12] J. Gil-Férez. *Categorical application to abstract algebraic logic*, Ph.D. Dissertation, University of Barcelona, December 2009. Chapters 2 and 5 published as [13] and [9] respectively.

[13] J. Gil-Férez. *Representations of structural closure operators*, Archive for Mathematical Logic, 50:45–73, 2011.

[14] J. Raftery. *Correspondences between Gentzen and Hilbert systems*, The Journal of Symbolic Logic 71:903–957, 2006.

[15] J. Rebagliato and V. Verdú. *On the algebraization of some Gentzen systems*, Fundamenta Informaticae (Special issue on Algebraic Logic and its Applications) 18:319–338, 1993.

Department of Probability, Logic and Statistics, University of Barcelona, Gran Via 585, 08007 Barcelona, Spain

E-mail address: jmfont@ub.edu , tommaso.moraschini@gmail.com