

LOGICS OF VARIETIES, LOGICS OF SEMILATTICES, AND CONJUNCTION

JOSEP MARIA FONT AND TOMMASO MORASCHINI

ABSTRACT. This paper starts with a general analysis of the problem of how to associate a logic with a given variety of algebras, and shows that it has a positive solution for two of the standard procedures of performing this association, and a negative one in the third. Then the paper focuses on the case of the “logics of semilattices”, which are defined as the logics related to the variety of semilattices via two of the standard procedures of abstract algebraic logic. We study their main properties, classify them in the Leibniz hierarchy and the Frege hierarchy (the two hierarchies of abstract algebraic logic), and study the poset they form (in particular, we find its least element and its two unique maximals, and prove it is atomless). Even if there is an infinity of such logics of semilattices, it is not known whether there are logics related to the variety of semilattices via the Leibniz reduction too; we discuss this issue and provide a partial solution to this problem. The final section studies one of the maximals of this poset, the conjunctive fragment of classical propositional logic; among other properties we give two new characterisations of this logic, one of them in terms of a property of the Leibniz operator.

INTRODUCTION

The main goal of this paper is to study, with the tools of abstract algebraic logic, a (rather large) class of logics that are associated with the variety of semilattices. Its starting point, however, is actually the general issue of studying the class of all logics associated with a given variety of algebras, of an arbitrary signature. To give the phrase “associated with” a precise, uniform meaning, we build upon the fact that in abstract algebraic logic there are three standard procedures to associate a class of algebras with a given logic \mathcal{L} : the class $\text{Alg}^*\mathcal{L}$ of the *Leibniz-reduced algebras* of \mathcal{L} , its *algebraic counterpart* $\text{Alg}\mathcal{L}$, which is the class of its Tarski-reduced algebras, and its *intrinsic variety* $\forall\mathcal{L}$ (Section 1 contains all needed definitions); note that the first two need not be varieties. Section 2 starts by posing the questions of whether, given a variety of algebras \mathbf{V} , there is a logic associated with \mathbf{V} through any of these three procedures. We provide the example of a variety which is not of the form $\text{Alg}^*\mathcal{L}$ for any logic \mathcal{L} (Example 2.1), but for each variety (of an arbitrary signature) \mathbf{V} we construct a logic $\mathcal{L}_{\mathbf{V}}$ (Definition 2.2) such that $\text{Alg}\mathcal{L}_{\mathbf{V}} = \forall\mathcal{L}_{\mathbf{V}} = \mathbf{V}$ (Theorem 2.5), thus giving an affirmative answer to the other two questions. We study the main properties of $\mathcal{L}_{\mathbf{V}}$, classify it under the two hierarchies of abstract algebraic logic, and show it is the weakest logic \mathcal{L} such that $\forall\mathcal{L} = \mathbf{V}$ (Corollary 2.10). Finally, we show (Corollary 2.12) that if there is a positive answer to the first question, then the logic $\mathcal{L}_{\mathbf{V}}$ will be such a positive answer as well, and indeed the weakest one. The

conclusion of this section is that “every variety is the variety of a logic” in two of the standard senses.

The rest of the paper deals with the class of logics associated with the variety of semilattices \mathbf{SL} in the second of the standard ways. We call *logics of semilattices*¹ the logics \mathcal{L} in the signature $\langle \cdot \rangle$ of type $\langle 2 \rangle$ such that $\mathbf{Alg} \mathcal{L} = \mathbf{SL}$. Semilattices form the ordered skeleton of a very large number of algebras arising in the field of logic; most of the times the semilattice operation of these algebras is taken to describe the behaviour of the conjunction of the logic, while in a few others it represents disjunction. Algebraically, semilattices are extremely well-behaved structures, with a pleasant equational theory and an easily solvable word problem. Thus, we find these structures to be of enough independent interest, and devote Sections 3 and 4 to the study of their corresponding logics.

One of these logics is, obviously, $\mathcal{L}_{\mathbf{SL}}$; we characterise logics of semilattices as the non-trivial extensions of $\mathcal{L}_{\mathbf{SL}}$, and show that their intrinsic variety is \mathbf{SL} as well (Theorem 3.3). We classify these logics in the two hierarchies of abstract algebraic logic, showing that they are never protoalgebraic nor truth-equational, but they are always selfextensional (Lemmas 3.5 and 3.7). We obtain characterisations of the Leibniz operator in an arbitrary semilattice (Lemma 3.8) and of the class of Leibniz-reduced matrices of $\mathcal{L}_{\mathbf{SL}}$ (Theorem 3.9). As an approximation to the problem of whether \mathbf{SL} is the class of Leibniz-reduced algebras of $\mathcal{L}_{\mathbf{SL}}$, which is still OPEN, we show that this class (obviously a subclass of \mathbf{SL}) contains all semilattices with sectionally finite height (i.e., the meet-semilattices with all elements having finite height), hence all finite semilattices in particular. Remark that a similar question (whether the class of distributive lattices is the class of Leibniz-reduced algebras of the fragment of classical logic with conjunction and disjunction) was answered in the negative in [10]. In a companion paper [13] we have applied the technique developed here to show that in every semilattice with sectionally finite height the Leibniz operator establishes a bijection between a certain family of subsets and the set of all congruences.

In Section 4 we study the poset of all logics of semilattices. Among several properties, we prove that its maximal elements are the fragments of classical logic with just conjunction (\mathcal{L}_{\wedge}) and with just disjunction (\mathcal{L}_{\vee}), that the latter is the only logic in the poset that is not below \mathcal{L}_{\wedge} (Theorem 4.2), and that this poset is atomless (Theorem 4.4) by identifying two infinite descending chains in it. Figure 2 on page 21 depicts the ordering relations in the relevant part of this poset.

Finally in Section 5 we study a particularly important logic of semilattices, namely the logic of classical conjunction \mathcal{L}_{\wedge} . It is a very simple yet powerful, and well-known logic, but we think we have been able to say something new about it. We prove that \mathcal{L}_{\wedge} is the unique logic whose only non-trivial Leibniz-reduced algebra (up to isomorphism) is the two-element semilattice (Theorem 5.5); this is a rather unusual feature, because, while any logic is characterised by the class of its Leibniz-reduced *matrices*, it is in general not possible to characterise it solely in terms of its class of Leibniz-reduced *algebras*. Moreover, we characterise this logic as the unique logic of semilattices whose Leibniz operator disconnects points on filters over arbitrary algebras (Theorem 5.6). We end the paper by characterising the full \mathbf{g} -models of this

¹An alternative obvious name was “semilattice-based logics”, but this has already been used in the algebraic logic literature [9, 16] to denote a related, but more general notion.

logic and displaying a finite presentation of a Gentzen system that is fully adequate for it (Theorem 5.8 and Corollary 5.9).

Among the most relevant literature on the topic of this paper we mention [1, 5, 7, 10, 14, 15, 17, 19]. The construction of $\forall\mathcal{L}$ was already considered by Rautenberg in [20], although he focused on the problem of the finite axiomatizability of logics associated with varieties in this way.

1. PRELIMINARIES

Here we present a brief survey of the main definitions and results of abstract algebraic logic we will make use of along the article; a systematic exposition can be found for example in [2, 3, 8, 11, 12, 22]. We begin by the definition of logic. In order to do this recall that a **closure operator** over a set A is a monotone function $C: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ such that $X \subseteq C(X) = C(C(X))$ for every $X \in \mathcal{P}(A)$ and that a **closure system** on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ closed under arbitrary intersections and such that $A \in \mathcal{C}$. It is well-known that the closed sets (fixed points) of a closure operator form a closure system, which will be denoted as \mathcal{C} , and that given a closure system \mathcal{C} one can construct a closure operator C by letting $C(X) = \bigcap \{Y \in \mathcal{C} \mid X \subseteq Y\}$ for every $X \in \mathcal{P}(A)$. These transformations are indeed inverse to one another.

Then, fixed an algebraic type \mathcal{L} , we denote by Fm the set of formulas over \mathcal{L} built up with countably many variables Var denoted by x, y, z , etc., and by \mathbf{Fm} the corresponding absolutely free algebra with countably many free generators. From now on we will assume that we are working with a fixed algebraic type and that $x \neq y$, unless explicitly warned.

By a **logic** \mathcal{L} we understand a closure operator $C_{\mathcal{L}}: \mathcal{P}(Fm) \rightarrow \mathcal{P}(Fm)$ which is structural in the sense that $\sigma(C_{\mathcal{L}}(\Gamma)) \subseteq C_{\mathcal{L}}(\sigma(\Gamma))$ for every $\Gamma \subseteq Fm$ and every endomorphism (or, equivalently, substitution) $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$. It is worth remarking that finitariness of the closure operator is not assumed. The set of closed sets of a logic (its **theories**) is denoted by $Th\mathcal{L}$. Given $\Gamma \cup \{\varphi\} \subseteq Fm$ we write $\Gamma \vdash_{\mathcal{L}} \varphi$ to denote the fact that $\varphi \in C_{\mathcal{L}}(\Gamma)$. Since \mathcal{L} will denote always an arbitrary logic, we will skip, in the formulation of our results, assumptions like “let \mathcal{L} be a logic”. Given two logics \mathcal{L} and \mathcal{L}' , we will write $\mathcal{L} \leq \mathcal{L}'$ if $C_{\mathcal{L}}(\Gamma) \subseteq C_{\mathcal{L}'}(\Gamma)$ for every $\Gamma \subseteq Fm$; in this case we say that \mathcal{L}' is an **extension** of \mathcal{L} , and also that \mathcal{L} is **weaker** than \mathcal{L}' ; this relation \leq is an ordering relation between logics. The **intersection** of a family of logics $\{\mathcal{L}_i\}_{i \in I}$ is the operator $C_{\mathcal{L}}$ defined, for every $\Gamma \subseteq Fm$, as $C_{\mathcal{L}}(\Gamma) := \bigcap_{i \in I} C_{\mathcal{L}_i}(\Gamma)$; it is easy to prove that this is a logic, and it is clearly the meet of the family with respect to the order \leq just defined.

The above general definition of logic allows for two pathological cases, which we shall need to avoid at some places in our analysis. More precisely, a logic \mathcal{L} is **inconsistent** when $Th\mathcal{L} = \{Fm\}$ and it is **almost inconsistent** when $Th\mathcal{L} = \{\emptyset, Fm\}$. A logic \mathcal{L} is **trivial** when it is either inconsistent or almost inconsistent.

Lemma 1.1. *A logic \mathcal{L} is trivial if and only if $x \vdash_{\mathcal{L}} y$. \(\boxtimes\)*

We now turn to describe how to construct algebra-based semantics for a logic. We will denote algebras with italic boldface capital letters $\mathbf{A}, \mathbf{B}, \mathbf{C}$, etc. (with universes A, B, C , etc. respectively). As in the case of logics, we will skip assumptions like “let \mathbf{A} be an algebra” in the formulation of our results. One of the natural ways of building models for a logic out of algebras is to consider the elements of the algebras

as truth values and select some of them as representing truth. More precisely, given a logic \mathcal{L} and an algebra \mathbf{A} , a set $F \subseteq A$ is a **deductive filter** of \mathcal{L} over \mathbf{A} when

$$\begin{aligned} & \text{if } \Gamma \vdash_{\mathcal{L}} \varphi, \text{ then for every homomorphism } h: \mathbf{Fm} \rightarrow \mathbf{A}, \\ & \text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F \end{aligned}$$

for every $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$. We denote by $\mathcal{F}i_{\mathcal{L}}(\mathbf{A})$ the set of deductive filters of \mathcal{L} over \mathbf{A} , which turns out to be a complete lattice when ordered under set-theoretical inclusion. A pair $\langle \mathbf{A}, F \rangle$ is a **matrix** when \mathbf{A} is an algebra and $F \subseteq A$, and a matrix $\langle \mathbf{A}, F \rangle$ is a **model** of a logic \mathcal{L} when $F \in \mathcal{F}i_{\mathcal{L}}(\mathbf{A})$.

Matrices of the form $\langle \mathbf{A}, F \rangle$ are associated with congruences of \mathbf{A} in a way independent from any logic. In order to explain how, let us fix some notation. Given an algebra \mathbf{A} we will denote by $\text{Co}(\mathbf{A})$ its lattice of congruences, and given $\theta \in \text{Co}(\mathbf{A})$ and $a \in A$ we will denote by $\llbracket a \rrbracket_{\theta}$ the congruence class of a modulo θ . Then $\theta \in \text{Co}(\mathbf{A})$ is **compatible** with the set F when

$$\text{if } a \in F \text{ and } \langle a, b \rangle \in \theta, \text{ then } b \in F$$

for every $a, b \in A$. It is easy to prove that given any $F \subseteq A$, there exists the largest congruence of \mathbf{A} compatible with F . We denote this congruence by ΩF and refer to it as the **Leibniz congruence** of F (over \mathbf{A}). Observe that $\Omega \emptyset = \Omega A = A \times A$. The matrix $\langle \mathbf{A}, F \rangle^* := \langle \mathbf{A}/\Omega F, F/\Omega F \rangle$ is the **Leibniz reduction** of $\langle \mathbf{A}, F \rangle$ and a matrix $\langle \mathbf{A}, F \rangle$ is **Leibniz-reduced** when $\Omega F = \text{Id}_{\mathbf{A}}$, the identity relation on \mathbf{A} . The definition of Leibniz congruence gives rise to a map $\Omega: \mathcal{P}(A) \rightarrow \text{Co}(\mathbf{A})$, called the **Leibniz operator**, whose behaviour over deductive filters of the logic captures interesting facts concerning the definability of truth and that of equivalence in selected classes of logics; this is one of the central topics studied in abstract algebraic logic. However, for the present aim, it is enough to remark the fact that the Leibniz congruence allows us to associate with a logic \mathcal{L} a special class of models and a special class of algebras:

$$\begin{aligned} \text{Mod}^* \mathcal{L} &:= \{ \langle \mathbf{A}, F \rangle : F \in \mathcal{F}i_{\mathcal{L}}(\mathbf{A}) \text{ and } \Omega F = \text{Id}_{\mathbf{A}} \} \\ \text{Alg}^* \mathcal{L} &:= \{ \mathbf{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{L}}(\mathbf{A}) \text{ such that } \Omega F = \text{Id}_{\mathbf{A}} \}. \end{aligned}$$

We will refer to matrices in $\text{Mod}^* \mathcal{L}$ as to **Leibniz-reduced models** of \mathcal{L} , and to algebras in $\text{Alg}^* \mathcal{L}$ as to **Leibniz-reduced algebras** of \mathcal{L} .

Another natural way of building algebra-based models for logics is achieved by using **generalised matrices**, or g-matrices for short. The idea that lies behind this construction is that of modelling the consequence relation of the logic instead of some notion of truth. More precisely, a **g-matrix** is a pair $\langle \mathbf{A}, \mathcal{C} \rangle$ where \mathbf{A} is an algebra and \mathcal{C} a closure system on its universe, and it is a **g-model** of a logic \mathcal{L} when, for every $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$,

$$\text{if } \Gamma \vdash_{\mathcal{L}} \varphi, \text{ then for every homomorphism } h: \mathbf{Fm} \rightarrow \mathbf{A}, h(\varphi) \in C(h[\Gamma]),$$

where C is the closure operator associated with \mathcal{C} . It is easy to see that $\langle \mathbf{A}, \mathcal{C} \rangle$ is a g-model of \mathcal{L} if and only if $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{L}}(\mathbf{A})$.

As is to be expected, there is a natural way to associate a congruence with a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$, as it is easy to prove that the intersection $\bigcap_{F \in \mathcal{C}} \Omega F$ is the largest congruence compatible with every $F \in \mathcal{C}$; we denote this congruence by $\tilde{\Omega} \mathcal{C}$, and refer to it as to the **Tarski congruence** of \mathcal{C} (over \mathbf{A}). It is worth remarking that, given a closure system \mathcal{C} on A , there is a strong connection between its Tarski congruence and its **Frege relation** $\mathcal{A} \mathcal{C} := \{ \langle a, b \rangle \in A \times A : C\{a\} = C\{b\} \}$, namely

that $\tilde{\Omega}\mathcal{C}$ is the largest congruence of \mathbf{A} below $\Lambda\mathcal{C}$. These constructions apply to logics as particular cases of g-matrices; then we write $\tilde{\Omega}\mathcal{L}$ and $\Lambda\mathcal{L}$ instead of $\tilde{\Omega}\mathcal{C}_{\mathcal{L}}$ and $\Lambda\mathcal{C}_{\mathcal{L}}$, respectively; the relation $\Lambda\mathcal{L}$ is also denoted by $\dashv\vdash_{\mathcal{L}}$. A logic \mathcal{L} is **selfextensional** when $\Lambda\mathcal{L}$ is a congruence, in which case $\Lambda\mathcal{L} = \tilde{\Omega}\mathcal{L}$.

The Tarski congruence allows us to associate with a logic \mathcal{L} another class of algebras

$$\text{Alg}\mathcal{L} := \{ \mathbf{A} : \text{there is a g-model } \langle \mathbf{A}, \mathcal{C} \rangle \text{ of } \mathcal{L} \text{ such that } \tilde{\Omega}\mathcal{C} = \text{Id}_{\mathbf{A}} \},$$

called the **algebraic counterpart** of \mathcal{L} , or also the class of **Tarski-reduced algebras** of \mathcal{L} . The third class of algebras associated with a logic \mathcal{L} we shall consider is its **intrinsic variety**

$$\mathbb{V}\mathcal{L} := \mathbb{V}(\mathbf{Fm}/\tilde{\Omega}\mathcal{L}),$$

where, for any class \mathbf{K} of algebras, $\mathbb{V}(\mathbf{K})$ denotes the variety generated by \mathbf{K} ; the Tarski congruence of a logic $\tilde{\Omega}\mathcal{L}$ has been always present, implicitly or explicitly, in algebraic logic, as in [20, 22], and the class $\mathbb{V}\mathcal{L}$ was also introduced in [20], with different notation.

The inclusions that hold in general between the three classes of algebras associated with a logic \mathcal{L} we have considered are $\text{Alg}^*\mathcal{L} \subseteq \text{Alg}\mathcal{L} \subseteq \mathbb{V}\mathcal{L}$; moreover, it is also possible to prove that $\mathbb{V}(\text{Alg}^*\mathcal{L}) = \mathbb{V}(\text{Alg}\mathcal{L}) = \mathbb{V}\mathcal{L}$ and that $\text{Alg}\mathcal{L} = \mathbb{P}_{\text{sd}}(\text{Alg}^*\mathcal{L})$, where \mathbb{P}_{sd} is the subdirect products operator. In general the three classes may be different; while for a large class of logics (which includes all protoalgebraic logics, but also many others) one can show that $\text{Alg}^*\mathcal{L} = \text{Alg}\mathcal{L}$, in general it is $\text{Alg}\mathcal{L}$ that deserves the title of “algebraic counterpart” of the logic \mathcal{L} , as is argued in [11, 12].

A logic \mathcal{L} is **fully selfextensional** when for any algebra \mathbf{A} , the relation $\Lambda\mathcal{F}i_{\mathcal{L}}\mathbf{A}$ is a congruence, in which case $\Lambda\mathcal{F}i_{\mathcal{L}}\mathbf{A} = \tilde{\Omega}\mathcal{F}i_{\mathcal{L}}\mathbf{A}$; equivalently, when for any $\mathbf{A} \in \text{Alg}\mathcal{L}$, $\Lambda\mathcal{F}i_{\mathcal{L}}\mathbf{A} = \text{Id}_{\mathbf{A}}$, which says that different elements generate different \mathcal{L} -filters.

We conclude this section with a brief remark on free algebras. Given a class of algebras \mathbf{K} , we will denote by $\mathbf{Fm}_{\mathbf{K}}$ the free algebra over \mathbf{K} with countably many free generators. Moreover we will denote by $\mathbf{Fm}_{\mathbf{K}}\{x_1, \dots, x_n\}$ the free algebra over \mathbf{K} with n distinct free generators. Since free algebras are quotients of formula algebras, we will denote their elements as $\llbracket \varphi \rrbracket$, $\llbracket \psi \rrbracket$ etc., omitting the subindex that would denote the corresponding congruence. The fact that $\mathbf{Fm}/\tilde{\Omega}\mathcal{L}$ is always the free algebra over $\mathbb{V}\mathcal{L}$ with countably many free generators implies the following fact, which we will use later on.

Lemma 1.2. *Let $\alpha, \beta \in \mathbf{Fm}$. If $\text{Alg}\mathcal{L} \models \alpha \approx \beta$, then $\alpha \dashv\vdash_{\mathcal{L}} \beta$. \(\square\)*

2. FROM VARIETIES TO LOGICS

We would like to begin our study by considering the general problem of constructing logics close to varieties (classes of algebras axiomatised by equations). The interest for this issue can be traced back to the work of Rautenberg [20], but no general study has been carried on until now. Corresponding to the three ways of associating a class of algebras with a logic \mathcal{L} , it is natural to ask the following three questions, given an arbitrary variety \mathbb{V} :

Question 1. *Is there a logic \mathcal{L} such that $\text{Alg}^*\mathcal{L} = \mathbb{V}$?*

Question 2. *Is there a logic \mathcal{L} such that $\text{Alg}\mathcal{L} = \mathbb{V}$?*

Question 3. *Is there a logic \mathcal{L} such that $\forall \mathcal{L} = \mathbb{V}$?*

It is clear that a positive answer to Question 1 would imply a positive one to Question 2, because one can show that $\text{Alg } \mathcal{L} = \mathbb{P}_{\text{SD}} \text{Alg}^* \mathcal{L}$ and varieties are closed under subdirect products. It is clear as well that a positive answer to Question 2 would imply a positive one to Question 3, since $\forall \mathcal{L} = \mathbb{V}(\text{Alg } \mathcal{L})$. However the next example shows that Question 1 cannot be answered in general in the positive.

Example 2.1. There is no logic \mathcal{L} for which $\text{Alg}^* \mathcal{L}$ is the variety SG of commutative semigroups. This class is axiomatised by the two equations

$$x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx (x \cdot y) \cdot z.$$

Consider the three-element null semigroup $\mathbf{3}$, i.e., the three-element semigroup on a set $\mathbf{3}$ with an element $0 \in \mathbf{3}$ such that $a \cdot b = 0$ for every $a, b \in \mathbf{3}$. We check that for each $F \subseteq \mathbf{3}$ the matrix $\langle \mathbf{3}, F \rangle$ is not reduced. If $F \in \{\mathbf{3}, \emptyset\}$, this is clear. Then suppose $F \notin \{\mathbf{3}, \emptyset\}$. It is easy to prove that for every $\{a, b\} \subseteq \mathbf{3}$ such that $a \neq b$, the equivalence relation with blocks $\{a, b\}$ and $\{c\}$ (where c is the third element of $\mathbf{3}$, different from a and b) is a congruence and is compatible with F for $F = \{a, b\}$ and for $F = \{c\}$. This easily yields that $\Omega F \neq \text{Id}_{\mathbf{3}}$ in all cases. \square

Even if Question 1 cannot be answered in general in the positive, we will prove that Questions 2 and 3 can indeed be so. In order to do this, we construct the logic which “approximates best” the variety \mathbb{V} in a certain sense:

Definition 2.2. Let \mathbb{V} be a variety. $\mathcal{L}_{\mathbb{V}}$ is the logic defined by the class of matrices $\{\langle \mathbf{A}, F \rangle : \mathbf{A} \in \mathbb{V}, F \subseteq A\}$.

In order to shorten notation we write $\Gamma \vdash_{\mathbb{V}} \varphi$ instead of $\Gamma \vdash_{\mathcal{L}_{\mathbb{V}}} \varphi$. The fact that $\mathcal{L}_{\mathbb{V}}$ is indeed the logic whose class of Leibniz-reduced algebras best approximates \mathbb{V} will be proved in Corollaries 2.10 and 2.12. In order to prove this, we need to go through some preliminary results; we begin by providing a characterisation of deductions from single formulas in $\mathcal{L}_{\mathbb{V}}$.

Lemma 2.3. *Let \mathbb{V} be a variety and $\alpha, \beta \in \text{Fm}$. The following conditions are equivalent:*

- (i) $\alpha \vdash_{\mathbb{V}} \beta$.
- (ii) $\alpha \dashv\vdash_{\mathbb{V}} \beta$.
- (iii) $\mathbb{V} \models \alpha \approx \beta$.

Proof. (i) \Rightarrow (iii) Assume $\alpha \vdash_{\mathbb{V}} \beta$ and take any homomorphism $h: \text{Fm} \rightarrow \mathbf{A}$ with $\mathbf{A} \in \mathbb{V}$. By definition $\langle \mathbf{A}, \{h(\alpha)\} \rangle$ is a model of $\mathcal{L}_{\mathbb{V}}$, therefore $h(\beta) \in \{h(\alpha)\}$, that is, $h(\alpha) = h(\beta)$. Thus, $\mathbb{V} \models \alpha \approx \beta$.

(iii) \Rightarrow (ii) Suppose that $\mathbb{V} \models \alpha \approx \beta$ and consider any matrix $\langle \mathbf{A}, F \rangle$ such that $\mathbf{A} \in \mathbb{V}$. Now, pick an homomorphism $h: \text{Fm} \rightarrow \mathbf{A}$ such that $h(\alpha) \in F$; since $\mathbb{V} \models \alpha \approx \beta$, we conclude that $h(\beta) \in F$. By definition of $\mathcal{L}_{\mathbb{V}}$ we conclude that $\alpha \vdash_{\mathbb{V}} \beta$. The proof that $\beta \vdash_{\mathbb{V}} \alpha$ is analogous.

(ii) \Rightarrow (i) is straightforward. \square

This lemma in particular implies that the relation $\dashv\vdash_{\mathbb{V}}$ is a congruence of the formula algebra. Therefore:

Corollary 2.4. *For every variety \mathbb{V} , the logic $\mathcal{L}_{\mathbb{V}}$ is selfextensional.* \square

The fact that deductions of \mathcal{L}_V from one premise correspond exactly to the equations that hold in V allows us to answer Questions 2 and 3 in the positive:

Theorem 2.5. *Let V be a variety.*

1. *If $A \in V$ is subdirectly irreducible, then $A \in \text{Alg}^* \mathcal{L}_V$.*
2. *$\forall \mathcal{L}_V = \text{Alg} \mathcal{L}_V = V$.*

Proof. 1. Consider any non-trivial subdirectly irreducible algebra $A \in V$. It is well known that $\langle \text{Co}(A) \setminus \{\text{Id}_A\}, \subseteq \rangle$ has a minimum element θ which is a principal congruence [6, Theorem II.8.4]. Thus, there are $a, b \in A$ such that $a \neq b$ and θ is the congruence generated by the pair $\langle a, b \rangle$. Then observe that by definition the matrix $\langle A, \{a\} \rangle$ is a model of \mathcal{L}_V . From the fact that $\Omega\{a\}$ is compatible with $\{a\}$ and that $b \neq a$, it follows that $\langle a, b \rangle \notin \Omega\{a\}$ and therefore $\theta \notin \Omega\{a\}$. This implies that $\Omega\{a\} = \text{Id}_A$. Hence we conclude that $\langle A, \{a\} \rangle$ is reduced and therefore that $A \in \text{Alg}^* \mathcal{L}_V$.

2. Since $\text{Alg} \mathcal{L}_V = \mathbb{P}_{\text{sd}} \text{Alg}^* \mathcal{L}_V$ and by point 1 $\text{Alg}^* \mathcal{L}_V$ contains every subdirectly irreducible member of V , we conclude that $V \subseteq \text{Alg} \mathcal{L}_V$. Then we turn to prove the other inclusion. If $\alpha \approx \beta$ is one of the equations defining V , by Lemma 2.3 we know that $\alpha \dashv\vdash_V \beta$, and from Corollary 2.4 it follows that $\langle \alpha, \beta \rangle \in \tilde{\Omega} \mathcal{L}$. But, since $\tilde{\Omega} \mathcal{L}$ is fully invariant, this means that $\forall \mathcal{L} \models \alpha \approx \beta$. Since $\forall \mathcal{L} = \forall(\text{Alg} \mathcal{L})$, we conclude that $\text{Alg} \mathcal{L} \models \alpha \approx \beta$ too. Therefore $\text{Alg} \mathcal{L}_V \subseteq V$, and we conclude that $\text{Alg} \mathcal{L}_V = V$. Finally, since $\forall \mathcal{L}_V$ is the closure of $\text{Alg} \mathcal{L}_V$ under \forall and $\text{Alg} \mathcal{L}_V$ is already a variety, $\forall \mathcal{L}_V = \text{Alg} \mathcal{L}_V$. \square

Corollary 2.6. *For every variety V , the logic \mathcal{L}_V is fully selfextensional and filter-distributive.*

Proof. From Definition 2.2 it follows that $\mathcal{F}i_{\mathcal{L}_V}(A) = \mathcal{P}(A)$ when $A \in V$, and after Theorem 2.5 this holds in any $A \in \text{Alg} \mathcal{L}_V$. This immediately implies that $\Lambda \mathcal{F}i_{\mathcal{L}_V}(A) = \text{Id}_A$ in these algebras, which says that \mathcal{L}_V is fully selfextensional, and that $\mathcal{F}i_{\mathcal{L}_V}(A)$ is a Boolean algebra, in particular a distributive lattice. Since for any algebra A the quotient $A/\tilde{\Omega} \mathcal{F}i_{\mathcal{L}_V}(A) \in \text{Alg} \mathcal{L}_V$ and the projection $\pi: A \rightarrow A/\tilde{\Omega} \mathcal{F}i_{\mathcal{L}_V}(A)$ extends to a lattice isomorphism between $\mathcal{F}i_{\mathcal{L}_V}(A)$ and $\mathcal{F}i_{\mathcal{L}_V}(A/\tilde{\Omega} \mathcal{F}i_{\mathcal{L}_V}(A))$, it follows that $\mathcal{F}i_{\mathcal{L}_V}(A)$ is distributive for any algebra A , that is, \mathcal{L}_V is filter-distributive. \square

Theorem 2.7. *For every non-trivial variety V , the logic \mathcal{L}_V is neither conjunctive nor disjunctive.*

Proof. We reason by contraposition. Suppose that \mathcal{L}_V is conjunctive. This means that there is a term-definable binary connective \wedge such that $x, y \vdash_V x \wedge y$, $x \wedge y \vdash_V x$ and $x \wedge y \vdash_V y$. Applying Lemma 2.9 to the last two deductions we obtain that V is a model of the equations $x \wedge y \approx x$ and $x \wedge y \approx y$. Hence we conclude that $V \models x \approx y$ and therefore that V is trivial.

Then consider the case in which \mathcal{L}_V is disjunctive. This means that there is a term-definable binary connective \vee such that \mathcal{L}_V is a model of the following Gentzen-style rules:

$$\frac{\Gamma, \alpha \triangleright \gamma \quad \Gamma, \beta \triangleright \gamma}{\Gamma, \alpha \vee \beta \triangleright \gamma} \quad \frac{\Gamma, \alpha \vee \beta \triangleright \gamma}{\Gamma, \alpha \triangleright \gamma} \quad \frac{\Gamma, \alpha \vee \beta \triangleright \gamma}{\Gamma, \beta \triangleright \gamma}$$

In particular this implies that $x \vdash_{\mathbf{V}} x \vee y$ and $y \vdash_{\mathbf{V}} x \vee y$. By Lemma 2.3 this yields that \mathbf{V} is a model of $x \approx x \vee y$ and $y \approx x \vee y$. Hence we conclude that $\mathbf{V} \models x \approx y$ and therefore that \mathbf{V} is trivial. \square

This result is particularly interesting in view of the large existing literature in algebraic logic that relates filter-distributivity with the property of being disjunctive; see [8, §2.5], and remark that by Corollary 2.14 the logic $\mathcal{L}_{\mathbf{V}}$ escapes the methods and results of [8].

Lemma 2.3 suggests the rather natural question of whether, instead of working with $\mathcal{L}_{\mathbf{V}}$, one could have defined a logic by a Hilbert calculus with the rules corresponding to the equations defining the variety \mathbf{V} and gain the same result. The next example shows that in general this is not the case.

Example 2.8. Consider the variety of semilattices \mathbf{SL} , which is axiomatised by the following equations: $x \approx x \cdot x$, $x \cdot y \approx y \cdot x$ and $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$, and consider the logic \mathcal{L} defined by the following six Hilbert-style rules:

$$x \dashv\vdash x \cdot x \quad x \cdot y \dashv\vdash y \cdot x \quad x \cdot (y \cdot z) \dashv\vdash (x \cdot y) \cdot z.$$

Even if one might guess that the algebraic counterpart of \mathcal{L} should be the variety of semilattices, it is not. For let $\langle \mathbf{Z}_3, + \rangle$ be the additive semigroup of integers modulo 3. It is easily proved that the matrix $\langle \mathbf{Z}_3, \{1, 2\} \rangle$ is reduced. Now we turn to prove that it is a model of \mathcal{L} . It is clearly a model of the rules $x \cdot y \dashv\vdash y \cdot x$ and $x \cdot (y \cdot z) \dashv\vdash (x \cdot y) \cdot z$, because \mathbf{Z}_3 is commutative and associative. Moreover it is a model of the rule $x \dashv\vdash x \cdot x$, because $1 + 1 = 2$ and $2 + 2 = 1$ while $0 + 0 = 0$ in \mathbf{Z}_3 . Therefore $\mathbf{Z}_3 \in \text{Alg}^* \mathcal{L} \subseteq \text{Alg} \mathcal{L}$; since $\mathbf{Z}_3 \notin \mathbf{SL}$, we conclude that $\text{Alg} \mathcal{L} \neq \mathbf{SL}$, and hence that $\mathcal{L} \neq \mathcal{L}_{\mathbf{SL}}$.

It is worth remarking that the fact that $\mathbf{Z}_3 \notin \mathbf{SL}$ yields also that $\forall \mathcal{L} \not\models x \approx x \cdot x$, against another natural supposition one may have done when looking at the rules defining \mathcal{L} . Thus we see that $\langle x, x \cdot x \rangle \in \mathbf{AL}$ while $\langle x, x \cdot x \rangle \notin \tilde{\mathcal{N}} \mathcal{L}$, which implies that \mathcal{L} is not selfextensional. \square

Another fundamental property of $\mathcal{L}_{\mathbf{V}}$ we will make use of in the next section is that its deductions are determined by the deductions from a single premise. More precisely, we say that a logic \mathcal{L} is **unitary** when for every $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$, if $\Gamma \vdash_{\mathcal{L}} \varphi$, then there is $\gamma \in \Gamma$ such that $\gamma \vdash_{\mathcal{L}} \varphi$.

Lemma 2.9. *For every variety \mathbf{V} , the logic $\mathcal{L}_{\mathbf{V}}$ is unitary.*

Proof. Suppose that $\Gamma \vdash_{\mathbf{V}} \varphi$. Then we consider the quotient projection $\pi: \mathbf{Fm} \rightarrow \mathbf{Fm}_{\mathbf{V}}$ and observe that $\langle \mathbf{Fm}_{\mathbf{V}}, \{[\gamma] : \gamma \in \Gamma\} \rangle$ is a model of $\mathcal{L}_{\mathbf{V}}$ by definition, because $\mathbf{Fm}_{\mathbf{V}} \in \mathbf{V}$ as \mathbf{V} is a variety. This implies that $[\varphi] = \pi(\varphi) \in \{[\gamma] : \gamma \in \Gamma\}$ and therefore that there is a formula $\gamma \in \Gamma$ such that $\mathbf{V} \models \gamma \approx \varphi$. By Lemma 2.3 this means that $\gamma \vdash_{\mathbf{V}} \varphi$. \square

We believe that the following corollaries justify the statement that $\mathcal{L}_{\mathbf{V}}$ is indeed the logic closest to \mathbf{V} from the point of view of the Leibniz operator.

Corollary 2.10. *Let \mathbf{V} be a variety. If $\forall \mathcal{L} = \mathbf{V}$, then $\mathcal{L}_{\mathbf{V}} \leq \mathcal{L}$ and consequently $\text{Alg}^* \mathcal{L} \subseteq \text{Alg}^* \mathcal{L}_{\mathbf{V}} \subseteq \mathbf{V}$. Thus, $\mathcal{L}_{\mathbf{V}}$ is the weakest logic whose intrinsic variety is \mathbf{V} . Moreover, $\mathcal{L}_{\mathbf{V}}$ is the weakest logic \mathcal{L} such that $\text{Alg} \mathcal{L} = \mathbf{V}$.*

Proof. Suppose that $\forall \mathcal{L} = \mathbf{V}$ and that $\Gamma \vdash_{\mathbf{V}} \varphi$. From Lemma 2.9 it follows that there is a formula $\gamma \in \Gamma$ such that $\gamma \vdash_{\mathbf{V}} \varphi$. By Lemma 2.3 this is equivalent to the

fact that $\mathbb{V} \models \gamma \approx \varphi$. By the fact that $\mathbb{V}\mathcal{L} = \mathbb{V}(\text{Alg}\mathcal{L})$ and Lemma 1.2, this implies that $\gamma \vdash_{\mathcal{L}} \varphi$ and hence that $\Gamma \vdash_{\mathcal{L}} \varphi$. This shows that $\mathcal{L}_{\mathbb{V}} \leq \mathcal{L}$, and this implies that $\text{Alg}^*\mathcal{L} \subseteq \text{Alg}^*\mathcal{L}_{\mathbb{V}} \subseteq \text{Alg}\mathcal{L}_{\mathbb{V}} = \mathbb{V}$. The final statement follows from the previous one because the assumption $\text{Alg}\mathcal{L} = \mathbb{V}$ implies that $\mathbb{V}\mathcal{L} = \mathbb{V}$. \square

The next example shows that the converse of Corollary 2.10 does not hold in general; nevertheless, in Theorem 3.3 we will show that this is the case for the variety of semilattices (disregarding the trivial logics).

Example 2.11. Consider the variety HA of Heyting algebras. From Theorem 2.5 it follows that $\text{Alg}\mathcal{L}_{\text{HA}} = \text{HA}$. Then let \mathcal{IPC} be the intuitionistic propositional logic. It is well-known that $\mathbb{V}\mathcal{IPC} = \text{HA}$, therefore, by Corollary 2.10, we have that $\mathcal{L}_{\text{HA}} \leq \mathcal{IPC}$. Let \mathcal{CPC} be the classical propositional logic and BA the variety of Boolean algebras. It is well-known that $\mathcal{IPC} \leq \mathcal{CPC}$ and $\mathbb{V}\mathcal{CPC} = \text{BA}$. Therefore we have that $\mathcal{L}_{\text{HA}} \leq \mathcal{CPC}$ and $\mathbb{V}\mathcal{CPC} = \text{BA} \neq \text{HA}$. \square

Corollary 2.12. *Let \mathbb{V} be a variety. There is a logic \mathcal{L} such that $\text{Alg}^*\mathcal{L} = \mathbb{V}$ if and only if $\text{Alg}^*\mathcal{L}_{\mathbb{V}} = \mathbb{V}$. In such a case, $\mathcal{L}_{\mathbb{V}}$ is the weakest of all such logics.*

Proof. If $\text{Alg}^*\mathcal{L} = \mathbb{V}$, then also $\text{Alg}\mathcal{L} = \mathbb{V}$, and therefore by Corollary 2.10 we have that $\mathcal{L}_{\mathbb{V}} \leq \mathcal{L}$. This yields that $\mathbb{V} = \text{Alg}^*\mathcal{L} \subseteq \text{Alg}^*\mathcal{L}_{\mathbb{V}}$. But from Theorem 2.5 it follows that $\text{Alg}^*\mathcal{L}_{\mathbb{V}} \subseteq \mathbb{V}$, which implies the equality. The converse implication is straightforward. In passing, we have also shown the last statement. \square

Thus, if there is some solution to Question 1, our logic $\mathcal{L}_{\mathbb{V}}$ is indeed one, and actually the weakest one. This considerably simplifies the task of answering this question, as we will see in the next section in the case of the variety of semilattices.

We have seen in Corollary 2.6 that $\mathcal{L}_{\mathbb{V}}$ occupies a good place in the Frege hierarchy. We conclude this section by observing that nevertheless, in the relevant cases, it lies outside of the Leibniz hierarchy.

Lemma 2.13. *For each variety \mathbb{V} , the logic $\mathcal{L}_{\mathbb{V}}$ has no theorems.*

Proof. Consider any algebra $\mathbf{A} \in \mathbb{V}$. Since by definition $\langle \mathbf{A}, \emptyset \rangle$ is a model of $\mathcal{L}_{\mathbb{V}}$, we conclude that $\mathcal{L}_{\mathbb{V}}$ has no theorem. \square

It may seem that we tailored our definition of $\mathcal{L}_{\mathbb{V}}$ in order to obtain Lemma 2.13, since we explicitly included $\langle \mathbf{A}, \emptyset \rangle$ among the models of $\mathcal{L}_{\mathbb{V}}$ for every $\mathbf{A} \in \mathbb{V}$. However, this is not the case, at least for non-trivial varieties. Actually, if \mathbb{V} is a non-trivial variety it is easy to prove that the logic \mathcal{L} defined by the class of matrices $\{\langle \mathbf{A}, F \rangle : \mathbf{A} \in \mathbb{V} \text{ and } F \subseteq A \text{ with } F \neq \emptyset\}$ coincides with $\mathcal{L}_{\mathbb{V}}$. In order to prove this, let $\mathbf{A} \in \mathbb{V}$ be non-trivial, which implies there are $a, b \in A$ with $a \neq b$. By definition we know that $\{a\}$ and $\{b\}$ are filters of \mathcal{L} . Since the intersection of filters is still a filter, we conclude that $\emptyset = \{a\} \cap \{b\}$ is a filter of \mathcal{L} too. This clearly yields that $\mathcal{L} = \mathcal{L}_{\mathbb{V}}$.

Corollary 2.14. *For each non-trivial variety \mathbb{V} , the logic $\mathcal{L}_{\mathbb{V}}$ is neither protoalgebraic nor truth-equational.*

Proof. Let \mathbb{V} be a non-trivial variety. From Theorem 2.5 we know that $\text{Alg}\mathcal{L}_{\mathbb{V}} = \mathbb{V}$. This implies in particular that $\mathcal{L}_{\mathbb{V}}$ is non-trivial. Since the unique protoalgebraic logic without theorems is the almost inconsistent one, from Lemma 2.13 it follows that $\mathcal{L}_{\mathbb{V}}$ is not protoalgebraic. Moreover truth-equational logics have theorems [18, Theorem 28], therefore again Lemma 2.13 shows that $\mathcal{L}_{\mathbb{V}}$ is not truth-equational. \square

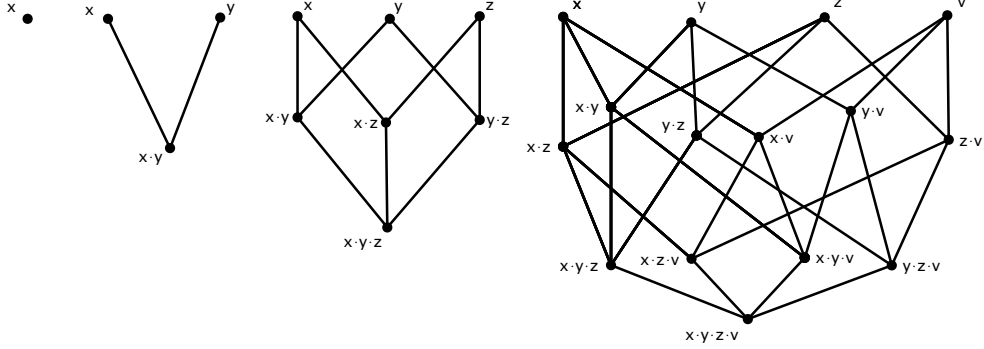


FIGURE 1. Free semilattices with 1, 2, 3, 4 free generators.

3. LOGICS OF SEMILATTICES

From now on and until the end of the paper we will work in the language of semilattices, i.e., the language $\langle \cdot \rangle$ of type $\langle 2 \rangle$. We have seen how to construct the logic closest to a certain variety \mathbf{V} . In this section and the next ones we will focus on logics related to a fixed variety, namely the variety of all semilattices, denoted by \mathbf{SL} . Recall (Example 2.8) that \mathbf{SL} is axiomatized by the following set of equations:

$$x \approx x \cdot x \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx (x \cdot y) \cdot z.$$

Now let us fix some notation and recall some basic facts about this class. We will assume all along the paper that we are working with meet-semilattices, in the sense that, given $\mathbf{A} \in \mathbf{SL}$ and $a, b \in A$, we will write $a \leq b$ as a shorthand of $a = a \cdot b$. We will also denote by $\mathbf{2} = \langle \{0, 1\}, \cdot \rangle$ the two-element semilattice with $0 < 1$ and by $\mathbf{1}$ the trivial one with domain $\{1\}$. The semilattice $\mathbf{2}$ will play a fundamental role in our analysis, since it is the only non-trivial subdirectly irreducible member of \mathbf{SL} and therefore $\mathbf{SL} = \mathbb{P}_{\text{sd}}\{\mathbf{1}, \mathbf{2}\}$ [7, Corollary 2.2.7].

Among semilattices, we will be particularly interested in free semilattices with a finite set of free generators $\mathbf{Fm}_{\mathbf{SL}}\{x_1, \dots, x_n\}$. It is well known that for every n , there is an isomorphism between $\mathbf{Fm}_{\mathbf{SL}}\{x_1, \dots, x_n\}$ and the power set semilattice $\langle \mathcal{P}\{x_1, \dots, x_n\} \setminus \{\emptyset\}, \cup \rangle$ which sends the equivalence class of a formula to the set of the variables occurring in it [7, Theorem 2.1.3]. This isomorphism is well defined since, given two formulas α and β , $\mathbf{SL} \models \alpha \approx \beta$ if and only if α and β have the same variables (this property will play an essential role in many of our reasonings). Figure 1 represents the four smallest free semilattices, ordered under the inverse of the inclusion relation since we will work with meet-semilattices. It is worth keeping in mind how $\mathbf{Fm}_{\mathbf{SL}}\{x, y\}$ looks like, since it will play a central role in the proofs of Theorems 5.5 and 5.6.

Let us now return to the main problem of the section, that of studying the wide class of logics that enjoy a special connection with the variety of semilattices. The next definition clarifies which connection we are talking about.

Definition 3.1. A logic \mathcal{L} is a **logic of semilattices** when $\text{Alg } \mathcal{L} = \mathbf{SL}$.

Thus, the logics of semilattices are the solutions to Question 2 for $\mathbf{V} = \mathbf{SL}$. From Theorem 2.5 it follows that a first example of a logic of semilattices is $\mathcal{L}_{\mathbf{SL}}$. The

other three paradigmatic examples of logics of semilattices are the $\{\wedge\}$ -fragment and the $\{\vee\}$ -fragment of classical propositional logic, and their intersection; we postpone the proof of this fact after that of Theorem 3.3, which provides a characterisation of the entire class. To begin with, recall that by Lemma 2.9 the logic \mathcal{L}_{SL} is unitary as are all the logics $\mathcal{L}_{\mathcal{V}}$. However, in this case we can be more precise, as deductions of \mathcal{L}_{SL} just depend on the variables occurring in their formulas. To see this we introduce the following notation, for $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$:

$$\text{Var}(\varphi) := \{x \in \text{Var} : x \text{ occurs in } \varphi\} \quad \text{Var}(\Gamma) := \bigcup_{\varphi \in \Gamma} \text{Var}(\varphi).$$

Lemma 3.2. *Let $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$. $\Gamma \vdash_{\text{SL}} \varphi$ if and only if there is $\gamma \in \Gamma$ such that $\text{Var}(\gamma) = \text{Var}(\varphi)$.*

Proof. First observe that for every pair of formulas α and β , it holds that $\text{SL} \vDash \alpha \approx \beta$ if and only if $\text{Var}(\alpha) = \text{Var}(\beta)$. Then assume that $\Gamma \vdash_{\text{SL}} \varphi$. By Lemma 2.9 there is $\gamma \in \Gamma$ such that $\gamma \vdash_{\text{SL}} \varphi$. From Lemma 2.3 it follows that this is equivalent to the fact that $\text{SL} \vDash \gamma \approx \varphi$ and therefore to $\text{Var}(\gamma) = \text{Var}(\varphi)$.

In order to prove the other direction, assume that $\text{Var}(\gamma) = \text{Var}(\varphi)$ for some $\gamma \in \Gamma$. As we remarked, this is equivalent to the fact that $\text{SL} \vDash \gamma \approx \varphi$. But by Lemma 2.3 this implies that $\gamma \vdash_{\text{SL}} \varphi$, and hence that $\Gamma \vdash_{\text{SL}} \varphi$. \square

The next result characterises logics of semilattices as non-trivial extensions of \mathcal{L}_{SL} . This is interesting for two reasons: firstly because it turns out that the algebraic counterpart of an extension of \mathcal{L}_{SL} is always a variety, secondly because it shows that the fact that a logic is a logic of semilattices can be characterised by a set of Hilbert-style rules, namely by taking as rules all deductions holding in \mathcal{L}_{SL} .

Theorem 3.3. *Let \mathcal{L} be non-trivial. The following conditions are equivalent:*

- (i) \mathcal{L} is a logic of semilattices; i.e., $\text{Alg } \mathcal{L} = \text{SL}$.
- (ii) $\mathbb{V} \mathcal{L} = \text{SL}$.
- (iii) $\mathcal{L}_{\text{SL}} \leq \mathcal{L}$.

Proof. The implication (i) \Rightarrow (ii) follows from the fact that $\mathbb{V}(\text{Alg } \mathcal{L}) = \mathbb{V} \mathcal{L}$ for any logic \mathcal{L} , and (ii) \Rightarrow (iii) is a particular case of the main property contained in Corollary 2.10. Finally we prove that (iii) \Rightarrow (i). From the fact that $\mathcal{L}_{\text{SL}} \leq \mathcal{L}$ it follows that $\text{Alg } \mathcal{L} \subseteq \text{Alg } \mathcal{L}_{\text{SL}}$. By Theorem 2.5 we know that $\text{Alg } \mathcal{L}_{\text{SL}} = \text{SL}$, therefore we conclude that $\text{Alg } \mathcal{L} \subseteq \text{SL}$. It only remains to prove that $\text{SL} \subseteq \text{Alg } \mathcal{L}$. Since $\text{Alg } \mathcal{L} = \mathbb{P}_{\text{SD}} \text{Alg}^* \mathcal{L}$ and $\text{SL} = \mathbb{P}_{\text{SD}} \{\mathbf{1}, \mathbf{2}\}$, it will be enough to prove that $\mathbf{1}, \mathbf{2} \in \text{Alg}^* \mathcal{L}$. Observe that the trivial algebra $\mathbf{1}$ belongs to the algebraic companion of every logic. Therefore our goal reduces to that of proving that $\mathbf{2} \in \text{Alg } \mathcal{L}$. Suppose towards a contradiction this is not the case; then neither $\langle \mathbf{2}, \{0\} \rangle$, nor $\langle \mathbf{2}, \{1\} \rangle$ is a model of \mathcal{L} . Note that (trivially) these matrices are reduced.

We first claim that $x \cdot y \vdash_{\mathcal{L}} y$. In order to prove this, observe that since $\langle \mathbf{2}, \{0\} \rangle$ is not a model of \mathcal{L} , there must be a deduction $\Gamma \vdash_{\mathcal{L}} \varphi$ and an homomorphism $h: \mathbf{Fm} \rightarrow \mathbf{2}$ such that $h[\Gamma] \subseteq \{0\}$ and $h(\varphi) = 1$. Then let $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ be the substitution defined as

$$\sigma(z) := \begin{cases} x & \text{if } h(z) = 0 \\ y & \text{otherwise} \end{cases}$$

for every variable z . By structurality of \mathcal{L} we know that $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$. Now observe that $h(z) = 1$ for every $z \in \text{Var}(\varphi)$, since $h(\varphi) = 1$. This yields that $\text{Var}(\sigma\varphi) = \{y\}$ and therefore, applying Lemma 3.2 and the fact that $\mathcal{L}_{\text{SL}} \leq \mathcal{L}$, that $\sigma\varphi \vdash_{\mathcal{L}} y$. We

conclude that $\sigma\Gamma \vdash_{\mathcal{L}} y$. Take any $\gamma \in \sigma\Gamma$; since $h(\gamma) = 0$ and $1 \cdot 1 = 1$, there is $z \in \text{Var}(\gamma)$ such that $h(z) = 0$ and therefore that $\{x\} \subseteq \text{Var}(\gamma) \subseteq \{x, y\}$. Applying Lemma 3.2 and the fact that $\mathcal{L}_{\text{SL}} \leq \mathcal{L}$, this yields that $\gamma \dashv\vdash_{\mathcal{L}} x$ or $\gamma \dashv\vdash_{\mathcal{L}} x \cdot y$. Therefore we conclude that $x, x \cdot y \vdash_{\mathcal{L}} \gamma$ for every $\gamma \in \sigma\Gamma$ and hence that $x, x \cdot y \vdash_{\mathcal{L}} y$. Finally we consider a new substitution $\sigma' : \mathbf{Fm} \rightarrow \mathbf{Fm}$ defined as

$$\sigma'(z) := \begin{cases} x \cdot y & \text{if } z = x \\ z & \text{otherwise} \end{cases}$$

for every variable z . By structurality we have that $x \cdot y, (x \cdot y) \cdot y \vdash_{\mathcal{L}} y$. Observe that $\text{SL} \models (x \cdot y) \cdot y \approx x \cdot y$; therefore by Lemma 2.3 we have that $x \cdot y \vdash_{\text{SL}} (x \cdot y) \cdot y$. By assumption we know that $\mathcal{L}_{\text{SL}} \leq \mathcal{L}$, and therefore $x \cdot y \vdash_{\mathcal{L}} (x \cdot y) \cdot y$. We conclude that $x \cdot y \vdash_{\mathcal{L}} y$.

We now claim that $x \vdash_{\mathcal{L}} x \cdot y$. As one can imagine, the argument is somehow dual to the previous one. Observe that since $\langle \mathbf{2}, \{1\} \rangle$ is not a model of \mathcal{L} , there must be a deduction $\Gamma \vdash_{\mathcal{L}} \varphi$ and an homomorphism $h : \mathbf{Fm} \rightarrow \mathbf{2}$ such that $h[\Gamma] \subseteq \{1\}$ and $h(\varphi) = 0$. Then let $\sigma : \mathbf{Fm} \rightarrow \mathbf{Fm}$ be the substitution defined as

$$\sigma(z) := \begin{cases} x & \text{if } h(z) = 1 \\ y & \text{otherwise} \end{cases}$$

for every variable z . By structurality of \mathcal{L} we know that $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$. Let $\gamma \in \Gamma$, since $h(\gamma) = 1$ we know that $h(z) = 1$ for every $z \in \text{Var}(\gamma)$. This yields in particular that $\text{Var}(\sigma\gamma) = x$ and therefore, by Lemma 3.2 and the fact that $\mathcal{L}_{\text{SL}} \leq \mathcal{L}$, that $x \vdash_{\mathcal{L}} \sigma\gamma$. We conclude that $x \vdash_{\mathcal{L}} \sigma\varphi$. Analogously, since $h(\varphi) = 0$ there is $z \in \text{Var}(\varphi)$ such that $h(z) = 0$. Then $\{y\} \subseteq \text{Var}(\sigma\varphi) \subseteq \{x, y\}$. Applying Lemma 3.2 and the fact that $\mathcal{L}_{\text{SL}} \leq \mathcal{L}$, this yields that $\sigma\varphi \dashv\vdash_{\mathcal{L}} y$ or $\sigma\varphi \dashv\vdash_{\mathcal{L}} x \cdot y$. Since \mathcal{L} is non-trivial, from Lemma 1.1 it follows that $\sigma\varphi \dashv\vdash_{\mathcal{L}} x \cdot y$. This yields that $x \vdash_{\mathcal{L}} x \cdot y$ and concludes the proof of the claim.

Now, pasting together the two claims, we obtain that $x \vdash_{\mathcal{L}} y$, and this, by Lemma 1.1, contradicts the assumption that \mathcal{L} is non-trivial. \square

This result allows us to construct several logics of semilattices, simply by considering extensions of \mathcal{L}_{SL} . The next example introduces three paradigmatic cases, which will play an important role along the paper. Other odder examples will be constructed in Section 4.

Example 3.4. Let \mathcal{L}_{\wedge} be the logic axiomatised by the following three Hilbert-style rules:

$$x \cdot y \vdash x \quad x \cdot y \vdash y \quad x, y \vdash x \cdot y.$$

Analogously let \mathcal{L}_{\vee} be the logic axiomatised by the Hilbert calculus defined by the following infinite set of rules:

$$\alpha \vdash \beta \text{ for every } \alpha, \beta \in \mathbf{Fm} \text{ such that } \text{Var}(\alpha) \subseteq \text{Var}(\beta).$$

It is not difficult to prove that \mathcal{L}_{\wedge} and \mathcal{L}_{\vee} are respectively the $\{\wedge\}$ -fragment and the $\{\vee\}$ -fragment of classical propositional logic [1, Theorems 2.1 and 3.1]; i.e., they are complete respectively with $\langle \mathbf{2}, \{1\} \rangle$ and $\langle \mathbf{2}, \{0\} \rangle$. Note that the matrix $\langle \mathbf{2}, \{0\} \rangle$ is isomorphic to the “dual” matrix $\langle \mathbf{2}^d, \{1\} \rangle$, where $\mathbf{2}^d$ is the algebra on $2 = \{0, 1\}$ obtained from $\mathbf{2}$ after shuffling 0 and 1; hence its semilattice operation is the usual operation of disjunction.

Keeping this in mind, it is clear that \mathcal{L}_{\wedge} and \mathcal{L}_{\vee} are non-trivial, and that for every $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$ we have the following properties:

- (i) $\Gamma \vdash_{\mathcal{L}_\wedge} \varphi \iff \text{Var}(\varphi) \subseteq \text{Var}(\Gamma)$;
- (ii) $\Gamma \vdash_{\mathcal{L}_\vee} \varphi \iff \text{Var}(\gamma) \subseteq \text{Var}(\varphi)$ for some $\gamma \in \Gamma$.

Since \mathcal{L}_\wedge and \mathcal{L}_\vee are non-trivial, from Theorem 3.3 it follows that, in order to prove that they are logics of semilattices, it will be enough to check that they extend \mathcal{L}_{SL} . But this is very easy: if $\Gamma \vdash_{\text{SL}} \varphi$, by Lemma 3.2 we know that there is $\gamma \in \Gamma$ such that $\text{Var}(\gamma) = \text{Var}(\varphi)$. From (i) and (ii) respectively it follows that $\gamma \vdash_{\mathcal{L}_\wedge} \varphi$ and $\gamma \vdash_{\mathcal{L}_\vee} \varphi$, thus showing that $\Gamma \vdash_{\mathcal{L}_\wedge} \varphi$ and $\Gamma \vdash_{\mathcal{L}_\vee} \varphi$.

The third example we would like to consider is the intersection of \mathcal{L}_\wedge and \mathcal{L}_\vee , which we will denote by $\mathcal{L}_{\wedge\vee}$. Since $\mathcal{L}_{\text{SL}} \leq \mathcal{L}_\wedge, \mathcal{L}_\vee$, it is clear that $\mathcal{L}_{\text{SL}} \leq \mathcal{L}_{\wedge\vee}$, and therefore, by Theorem 3.3, $\mathcal{L}_{\wedge\vee}$ is a logic of semilattices. \square

Now let us spend a few words on the classification of logics of semilattices into the two hierarchies of abstract algebraic logic. Quite surprisingly, it turns out that all of them lie outside the Leibniz hierarchy:

Lemma 3.5. *If \mathcal{L} is a logic of semilattices, then \mathcal{L} has no theorems.*

Proof. Suppose towards a contradiction that $\text{Alg}\mathcal{L} = \text{SL}$ and that there is $\varphi \in \text{Fm}$ such that $\emptyset \vdash_{\mathcal{L}} \varphi$. Then we would have $\emptyset \vdash_{\mathcal{L}} \sigma\varphi$ where $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ is the substitution sending every variable to x . Since $\text{SL} \models \sigma\varphi \approx x$, by Lemma 1.2 we would have that $\emptyset \vdash_{\mathcal{L}} x$. But this yields that \mathcal{L} is the inconsistent logic, whose algebraic counterpart is the trivial variety $\mathbb{I}\{1\}$, against the assumption. \square

As with Corollary 2.14, this immediately implies:

Corollary 3.6. *If \mathcal{L} is a logic of semilattices, then it is neither proalgebraic, nor truth-equational.* \square

Even if logics of semilattices do not belong to the Leibniz hierarchy, it is possible to prove that they belong at least to the lowest level of the Frege hierarchy, i.e., that their interderivability relation is a congruence on the formula algebra:

Lemma 3.7. *If \mathcal{L} is a logic of semilattices, then it is selfextensional.*

Proof. Consider $\alpha, \beta \in \text{Fm}$ such that $\alpha \dashv\vdash_{\mathcal{L}} \beta$. We claim that $\alpha \cdot \gamma \dashv\vdash_{\mathcal{L}} \beta \cdot \gamma$ for every $\gamma \in \text{Fm}$. Let $\gamma \in \text{Fm}$ and observe that in α it occurs at least one variable x and in β a variable y (possibly equal). Then consider the substitution $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ defined as

$$\sigma(z) := \begin{cases} x \cdot \gamma & \text{if } z = x \\ y \cdot \gamma & \text{if } z = y \\ z & \text{otherwise} \end{cases}$$

for every variable z . Clearly $\text{SL} \models \alpha \cdot \gamma \approx \sigma\alpha$ and $\text{SL} \models \beta \cdot \gamma \approx \sigma\beta$. By Lemma 1.2 we conclude that $\alpha \cdot \gamma \dashv\vdash_{\mathcal{L}} \sigma\alpha$ and $\beta \cdot \gamma \dashv\vdash_{\mathcal{L}} \sigma\beta$. Now, recall that by assumption $\alpha \dashv\vdash_{\mathcal{L}} \beta$, therefore by structurality we have that $\sigma\alpha \dashv\vdash_{\mathcal{L}} \sigma\beta$. This concludes the proof of the claim.

It is now easy to prove that \mathcal{L} is selfextensional. For let $\alpha, \alpha', \beta, \beta' \in \text{Fm}$ be such that $\alpha \dashv\vdash_{\mathcal{L}} \alpha'$ and $\beta \dashv\vdash_{\mathcal{L}} \beta'$. We have that

$$\alpha \cdot \beta \dashv\vdash_{\mathcal{L}} \alpha' \cdot \beta \dashv\vdash_{\mathcal{L}} \beta \cdot \alpha' \dashv\vdash_{\mathcal{L}} \beta' \cdot \alpha' \dashv\vdash_{\mathcal{L}} \alpha' \cdot \beta'$$

where the first and the third interderivations follow from the claim, while the second and the fourth follow from Lemma 1.2. \square

We now retake Question 1 in the particular case of the variety of semilattices:

Question 4. *Is there a logic (of semilattices) \mathcal{L} such that $\text{Alg}^*\mathcal{L} = \text{SL}$?*

The expression “of semilattices” is written between parentheses because this assumption can be taken or skipped without changing the question, since $\text{Alg}\mathcal{L} = \mathbb{P}_{\text{SD}}\text{Alg}^*\mathcal{L}$. By Corollary 2.12 and Theorem 3.3, Question 4 amounts to the following one.

Question 5. *Is it true that $\text{SL} \subseteq \text{Alg}^*\mathcal{L}_{\text{SL}}$? That is, is it true that in each semilattice \mathbf{A} there is some $F \subseteq A$ such that $\Omega F = \text{Id}_{\mathbf{A}}$?*

This problem is still OPEN. An approximation to its solution requires improving our knowledge of $\text{Alg}^*\mathcal{L}_{\text{SL}}$. Here we will prove that $\text{Alg}^*\mathcal{L}_{\text{SL}}$ contains every semilattice with sectionally finite height (see Definition 3.10), hence in particular every finite semilattice. This quest can be simplified thanks to the fact that the behaviour of the Leibniz congruence in the context of semilattices can be easily characterised in a way inspired in the characterisation in [10] of ΩF for a filter F of the $\{\wedge, \vee\}$ -fragment of classical propositional logic.

Lemma 3.8. *Let $\mathbf{A} \in \text{SL}$, $F \subseteq A$ and $a, b \in A$. $\langle a, b \rangle \in \Omega F$ if and only if $[a \cdot c \in F \text{ if and only if } b \cdot c \in F]$ for every $c \in A$.*

Proof. It is easy to prove that the relation $R \subseteq A \times A$, defined as $\langle a, b \rangle \in R$ if and only if $[a \cdot c \in F \Leftrightarrow b \cdot c \in F]$ for every $c \in A$, is a congruence over \mathbf{A} . Then we check that it is compatible with F . For this, let $a \in F$ and $\langle a, b \rangle \in R$. Observe that $a \cdot a = a \in F$ and therefore, since $\langle a, b \rangle \in R$, that $b \cdot a \in F$. Moreover it holds that $\langle b \cdot a, b \rangle \in R$, since R is a congruence. Therefore, from the fact that $(b \cdot a) \cdot b = b \cdot a \in F$ and $\langle b \cdot a, b \rangle \in R$, it follows that $b = b \cdot b \in F$. We conclude that R is compatible with F .

It only remains to prove that R is the largest congruence over \mathbf{A} compatible with F . For this, let $\theta \in \text{Co}(\mathbf{A})$ be compatible with F and $\langle a, b \rangle \in \theta$. Then $\langle a \cdot c, b \cdot c \rangle \in \theta$ for any $c \in A$. Since θ is compatible with F , this yields that $a \cdot c \in F$ if and only if $b \cdot c \in F$. We conclude that $\langle a, b \rangle \in R$ and therefore that $\theta \subseteq R$. \square

This result allows us to characterise Leibniz-reduced models of \mathcal{L}_{SL} and therefore semilattices that belong to $\text{Alg}^*\mathcal{L}_{\text{SL}}$.

Theorem 3.9. *Let $\mathbf{A} \in \text{SL}$ and $F \subseteq A$. $\langle \mathbf{A}, F \rangle \in \text{Mod}^*\mathcal{L}_{\text{SL}}$ if and only if for every $a, b \in A$ such that $a < b$, there is $d \in A$ such that $d \leq b$ and either $[a \cdot d \in F \text{ and } d \notin F]$ or $[a \cdot d \notin F \text{ and } d \in F]$.*

Proof. For the “only if” direction, assume $\Omega F = \text{Id}_{\mathbf{A}}$ and let $a, b \in A$ be such that $a < b$. Since in particular $a \neq b$, by Lemma 3.8 there is $c \in A$ such that either $[a \cdot c \in F \text{ and } b \cdot c \notin F]$ or $[a \cdot c \notin F \text{ and } b \cdot c \in F]$. Put $d := b \cdot c$. Since $a \leq b$, we have that $a \cdot d = a \cdot (b \cdot c) = a \cdot c$. Therefore either $[a \cdot d \in F \text{ and } d \notin F]$ or $[a \cdot d \notin F \text{ and } d \in F]$, as required.

For the “if” direction we reason by contraposition and take $a, b \in A$ such that $a \neq b$. We can assume, without loss of generality, that $a \cdot b < b$. Therefore, by the assumption, there is $d \in A$ such that $d \leq b$ and either $[(a \cdot b) \cdot d \in F \text{ and } d \notin F]$ or $[(a \cdot b) \cdot d \notin F \text{ and } d \in F]$. Since $d = b \cdot d$, by Lemma 3.8 this yields that $\langle a \cdot b, b \rangle \notin \Omega F$. Since ΩF is a congruence, this implies that $\langle a, b \rangle \notin \Omega F$. This shows that $\Omega F \neq \text{Id}_{\mathbf{A}}$. \square

Thus, the answer to Question 4 would be positive if for every $\mathbf{A} \in \mathbf{SL}$ there is a subset $F \subseteq A$ satisfying the condition expressed in Lemma 3.9, and negative otherwise. As we mentioned above, in general this is still an open problem, and no characterisation of the class $\text{Alg}^* \mathcal{L}_{\mathbf{SL}}$ has been found that makes no reference to the subset F , unlike in [10]. However, we will show that such a subset exists for a wide class of semilattices. In order to do this, we need to introduce some more notation. Let $\mathbf{A} \in \mathbf{SL}$. Given $a, b \in A$ with $a \leq b$, we put $[a, b] := \{c \in A : a \leq c \leq b\}$, and we write $a \prec b$ when $a < b$ and there is no $c \in A$ such that $a < c < b$. If $\mathbf{A} \in \mathbf{SL}$ and $a \in A$, the **height** of a , denoted by $\mathcal{H}(a)$, is the maximum length of the chains in \mathbf{A} having a as their top; this maximum, of course, might not exist, in which case we say it is infinite. But we are going to use this notion only in cases where it is finite:

Definition 3.10. Let $\mathbf{A} \in \mathbf{SL}$. \mathbf{A} has **sectionally finite height** when all its elements have finite height, that is, when $\mathcal{H}(a)$ is finite for all $a \in A$.

Of course every finite semilattice has sectionally finite height. In order to prove that semilattices with sectionally finite height belong to $\text{Alg}^* \mathcal{L}_{\mathbf{SL}}$, we now draw a “rainbow” on each of them:

Definition 3.11. Let $\mathbf{A} \in \mathbf{SL}$ have sectionally finite height. $R(\mathbf{A}) := \{a \in A : \mathcal{H}(a) = 2n + 1 \text{ for some } n \in \mathbb{N}\}$.

Then $R(\mathbf{A})$ is exactly the kind of subset we are looking for:

Theorem 3.12. *If $\mathbf{A} \in \mathbf{SL}$ has sectionally finite height, then $\Omega R(\mathbf{A}) = \text{Id}_{\mathbf{A}}$.*

Proof. We reason towards a contradiction: suppose that $\Omega R(\mathbf{A}) \neq \text{Id}_{\mathbf{A}}$. Then there are two different $b, c \in A$ such that $\langle b, c \rangle \in \Omega R(\mathbf{A})$. Put $a := b \cdot c$. We can assume, without loss of generality, that $a < b$. Moreover, from the fact that $\langle b, c \rangle \in \Omega R(\mathbf{A})$ and that $\Omega R(\mathbf{A})$ is a congruence, it follows that $\langle a, b \rangle \in \Omega R(\mathbf{A})$. Since $\Omega R(\mathbf{A})$ is compatible with $R(\mathbf{A})$, this yields that $a \in R(\mathbf{A})$ if and only if $b \in R(\mathbf{A})$.

Consider the case in which $a, b \in R(\mathbf{A})$. Even though $a = a \cdot b$, we keep writing $a \cdot b$ instead of a in order to make the general construction clearer. By the fact that \mathbf{A} has sectionally finite height and that $a \cdot b < b$, we know that there is $a_1 \in [a \cdot b, b]$ such that $a_1 \prec b$. Since $\langle a \cdot b, b \rangle \in \Omega R(\mathbf{A})$, it holds that $[a \cdot b, b] \subseteq b / \Omega R(\mathbf{A})$ and therefore, in particular, that $\langle a_1, b \rangle \in \Omega R(\mathbf{A})$. Since $\Omega R(\mathbf{A})$ is compatible with $R(\mathbf{A})$, this yields that $a_1 \in R(\mathbf{A})$. We conclude that $\mathcal{H}(a_1) = 2n + 1$ for some $n \in \mathbb{N}$. Since $b \in R(\mathbf{A})$ and $a_1 < b$, this implies that $\mathcal{H}(b) = 2k + 1$ for some $k > n$. Therefore there is $b_1 \in A$ such that $\mathcal{H}(b_1) = \mathcal{H}(a_1)$ and $c \in A$ such that $b_1 < c < b$.

In particular this yields that $b_1 \neq a_1$, since $a_1 \prec b$. From the fact that $\mathcal{H}(b_1) = \mathcal{H}(a_1)$ it follows that $a_1 \cdot b_1 < b_1$. Now recall that $\langle a_1, b \rangle \in \Omega R(\mathbf{A})$, therefore we have that $\langle a_1 \cdot b_1, b_1 \rangle = \langle a_1 \cdot b_1, b \cdot b_1 \rangle \in \Omega R(\mathbf{A})$. We conclude that $[a_1 \cdot b_1, b_1] \subseteq b_1 / \Omega R(\mathbf{A})$. This allows us to repeat exactly the same argument and construct an element b_2 such that $b_2 < b_1$. Repeating this process we build an infinite descending chain $\dots < b_n < \dots < b_2 < b_1 < b$. But this contradicts the assumption that \mathbf{A} has locally finite height. Therefore we are done.

It only remains to prove the case in which $a, b \notin R(\mathbf{A})$, but this is dual to the previous one in the sense that it can be carried on in the same way but working with elements whose height is even instead of odd. \square

Corollary 3.13. *If $\mathbf{A} \in \mathbf{SL}$ has sectionally finite height, then $\mathbf{A} \in \text{Alg}^* \mathcal{L}_{\mathbf{SL}}$.*

Proof. Let $\mathbf{A} \in \mathbf{SL}$ have sectionally finite height. Then observe that, by definition of $\mathcal{L}_{\mathbf{SL}}$, the matrix $\langle \mathbf{A}, R(\mathbf{A}) \rangle$ is a model of $\mathcal{L}_{\mathbf{SL}}$. By Theorem 3.12 we conclude that $\langle \mathbf{A}, R(\mathbf{A}) \rangle \in \text{Mod}^* \mathcal{L}_{\mathbf{SL}}$ and therefore that $\mathbf{A} \in \text{Alg}^* \mathcal{L}_{\mathbf{SL}}$. \square

In particular $\text{Alg}^* \mathcal{L}_{\mathbf{SL}}$ contains all finite semilattices, but also many others that are not.

The above construction of rainbows in semilattices with sectionally finite height has been further exploited in [13] in order to prove that in these semilattices the Leibniz operator gives rise to a bijection between some peculiar subsets of the algebra (called “clouds”) and the set of all its congruences.

4. ORDERING THE LOGICS

Now that we learned some basic properties of the logics of semilattices, we would like to take a look at the poset they form:

$$\mathbf{Log}(\mathbf{SL}) := \langle \{ \mathcal{L} : \mathcal{L} \text{ is a logic of semilattices} \}, \leq \rangle.$$

From Theorem 3.3 we know that $\mathcal{L}_{\mathbf{SL}}$ is the minimum of $\mathbf{Log}(\mathbf{SL})$, and that this set is closed under meets (i.e., intersections) of arbitrary non-empty families. In order to work within the context of logics of semilattices, it is useful to know how to axiomatise our paradigmatic examples with respect to $\mathcal{L}_{\mathbf{SL}}$. Given a logic \mathcal{L} and a rule $\Gamma \vdash \varphi$, we will write $\mathcal{L} + [\Gamma \vdash \varphi]$ to denote the weakest logic extending \mathcal{L} in which the deduction $\Gamma \vdash \varphi$ holds (it is easy to see that such a logic exists).

Lemma 4.1.

1. $\mathcal{L}_{\wedge} = \mathcal{L}_{\mathbf{SL}} + [x \cdot y \vdash x] + [x, y \vdash x \cdot y]$.
2. $\mathcal{L}_{\vee} = \mathcal{L}_{\mathbf{SL}} + [x \vdash x \cdot y]$.
3. $\mathcal{L}_{\wedge \vee} = \mathcal{L}_{\mathbf{SL}} + [x, y \cdot z \vdash x \cdot y]$.

Proof. 1 is an easy consequence of Lemma 3.2 and 3 is proved in Example 2.1 of [20]. To prove 2, put $\mathcal{L}' := \mathcal{L}_{\mathbf{SL}} + [x \vdash x \cdot y]$. In Example 3.4 we proved that \mathcal{L}_{\vee} is a logic of semilattices, therefore from Theorem 3.3 it follows that $\mathcal{L}_{\mathbf{SL}} \leq \mathcal{L}_{\vee}$. Moreover the deduction $x \vdash x \cdot y$ holds in \mathcal{L}_{\vee} , therefore we conclude that $\mathcal{L}' \leq \mathcal{L}_{\vee}$. Then we turn to prove the reverse inequality, so suppose that $\Gamma \vdash_{\mathcal{L}_{\vee}} \varphi$. From property (ii) of Example 3.4 it follows that there is $\gamma \in \Gamma$ such that $\text{Var}(\gamma) \subseteq \text{Var}(\varphi)$. If $\text{Var}(\gamma) = \text{Var}(\varphi)$, by Lemma 3.2 we know that $\gamma \dashv\vdash_{\mathbf{SL}} \varphi$ and therefore $\Gamma \vdash_{\mathcal{L}'} \varphi$. Now assume that $\text{Var}(\gamma) \subsetneq \text{Var}(\varphi)$ and put $\{x_1, \dots, x_k\} := \text{Var}(\varphi) \setminus \text{Var}(\gamma)$. Consider the substitution $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ such that

$$\sigma(z) := \begin{cases} x_1 \cdot (\dots (x_{k-2} \cdot (x_{k-1} \cdot x_k)) \dots) & \text{if } z = y \\ \gamma & \text{otherwise} \end{cases}$$

for every variable z . By structurality $\sigma x \vdash_{\mathcal{L}'} \sigma(x \cdot y)$, that is, $\gamma \vdash_{\mathcal{L}'} \gamma \cdot \sigma y$, and by Lemma 3.2 we have that $\gamma \cdot \sigma y \dashv\vdash_{\mathbf{SL}} \varphi$, therefore we conclude that $\gamma \vdash_{\mathcal{L}'} \varphi$, which implies $\Gamma \vdash_{\mathcal{L}'} \varphi$. \square

We are now ready to characterise the maximal logics of semilattices, which turn out to be exactly the logics of classical conjunction and disjunction.

Theorem 4.2.

1. \mathcal{L}_{\wedge} and \mathcal{L}_{\vee} are the only maximal elements of $\mathbf{Log}(\mathbf{SL})$.
2. If $\mathcal{L} \in \mathbf{Log}(\mathbf{SL})$ and $\mathcal{L} \not\leq \mathcal{L}_{\wedge}$, then $\mathcal{L} = \mathcal{L}_{\vee}$.

3. If $\mathcal{L} \in \mathbf{Log}(\mathbf{SL})$ and $\mathcal{L}_{\wedge\vee} < \mathcal{L}$, then either $\mathcal{L} = \mathcal{L}_{\wedge}$ or $\mathcal{L} = \mathcal{L}_{\vee}$.

Proof. 1. We begin by proving that \mathcal{L}_{\vee} is maximal in $\mathbf{Log}(\mathbf{SL})$. For this suppose, towards a contradiction, that there is $\mathcal{L} \in \mathbf{Log}(\mathbf{SL})$ such that $\mathcal{L}_{\vee} < \mathcal{L}$. Then there is a deduction $\Gamma \vdash_{\mathcal{L}} \varphi$ such that $\Gamma \not\vdash_{\mathcal{L}_{\vee}} \varphi$. From property (ii) of Example 3.4 it follows that $\text{Var}(\gamma) \not\subseteq \text{Var}(\varphi)$ for every $\gamma \in \Gamma$. Now consider the substitution $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ such that

$$\sigma(z) := \begin{cases} x & \text{if } z \notin \text{Var}(\varphi) \\ y & \text{otherwise} \end{cases}$$

for every variable z . By structurality we have that $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$. Observe that, by Lemma 1.2, we have that $\sigma\varphi \dashv\vdash_{\mathcal{L}} y$, and that for every $\gamma \in \Gamma$ either $\sigma\gamma \dashv\vdash_{\mathcal{L}} x \cdot y$ or $\sigma\gamma \dashv\vdash_{\mathcal{L}} x$, depending on whether $y \in \text{Var}(\sigma\gamma)$ or not. Since \mathcal{L} is a logic of semilattices, it is non-trivial and therefore, by Lemma 1.1, there is at least one $\gamma \in \Gamma$ such that $\sigma\gamma \dashv\vdash_{\mathcal{L}} x \cdot y$. We conclude that $x, x \cdot y \vdash_{\mathcal{L}} y$. Then consider a new substitution $\sigma': \mathbf{Fm} \rightarrow \mathbf{Fm}$ such that

$$\sigma'(z) := \begin{cases} x \cdot y & \text{if } z = x \\ z & \text{otherwise} \end{cases}$$

for every variable z . By structurality and Lemma 1.2 we conclude that $x \cdot y \vdash_{\mathcal{L}} y$. Since $\mathcal{L}_{\vee} \leq \mathcal{L}$, by Lemma 1.1 we conclude that \mathcal{L} is trivial against the assumption that it is a logic of semilattices. Thus, \mathcal{L}_{\vee} is maximal in $\mathbf{Log}(\mathbf{SL})$. The rest of point 1 will be proved by using point 2.

2. Assume that $\mathcal{L} \in \mathbf{Log}(\mathbf{SL})$ is such that $\mathcal{L} \not\leq \mathcal{L}_{\wedge}$. Then there is a deduction $\Gamma \vdash_{\mathcal{L}} \varphi$ such that $\Gamma \not\vdash_{\mathcal{L}_{\wedge}} \varphi$. From property (i) of Example 3.4 it follows that there is a variable $x \in \text{Var}(\varphi)$ such that $x \notin \text{Var}(\Gamma)$. Then consider the substitution $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ such that

$$\sigma(z) := \begin{cases} x & \text{if } z = x \\ y & \text{otherwise} \end{cases}$$

for every variable z . By structurality this yields that $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$. From Lemma 1.2, it follows that $y \dashv\vdash_{\mathcal{L}} \sigma\Gamma$ and either $\sigma\varphi \dashv\vdash_{\mathcal{L}} x$ or $\sigma\varphi \dashv\vdash_{\mathcal{L}} x \cdot y$, depending on whether $y \in \text{Var}(\sigma\varphi)$ or not. Since \mathcal{L} is non-trivial, by Lemma 1.1 we conclude that $\sigma\varphi \dashv\vdash_{\mathcal{L}} x \cdot y$ and therefore that $y \vdash_{\mathcal{L}} x \cdot y$. From Lemma 4.1 it follows that $\mathcal{L}_{\vee} \leq \mathcal{L}$ and then, since \mathcal{L}_{\vee} is maximal by point 1, that $\mathcal{L} = \mathcal{L}_{\vee}$.

This fact allows us to complete the proof of point 1. We begin by proving that \mathcal{L}_{\wedge} is a maximal element of $\mathbf{Log}(\mathbf{SL})$. Assume, towards a contradiction, that there is $\mathcal{L} \in \mathbf{Log}(\mathbf{SL})$ such that $\mathcal{L}_{\wedge} < \mathcal{L}$. Then in particular $\mathcal{L} \not\leq \mathcal{L}_{\wedge}$, and by point 2 this would imply that $\mathcal{L} = \mathcal{L}_{\vee}$, and hence that $\mathcal{L}_{\wedge} < \mathcal{L}_{\vee}$, which is clearly false. Hence we conclude that \mathcal{L}_{\wedge} is maximal too. That there are no maximals in $\mathbf{Log}(\mathbf{SL})$ other than \mathcal{L}_{\wedge} and \mathcal{L}_{\vee} is proved in the same way.

3. Observe that as a consequence of point 2, there is no logic strictly between $\mathcal{L}_{\wedge\vee}$ and \mathcal{L}_{\vee} , so that our goal reduces to proving that there is no logic strictly between $\mathcal{L}_{\wedge\vee}$ and \mathcal{L}_{\wedge} . Suppose that there is $\mathcal{L} \in \mathbf{Log}(\mathbf{SL})$ such that $\mathcal{L}_{\wedge\vee} < \mathcal{L} \leq \mathcal{L}_{\wedge}$. In particular this implies that there is a deduction $\Gamma \vdash \varphi$ which holds in \mathcal{L} but not in $\mathcal{L}_{\wedge\vee}$.

We claim that $\text{Var}(\gamma) \not\subseteq \text{Var}(\varphi)$ for every $\gamma \in \Gamma$. In order to prove this we reason towards a contradiction, so suppose there is $\gamma_1 \in \Gamma$ such that $\text{Var}(\gamma_1) \subseteq \text{Var}(\varphi)$. Then let $\{x_1, \dots, x_k\} := \text{Var}(\varphi) \setminus \text{Var}(\gamma_1)$. From condition (i) of Example 3.4 and the fact that $\Gamma \vdash_{\mathcal{L}_{\wedge}} \varphi$, it follows that there is $\psi_1 \in \Gamma$ such that $x_1 \in \text{Var}(\psi_1)$. By

Lemma 1.2, this yields that $\psi_1 \dashv\vdash_{\mathcal{L}_{\wedge}} x_1 \cdot \psi_1$. Now, recall from Lemma 4.1 that $x, y \cdot z \vdash_{\mathcal{L}_{\wedge}} x \cdot y$. Then we conclude that $\gamma_1, x_1 \cdot \psi_1 \vdash_{\mathcal{L}_{\wedge}} \gamma_1 \cdot x_1$ and therefore that $\Gamma \vdash_{\mathcal{L}_{\wedge}} \gamma_1 \cdot x_1$. Then we let $\gamma_2 := \gamma_1 \cdot x_1$ and consider $\psi_2 \in \Gamma$ such that $x_2 \in \text{Var}(\psi_2)$. The same argument yields $\Gamma \vdash_{\mathcal{L}_{\wedge}} \gamma_2 \cdot x_2$. Iterating this process we obtain that $\Gamma \vdash_{\mathcal{L}_{\wedge}} \gamma_{k-1} \cdot x_k$, where $\gamma_{k-1} \cdot x_k \dashv\vdash_{\mathcal{L}_{\wedge}} \varphi$ by Lemma 1.2, against the assumption. This concludes the proof of our claim.

Then we consider the substitution $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ defined as

$$\sigma(z) := \begin{cases} x & \text{if } z \in \text{Var}(\varphi) \\ y & \text{otherwise} \end{cases}$$

for every variable z . By structurality we have that $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$. From Lemma 1.2, it follows that $\sigma\varphi \dashv\vdash_{\mathcal{L}} x$. Moreover our claim implies that $\{y\} \subseteq \text{Var}(\sigma\gamma) \subseteq \{x, y\}$ for every $\gamma \in \Gamma$ and that either $\sigma\gamma \dashv\vdash_{\mathcal{L}} x \cdot y$ or $\sigma\gamma \dashv\vdash_{\mathcal{L}} y$, depending on whether $x \in \text{Var}(\sigma\gamma)$ or not. We conclude that $y, x \cdot y \vdash_{\mathcal{L}} x$. Then consider a new substitution $\sigma': \mathbf{Fm} \rightarrow \mathbf{Fm}$ such that

$$\sigma'(z) := \begin{cases} x \cdot y & \text{if } z = y \\ z & \text{otherwise} \end{cases}$$

for every variable z . By structurality and Lemma 1.2 we conclude that $x \cdot y \vdash_{\mathcal{L}} x$. Therefore by Lemma 4.1, in order to prove that $\mathcal{L} = \mathcal{L}_{\wedge}$, it is enough to check that $x, y \vdash_{\mathcal{L}} x \cdot y$. But this follows easily from the fact that $x, y \cdot z \vdash_{\mathcal{L}_{\wedge}} x \cdot y$, therefore we are done. \square

Our next goal is to prove that $\mathbf{Log}(\text{SL})$ is **atomless**, i.e., that there is no $\mathcal{L} \in \mathbf{Log}(\text{SL})$ such that $\mathcal{L}_{\text{SL}} < \mathcal{L}$ and for every $\mathcal{L}' \in \mathbf{Log}(\text{SL})$ if $\mathcal{L}_{\text{SL}} < \mathcal{L}' \leq \mathcal{L}$, then $\mathcal{L}' = \mathcal{L}$. In order to do this, let us fix for the rest of the section an enumeration of the set of variables of our language $\text{Var} = \{x_0, x_1, x_2, \dots\}$. For each natural number n , we define a set of formulas and a logic:

$$\begin{aligned} W(n) &:= \{\varphi \in \mathbf{Fm} : \text{Var}(\varphi) \subsetneq \{x_0, \dots, x_n\} \text{ or } \text{Var}(\varphi) = \{x_0, \dots, x_{n+1}\}\} \\ \mathcal{W}_n &:= \mathcal{L}_{\text{SL}} + [W(n) \vdash x_0 \cdot (x_1 \cdot \dots \cdot (x_{n-1} \cdot x_n) \dots)] \end{aligned} \quad (1)$$

Observe that, since the logics \mathcal{W}_n are extensions of \mathcal{L}_{SL} , the set of premises $W(n)$ of the deduction (1) above can be finitised by selecting its formulas in which no variable occurs more than once. Observe in particular that \mathcal{W}_0 is the weakest logic extending \mathcal{L}_{SL} in which the deduction $x_0 \cdot x_1 \vdash x_0$ holds; this implies that \mathcal{W}_0 is strictly weaker than \mathcal{L}_{\wedge} , but not weaker than \mathcal{L}_{\vee} .

Lemma 4.3. *Let $\mathcal{L} \in \mathbf{Log}(\text{SL})$ and let $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$ be such that $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Gamma \not\vdash_{\text{SL}} \varphi$. Then $\mathcal{W}_{k-1} \leq \mathcal{L}$, where $k := |\text{Var}(\varphi)|$.*

Proof. We start by choosing any $\psi \in \mathbf{Fm}$ such that $\text{Var}(\psi) = \{x_0, \dots, x_k\}$ and consider a substitution $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ which maps bijectively $\text{Var}(\varphi)$ into $\{x_0, \dots, x_{k-1}\}$ and each variable $y \notin \text{Var}(\varphi)$ to ψ . By structurality $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$. From Lemma 1.2, it follows that $\sigma\varphi \dashv\vdash_{\mathcal{L}} x_0 \cdot (x_1 \cdot \dots \cdot (x_{k-2} \cdot x_{k-1}) \dots)$. Moreover observe that, since $\Gamma \not\vdash_{\text{SL}} \varphi$, by Lemma 3.2 there is no $\gamma \in \Gamma$ such that $\text{Var}(\gamma) = \text{Var}(\varphi)$. This yields that for every $\gamma \in \Gamma$ either $\text{Var}(\sigma\gamma) = \{x_0, \dots, x_k\}$ or $\text{Var}(\sigma\gamma) \subsetneq \{x_0, \dots, x_{k-1}\}$. Therefore we conclude that $\sigma\Gamma \subseteq W(k-1)$. But this yields $W(k-1) \vdash_{\mathcal{L}} x_0 \cdot (x_1 \cdot \dots \cdot (x_{k-2} \cdot x_{k-1}) \dots)$ and therefore that $\mathcal{W}_{k-1} \leq \mathcal{L}$. \square

We are now ready to prove that $\mathbf{Log}(\text{SL})$ is atomless.

Theorem 4.4.

1. If $n < m$, then $\mathcal{W}_m < \mathcal{W}_n$.
2. If $\mathcal{L} \in \mathbf{Log}(\mathbf{SL})$ and $\mathcal{L}_{\mathbf{SL}} < \mathcal{L}$, then $\mathcal{W}_n < \mathcal{L}$ for some n ; that is, $\mathcal{L}_{\mathbf{SL}} = \bigcap_{n \in \omega} \mathcal{W}_n$.
3. $\mathbf{Log}(\mathbf{SL})$ is atomless.

Proof. 1. Let $n < m$. In order to prove that $\mathcal{W}_m \leq \mathcal{W}_n$, consider the substitution $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ such that

$$\sigma(z) := \begin{cases} z \cdot (x_{n+1} \cdot (x_{n+2} \cdot \dots \cdot (x_{m-1} \cdot x_m) \dots)) & \text{if } z \in \{x_0, \dots, x_n\} \\ z \cdot x_{m+1} & \text{otherwise} \end{cases}$$

for every variable z . By structurality, $\sigma W(n) \vdash_{\mathcal{W}_n} \sigma(x_0 \cdot (x_1 \cdot \dots \cdot (x_{n-1} \cdot x_n) \dots))$. By Lemma 1.2 we know that

$$\sigma(x_0 \cdot (x_1 \cdot \dots \cdot (x_{n-1} \cdot x_n) \dots)) \dashv\vdash_{\mathcal{W}_n} x_0 \cdot (x_1 \cdot \dots \cdot (x_{m-1} \cdot x_m) \dots).$$

Moreover it is easy to check that $\sigma W(n) \subseteq W(m)$ and therefore that $W(m) \vdash_{\mathcal{W}_n} x_0 \cdot (x_1 \cdot \dots \cdot (x_{m-1} \cdot x_m) \dots)$. This shows that $\mathcal{W}_m \leq \mathcal{W}_n$, and it only remains to prove that $\mathcal{W}_m \neq \mathcal{W}_n$. In order to do this, take $F := \{\llbracket \gamma \rrbracket \in \mathbf{Fm}_{\mathbf{SL}} : \gamma \in W(n)\}$. We will prove that $\langle \mathbf{Fm}_{\mathbf{SL}}, F \rangle$ is a model of \mathcal{W}_m . Clearly $\langle \mathbf{Fm}_{\mathbf{SL}}, F \rangle$ is a model of $\mathcal{L}_{\mathbf{SL}}$, therefore it will be enough to check that it is a model of the rule $W(m) \vdash x_0 \cdot (x_1 \cdot \dots \cdot (x_{m-1} \cdot x_m) \dots)$.

Suppose the contrary. Then there is an homomorphism $h: \mathbf{Fm} \rightarrow \mathbf{Fm}_{\mathbf{SL}}$ such that $h[W(m)] \subseteq F$ and $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_{m-1} \cdot x_m) \dots)) \notin F$. Since $m \neq 0$, observe that $\{x_0, \dots, x_m\} \subseteq W(m)$. Hence $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_{m-1} \cdot x_m) \dots))$ belongs to the subsemilattice of $\mathbf{Fm}_{\mathbf{SL}}$ generated by $h[W(m)]$. In particular, since $h[W(m)] \subseteq F$, this implies that $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_{m-1} \cdot x_m) \dots))$ belongs to the subsemilattice of $\mathbf{Fm}_{\mathbf{SL}}$ generated by F . But, from the definition of F , it follows that this subsemilattice has universe $F \cup \{\llbracket x_0 \cdot (x_1 \cdot \dots \cdot (x_{n-1} \cdot x_n) \dots) \rrbracket\}$. Therefore we conclude that $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_{m-1} \cdot x_m) \dots)) = \llbracket x_0 \cdot (x_1 \cdot \dots \cdot (x_{n-1} \cdot x_n) \dots) \rrbracket$, since $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_{m-1} \cdot x_m) \dots)) \notin F$.

In particular, this implies that for every $\llbracket x_k \rrbracket \in \{\llbracket x_0 \rrbracket, \dots, \llbracket x_n \rrbracket\}$, there is $x^k \in \{x_0, \dots, x_m\}$ such that $\llbracket x_0 \cdot (x_1 \cdot \dots \cdot (x_{n-1} \cdot x_n) \dots) \rrbracket \leq h(x^k) \leq \llbracket x_k \rrbracket$. This clearly yields that $h(x^0 \cdot (x^1 \cdot \dots \cdot (x^{n-1} \cdot x^n) \dots)) = \llbracket x_0 \cdot (x_1 \cdot \dots \cdot (x_{n-1} \cdot x_n) \dots) \rrbracket \notin F$. But, since $n < m$, we have that $x^0 \cdot (x^1 \cdot \dots \cdot (x^{n-1} \cdot x^n) \dots) \in W(m)$ against the assumption that $h[W(m)] \subseteq F$. Therefore we conclude that $\langle \mathbf{Fm}_{\mathbf{SL}}, F \rangle$ is a model of \mathcal{W}_m .

The fact that $\langle \mathbf{Fm}_{\mathbf{SL}}, F \rangle$ is not a model of $W(n)$ follows from the fact that $\pi W(n) = F$ and $\pi(x_0 \cdot (x_1 \cdot \dots \cdot (x_{n-1} \cdot x_n) \dots)) \notin F$, where $\pi: \mathbf{Fm} \rightarrow \mathbf{Fm}_{\mathbf{SL}}$ is the projection onto the quotient. We conclude that $\mathcal{W}_m \neq \mathcal{W}_n$ and therefore we are done.

2. Assume $\mathcal{L} \in \mathbf{Log}(\mathbf{SL})$ is such that $\mathcal{L}_{\mathbf{SL}} < \mathcal{L}$. Then there is a deduction $\Gamma \vdash_{\mathcal{L}} \varphi$ such that $\Gamma \not\vdash_{\mathcal{L}_{\mathbf{SL}}} \varphi$. From Lemma 4.3 it follows that $\mathcal{W}_{|\text{Var}(\varphi)|-1} \leq \mathcal{L}$. Now, by point 1, we conclude that $\mathcal{W}_{|\text{Var}(\varphi)|} < \mathcal{L}$.

3 is an easy consequence of points 1 and 2. \square

Theorem 4.2 implies that each logic of semilattices that is strictly weaker than \mathcal{L}_{\vee} is strictly weaker than \mathcal{L}_{\wedge} too. We will show that the behaviour of \mathcal{L}_{\vee} and \mathcal{L}_{\wedge} is not analogous, by constructing several logics of semilattices strictly weaker than \mathcal{L}_{\wedge} which are not weaker than \mathcal{L}_{\vee} . A first example of such logics is \mathcal{W}_0 ; therefore, each logic of semilattices that extends \mathcal{W}_0 will be weaker than \mathcal{L}_{\wedge} and not weaker than \mathcal{L}_{\vee} . We will see that there is an infinite descending chain of logics (of semilattices)

between \mathcal{L}_\wedge and \mathcal{W}_0 . In order to do this, for each natural number n we define a set of formulas and a corresponding logic:

$$\begin{aligned} R(n) &:= \{\varphi \in Fm : Var(\varphi) \not\subseteq \{x_0, \dots, x_{n+1}\}\} \\ \mathcal{R}_n &:= \mathcal{W}_0 + [R(n) \vdash x_0 \cdot (x_1 \cdot \dots \cdot (x_n \cdot x_{n+1}) \dots)] \end{aligned} \quad (2)$$

Also in this case the set of premisses $R(n)$ of the deduction (2) can be finitised by selecting the formulas in which no variable occurs more than once.

Theorem 4.5.

1. $\mathcal{R}_0 = \mathcal{L}_\wedge$.
2. If $n < m$, then $\mathcal{R}_m < \mathcal{R}_n$.

Proof. 1 is an easy exercise, therefore we turn to prove 2. For $n < m$ consider the substitution $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ defined as

$$\sigma(z) := \begin{cases} z \cdot (x_{n+2} \cdot (x_{n+3} \cdot \dots \cdot (x_m \cdot x_{m+1}) \dots)) & \text{if } z \in \{x_0, \dots, x_{n+1}\} \\ z & \text{otherwise} \end{cases}$$

for every variable z . By structurality, $\sigma R(n) \vdash_{\mathcal{R}_n} \sigma(x_0 \cdot (x_1 \cdot \dots \cdot (x_n \cdot x_{n+1}) \dots))$. By Lemma 1.2, we know that

$$\sigma(x_0 \cdot (x_1 \cdot \dots \cdot (x_n \cdot x_{n+1}) \dots)) \dashv\vdash_{\mathcal{R}_n} x_0 \cdot (x_1 \cdot \dots \cdot (x_m \cdot x_{m+1}) \dots).$$

Moreover it is easy to prove that $\sigma R(m) \subseteq R(n)$. We conclude that $R(m) \vdash_{\mathcal{R}_n} x_0 \cdot (x_1 \cdot \dots \cdot (x_m \cdot x_{m+1}) \dots)$ and therefore that $\mathcal{R}_m \leq \mathcal{R}_n$.

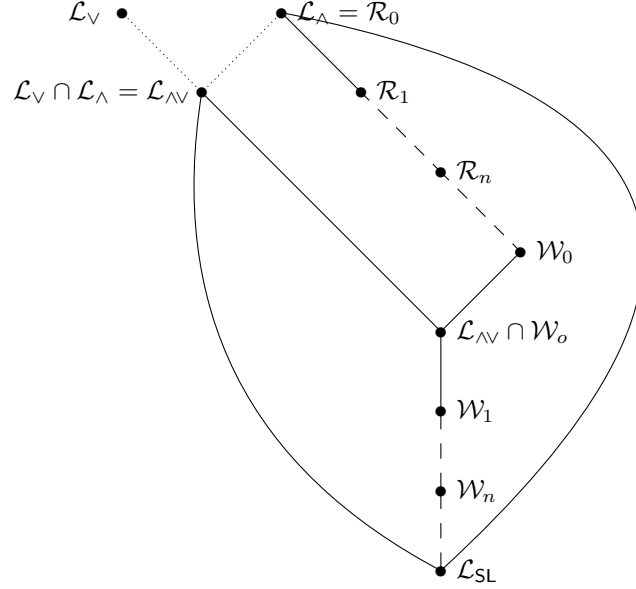
It only remains to prove that $\mathcal{R}_m \neq \mathcal{R}_n$. In order to do this, take $F := \{\llbracket \gamma \rrbracket \in Fm_{\text{SL}} : \gamma \in R(n)\}$. We will prove that $\langle \mathbf{Fm}_{\text{SL}}, F \rangle$ is a model of \mathcal{R}_m . Clearly $\langle \mathbf{Fm}_{\text{SL}}, F \rangle$ is a model of \mathcal{W}_0 , therefore it will be enough to check that it is a model of the rule $R(m) \vdash x_0 \cdot (x_1 \cdot \dots \cdot (x_m \cdot x_{m+1}) \dots)$.

Suppose the contrary, so that there is an homomorphism $h: \mathbf{Fm} \rightarrow \mathbf{Fm}_{\text{SL}}$ such that $h[R(m)] \subseteq F$ and $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_m \cdot x_{m+1}) \dots)) \notin F$. Then observe that $\{x_0, \dots, x_{m+1}\} \subseteq R(m)$. Hence $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_m \cdot x_{m+1}) \dots))$ belongs to the subsemilattice of \mathbf{Fm}_{SL} generated by $h[R(m)]$. In particular, since $h[R(m)] \subseteq F$, this implies that $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_m \cdot x_{m+1}) \dots))$ belongs to the subsemilattice of \mathbf{Fm}_{SL} generated by F . But, from the definition of F , it follows that this subsemilattice has universe $F \cup \{\llbracket x_0 \cdot (x_1 \cdot \dots \cdot (x_n \cdot x_{n+1}) \dots) \rrbracket\}$. Therefore we conclude that $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_m \cdot x_{m+1}) \dots)) = \llbracket x_0 \cdot (x_1 \cdot \dots \cdot (x_n \cdot x_{n+1}) \dots) \rrbracket$, since $h(x_0 \cdot (x_1 \cdot \dots \cdot (x_m \cdot x_{m+1}) \dots)) \notin F$.

In particular, this implies that for every $\llbracket x_k \rrbracket \in \{\llbracket x_0 \rrbracket, \dots, \llbracket x_{n+1} \rrbracket\}$, there is $x^k \in \{x_0, \dots, x_{m+1}\}$ such that $\llbracket x_0 \cdot (x_1 \cdot \dots \cdot (x_n \cdot x_{n+1}) \dots) \rrbracket \leq h(x^k) \leq \llbracket x_k \rrbracket$. This clearly yields that $h(x^0 \cdot (x^1 \cdot \dots \cdot (x^n \cdot x^{n+1}) \dots)) = \llbracket x_0 \cdot (x_1 \cdot \dots \cdot (x_n \cdot x_{n+1}) \dots) \rrbracket \notin F$. But, since $n < m$, we have that $x^0 \cdot (x^1 \cdot \dots \cdot (x^n \cdot x^{n+1}) \dots) \in R(m)$ against the assumption that $h[R(m)] \subseteq F$. Therefore we conclude that $\langle \mathbf{Fm}_{\text{SL}}, F \rangle$ is a model of \mathcal{R}_m .

The fact that $\langle \mathbf{Fm}_{\text{SL}}, F \rangle$ is not a model of $R(n)$, follows from the fact that $\pi R(n) = F$ and $\pi(x_0 \cdot (x_1 \cdot \dots \cdot (x_n \cdot x_{n+1}) \dots)) \notin F$, where $\pi: \mathbf{Fm} \rightarrow \mathbf{Fm}_{\text{SL}}$ is the projection onto the quotient. We conclude that $\mathcal{R}_m \neq \mathcal{R}_n$ and therefore we are done. \square

Figure 2 presents a partial picture of $\mathcal{L}\text{og}(\text{SL})$. The dotted lines indicate that there is no logic of semilattices strictly between their edges, while dashed and solid


 FIGURE 2. Order relations in a part of $\mathcal{L}og(\mathbf{SL})$.

lines admit the presence of other logics; in particular the dashed lines indicate the location of the two infinite families of logics constructed in this section.

5. THE LOGIC OF CONJUNCTION

In the previous sections we assumed a very general perspective, embracing every logic whose algebraic counterpart is the variety of semilattices. Now we would like to focus on a concrete such logic, namely the logic of classical conjunction \mathcal{L}_\wedge . Our first aim will be to prove that \mathcal{L}_\wedge is intrinsically bivalent. In order to do this let us record that deductive filters of \mathcal{L}_\wedge coincide with subsets for which \cdot behaves like a comma.

Lemma 5.1. *Let $F \subseteq A$. $F \in \mathcal{Fi}_{\mathcal{L}_\wedge}(A)$ if and only if $[a, b \in F \iff a \cdot b \in F]$ for every $a, b \in A$. \square*

This fact implies that the Leibniz congruence literally “cuts into two parts” the models of the logic, so that all Leibniz-reduced algebras will forcefully be two valued.

Lemma 5.2. *Let $F \in \mathcal{Fi}_{\mathcal{L}_\wedge}(A)$. For every $a, b \in A$ it holds that $\langle a, b \rangle \in \Omega F$ if and only if $[a, b \in F \text{ or } a, b \notin F]$.*

Proof. Let $R \subseteq A \times A$ be the relation defined as $\langle a, b \rangle \in R$ if and only if $[a, b \in F \text{ or } a, b \notin F]$. It is clear the R is an equivalence relation and that it is the maximal one compatible with F . Therefore, in order prove that $R = \Omega F$, it only remains to check that R is a congruence. But this easily follows from Lemma 5.1. \square

From this result it is possible to characterise the class of Leibniz-reduced models of \mathcal{L}_\wedge , the interesting point here being that \mathcal{L}_\wedge represents a case of extreme difference

between the class of Leibniz-reduced algebras and its algebraic counterpart, since one of them seems to be extremely poor.²

Corollary 5.3.

1. $\text{Alg}^*\mathcal{L}_\wedge = \mathbb{I}\{\mathbf{1}, \mathbf{2}\}$ and $\text{Alg}\mathcal{L}_\wedge = \text{SL}$.
2. $\text{Mod}^*\mathcal{L}_\wedge = \mathbb{I}\{\langle \mathbf{1}, \emptyset \rangle, \langle \mathbf{1}, \{1\} \rangle, \langle \mathbf{2}, \{1\} \rangle\}$.

Proof. 1. Recall from Example 3.4 that $\text{Alg}\mathcal{L}_\wedge = \text{SL}$, so that $\text{Alg}^*\mathcal{L}_\wedge \subseteq \text{SL}$. By Lemma 5.2 each member of $\text{Alg}^*\mathcal{L}_\wedge$ has at most two elements. Since $\mathbf{2}$ is (up to isomorphism) the only two-element semilattice, this yields that $\text{Alg}^*\mathcal{L}_\wedge \subseteq \mathbb{I}\{\mathbf{1}, \mathbf{2}\}$. Moreover $\mathbf{1} \in \text{Alg}^*\mathcal{L}$ for every logic \mathcal{L} , and it is straightforward to prove that $\langle \mathbf{2}, \{1\} \rangle$ is a reduced model of \mathcal{L}_\wedge and therefore that $\mathbf{2} \in \text{Alg}^*\mathcal{L}_\wedge$. We conclude that $\text{Alg}^*\mathcal{L}_\wedge = \mathbb{I}\{\mathbf{1}, \mathbf{2}\}$.

2 follows easily from 1 and Lemma 5.1. ⊠

Keeping in mind the fact that \mathcal{L}_\wedge is the $\{\wedge\}$ -fragment of classical propositional logic, it is easy to prove a general, and somehow surprising completeness result for it.

Lemma 5.4. *The logic \mathcal{L}_\wedge is complete with respect to any matrix $\langle \mathbf{A}, F \rangle$ such that \mathbf{A} is non trivial and $F \in \text{Fi}_{\mathcal{L}_\wedge}(\mathbf{A}) \setminus \{\emptyset, A\}$.*

Proof. The $\{\wedge\}$ -fragment of classical logic is, by definition, complete with respect to the matrix $\langle \mathbf{2}, \{1\} \rangle$. But by using Lemma 5.2 it is easy to prove that $\langle \mathbf{A}, F \rangle^* \cong \langle \mathbf{2}, \{1\} \rangle$. Since a matrix and its reduction via the Leibniz congruence determine the same logic, we conclude that \mathcal{L}_\wedge will be complete with respect to the matrix $\langle \mathbf{A}, F \rangle$. ⊠

We are now ready to state our desired result which characterises \mathcal{L}_\wedge from its class of Leibniz-reduced algebras, namely as the unique logic which has a purely bivalent algebra-based semantics from the point of view of the Leibniz operator.

Theorem 5.5. *\mathcal{L}_\wedge is the only logic \mathcal{L} such that $\text{Alg}^*\mathcal{L} = \mathbb{I}\{\mathbf{1}, \mathbf{2}\}$.*

Proof. From point 1 of Corollary 5.3 we know that $\text{Alg}^*\mathcal{L}_\wedge = \mathbb{I}\{\mathbf{1}, \mathbf{2}\}$. Therefore it will be enough to check that for every logic \mathcal{L} if $\text{Alg}^*\mathcal{L} = \mathbb{I}\{\mathbf{1}, \mathbf{2}\}$, then $\mathcal{L} = \mathcal{L}_\wedge$. In order to do this, let \mathcal{L} be such that $\text{Alg}^*\mathcal{L} = \mathbb{I}\{\mathbf{1}, \mathbf{2}\}$. First observe that the matrix $\langle \mathbf{1}, \{1\} \rangle$ is a reduced model of every logic. Moreover, since $\text{Alg}\mathcal{L} = \mathbb{P}_{\text{sd}}\text{Alg}^*\mathcal{L} = \text{SL}$, we can apply Lemma 3.5 and get that $\langle \mathbf{1}, \emptyset \rangle \in \text{Mod}^*\mathcal{L}$ too. It is clear that there cannot be other matrices whose algebraic reduct is $\mathbf{1}$.

The only reduced matrices whose algebraic reduct is $\mathbf{2}$ are $\langle \mathbf{2}, \{0\} \rangle$ and $\langle \mathbf{2}, \{1\} \rangle$. Since $\mathbf{2} \in \text{Alg}^*\mathcal{L}$ we know that at least one of these should be a model of \mathcal{L} . Then suppose towards a contradiction that $\langle \mathbf{2}, \{0\} \rangle$ is a model of \mathcal{L} . Recall from Figure 1 how $\mathbf{Fm}_{\text{SL}}\{x, y\}$ looks like. We consider the epimorphisms $f, g: \mathbf{Fm}_{\text{SL}}\{x, y\} \rightarrow \mathbf{2}$

²The contents of the following corollary was stated without proof by Rautenberg in the example in page 68 of [21].

defined as

$$f(a) := \begin{cases} 1 & \text{if } a = \llbracket x \rrbracket \\ 0 & \text{if } a \in \{\llbracket y \rrbracket, \llbracket x \cdot y \rrbracket\} \end{cases}$$

$$g(a) := \begin{cases} 1 & \text{if } a = \llbracket y \rrbracket \\ 0 & \text{if } a \in \{\llbracket x \rrbracket, \llbracket x \cdot y \rrbracket\}. \end{cases}$$

for every $a \in \mathbf{Fm}_{\mathbf{SL}}\{x, y\}$. Since inverse images of deductive filters under homomorphisms are deductive filters and the family of deductive filters is closed under intersections we conclude that $\{\llbracket x \cdot y \rrbracket\} \in \mathcal{Fi}_{\mathcal{L}}(\mathbf{Fm}_{\mathbf{SL}}\{x, y\})$. Since $\Omega\{\llbracket x \cdot y \rrbracket\} = \text{Id}_{\mathbf{Fm}_{\mathbf{SL}}\{x, y\}}$, we would have $\mathbf{Fm}_{\mathbf{SL}}\{x, y\} \in \text{Alg}^*\mathcal{L}$ against the assumption that $\text{Alg}^*\mathcal{L} = \mathbb{I}\{\mathbf{1}, \mathbf{2}\}$. Therefore we conclude that the only reduced model of \mathcal{L} , whose algebraic reduct is $\mathbf{2}$, is $\langle \mathbf{2}, \{1\} \rangle$. This yields that $\text{Mod}^*\mathcal{L} = \mathbb{I}\{\langle \mathbf{1}, \emptyset \rangle, \langle \mathbf{1}, \{1\} \rangle, \langle \mathbf{2}, \{1\} \rangle\}$. Since a logic is characterised by its Leibniz-reduced models, from Corollary 5.3 it follows that $\mathcal{L} = \mathcal{L}_{\wedge}$. \square

We would like to turn back now to the more general context of logics of semilattices and ask if it is possible to characterise \mathcal{L}_{\wedge} among them. Theorem 4.2 provides a first step in this direction by telling us that \mathcal{L}_{\wedge} is a maximal element in $\mathbf{Log}(\mathbf{SL})$, but the next result provides a more precise characterisation, namely a characterisation in terms of a property of the Leibniz operator. In order to do this, let us fix some terminology: given two complete lattices \mathbf{A} and \mathbf{B} , we say that a map $\alpha: A \rightarrow B$ **disconnects points** if for every $a, b \in A \setminus \{\perp, \top\}$ such that $a \neq b$, the images $\alpha(a)$ and $\alpha(b)$ are incomparable in \mathbf{B} . The meaning of the expression “disconnects points” is intended to remark that what α is doing is just breaking all non-trivial order connections in \mathbf{A} and mapping it (except the bounds) to a totally order-disconnected set.

Theorem 5.6. *Let \mathcal{L} be a logic of semilattices. The following conditions are equivalent:*

- (i) $\mathcal{L} = \mathcal{L}_{\wedge}$.
- (ii) $\Omega: \mathcal{Fi}_{\mathcal{L}}(\mathbf{A}) \rightarrow \text{Co}(\mathbf{A})$ disconnects points, for every algebra \mathbf{A} ;
- (iii) $\Omega: \text{Th}\mathcal{L} \rightarrow \text{Co}(\mathbf{Fm})$ disconnects points.

Proof. (i) \Rightarrow (ii) Let \mathbf{A} be an algebra; we have to prove that Ω disconnects points on $\mathcal{Fi}_{\mathcal{L}_{\wedge}}(\mathbf{A})$. From Lemma 5.2 it follows that the minimum of $\mathcal{Fi}_{\mathcal{L}_{\wedge}}(\mathbf{A})$ is \emptyset . Then pick $F, G \in \mathcal{Fi}_{\mathcal{L}_{\wedge}}(\mathbf{A}) \setminus \{\emptyset, A\}$. First we claim that $F \neq A \setminus G$. Suppose the contrary: since $G \notin \{\emptyset, A\}$, there are $a, b \in A$ such that $a \in G$ and $b \notin G$. By Lemma 5.1 this is to say that $a \cdot b \notin G$. Then $a \cdot b \in F$ and, by Lemma 5.1, $a \in F$ against the assumption that $F = A \setminus G$.

To prove that Ω disconnects points on $\mathcal{Fi}_{\mathcal{L}_{\wedge}}(\mathbf{A})$ we have to show that if $\Omega F \subseteq \Omega G$, then $F = G$. We reason towards a contradiction, so we assume that $\Omega F \subseteq \Omega G$ and that $F \neq G$. We have two cases:

1. $a \in F$ and $a \notin G$ for some $a \in A$;
2. $a \notin F$ and $a \in G$ for some $a \in A$.

We begin by case 1. Pick $b \in F$, by Lemma 5.2 we have that $\langle a, b \rangle \in \Omega F$ and therefore $\langle a, b \rangle \in \Omega G$ which, applying another time Lemma 5.2, yields $b \notin G$. This proves that $F \subseteq A \setminus G$. But we claimed that $F \neq A \setminus G$, therefore there is $c \in A$ such that $c \notin G$ and $c \notin F$. Now, since $G \neq \emptyset$, there is $d \in A$ such that $d \in G$ and

consequently $d \notin F$. By Lemma 5.2 we conclude that $\langle c, d \rangle \in \Omega F$ and $\langle c, d \rangle \notin \Omega G$, against the assumption that $\Omega F \subseteq \Omega G$. The proof of case 2 is somehow dual. By an analogous argument one can check that $A \setminus F \subseteq G$. Since we know that $A \setminus F \neq G$, there is $c \in A$ such that $c \in F$ and $c \in G$. Moreover, since $G \neq A$, there is $d \in A$ such that $d \notin G$ and therefore $d \in F$. By Lemma 5.2 this yields that $\langle c, d \rangle \in \Omega F$ and $\langle c, d \rangle \notin \Omega G$ against the assumption that $\Omega F \subseteq \Omega G$. We conclude that Ω disconnects points on $\mathcal{F}i_{\mathcal{L}_\wedge}(A)$.

(ii) \Rightarrow (iii) is straightforward; therefore we turn to prove (iii) \Rightarrow (i). Recall first, from Lemma 3.7, that \mathcal{L} is selfextensional. This easily yields that $\mathbf{Fm}_{\mathbf{SL}}\{x, y\} = \mathbf{Fm}\{x, y\}/\tilde{\Omega}\mathcal{L}\{x, y\}$, where $\mathbf{Fm}\{x, y\}$ and $\mathcal{L}\{x, y\}$ are respectively the formula algebra with variables $\{x, y\}$ and the restriction of \mathcal{L} to it. Now, since \mathcal{L} is a logic of semilattices, by Theorem 4.2 it will be enough to prove that $\mathcal{L}_\wedge \leq \mathcal{L}$. In order to do this, we will prove that the three rules of the Hilbert calculus defining \mathcal{L}_\wedge (Example 3.4) hold in \mathcal{L} too. In doing this, we shall keep in mind how $\mathbf{Fm}_{\mathbf{SL}}\{x, y\}$ looks like (see Figure 1).

We begin by proving that $x, y \vdash_{\mathcal{L}} x \cdot y$. We reason towards a contradiction, so suppose that $x, y \not\vdash_{\mathcal{L}} x \cdot y$. This clearly yields that $\{\llbracket x \rrbracket, \llbracket y \rrbracket\} \in \mathcal{F}i_{\mathcal{L}}(\mathbf{Fm}_{\mathbf{SL}}\{x, y\})$. From $x, y \not\vdash_{\mathcal{L}} x \cdot y$ it follows that $x \not\vdash_{\mathcal{L}} x \cdot y$ and from the fact that \mathcal{L} is non-trivial and Lemma 1.1 that $x \not\vdash_{\mathcal{L}} y$. We conclude that $\{\llbracket x \rrbracket\} \in \mathcal{F}i_{\mathcal{L}}(\mathbf{Fm}_{\mathbf{SL}}\{x, y\})$. Then choose an epimorphism $h: \mathbf{Fm} \rightarrow \mathbf{Fm}_{\mathbf{SL}}\{x, y\}$. Then let $\Gamma := h^{-1}[\{\llbracket x \rrbracket, \llbracket y \rrbracket\}]$ and $\Gamma' := h^{-1}[\{\llbracket x \rrbracket\}]$. Since inverse images of filters under homomorphisms are still deductive filters, we know that $\Gamma, \Gamma' \in \mathcal{Th}\mathcal{L} \setminus \{\emptyset, \mathbf{Fm}\}$. Moreover, since the Leibniz operator commutes with inverse images of epimorphisms, we have that

$$\Omega\Gamma = h^{-1}\Omega\{\llbracket x \rrbracket, \llbracket y \rrbracket\} = h^{-1}\text{Id}_{\mathbf{Fm}_{\mathbf{SL}}\{x, y\}} \subseteq h^{-1}\Omega\{\llbracket x \rrbracket\} = \Omega\Gamma'.$$

This fact, together with $\Gamma \neq \Gamma'$, implies that Ω does not disconnect points over $\mathcal{Th}\mathcal{L}$, against the assumption.

It only remains to prove that $x \cdot y \vdash_{\mathcal{L}} x$, since the proof of the dual rule is analogous. Also this time we reason towards a contradiction, so suppose that $x \cdot y \not\vdash_{\mathcal{L}} x$. Observe that by Lemma 1.2 we have that $x \cdot y \dashv\vdash_{\mathcal{L}} y \cdot x$. This yields that $x \cdot y \not\vdash_{\mathcal{L}} y$ and therefore that $\{\llbracket x \cdot y \rrbracket\} \in \mathcal{F}i_{\mathcal{L}}(\mathbf{Fm}_{\mathbf{SL}}\{x, y\})$. Now, since \mathcal{L} is non-trivial, by Lemma 1.1 we know that $x \not\vdash_{\mathcal{L}} y$. This yields that $\text{Fi}_{\mathcal{L}}^{\mathbf{Fm}_{\mathbf{SL}}\{x, y\}}\{\llbracket x \rrbracket\} \notin \{\emptyset, \mathbf{Fm}_{\mathbf{SL}}\{x, y\}\}$. Now let $\Gamma := h^{-1}[\{\llbracket x \cdot y \rrbracket\}]$ and $\Gamma' := h^{-1}[\text{Fi}_{\mathcal{L}}^{\mathbf{Fm}_{\mathbf{SL}}\{x, y\}}\{\llbracket x \rrbracket\}]$. As before we have that $\Gamma, \Gamma' \in \mathcal{Th}\mathcal{L} \setminus \{\emptyset, \mathbf{Fm}\}$. Moreover, since the Leibniz operator commutes with inverse images of epimorphisms, we have that

$$\Omega\Gamma = h^{-1}\Omega\{\llbracket x \cdot y \rrbracket\} = h^{-1}\text{Id}_{\mathbf{Fm}_{\mathbf{SL}}\{x, y\}} \subseteq h^{-1}\Omega\text{Fi}_{\mathcal{L}}^{\mathbf{Fm}_{\mathbf{SL}}\{x, y\}}\{\llbracket x \rrbracket\} = \Omega\Gamma'.$$

This fact, together with $\Gamma \neq \Gamma'$, implies that Ω does not disconnect points over $\mathcal{Th}\mathcal{L}$, against the assumption as well. \boxtimes

Blok and Pigozzi suggest in [4, p. 15] a philosophical reading of the relation between a theory and its corresponding Leibniz congruence. More precisely they propose to think of a theory T as of a state of knowledge and of ΩT as identifying the terms that have the same properties according to the knowledge in T . From this point of view, applied to matrices in general, the fact that the Leibniz operator disconnects points acquires a more intuitive meaning: it tells us that each pair of (non-trivial) different states of knowledge leads to two incomparable visions of the world as far as their ability to identify terms by their properties known to them is concerned.

Another fact about \mathcal{L}_\wedge that is worth remarking is that its lattice of theories enjoys a curious structure, namely that it is isomorphic to a power set lattice. In order to show this let $\lambda: \mathcal{P}(Var) \rightarrow \mathcal{P}(Fm)$ be defined as $\lambda(X) := \{\varphi \in Fm : Var(\varphi) \subseteq X\}$ for every $X \subseteq Var$.

Lemma 5.7. $\lambda: \mathcal{P}(Var) \rightarrow Th\mathcal{L}_\wedge$ is a lattice isomorphism (with respect to the subset relation).

Proof. The fact that λ is well-defined follows from property (i) of Example 3.4. Now, observe that λ is clearly monotone and injective, therefore it only remains to prove that it is surjective. Then take any $T \in Th\mathcal{L}_\wedge$ and $X := \{x \in Var : x \in Var(T)\}$. Clearly we have $T \subseteq \lambda(X)$. The reverse inclusion follows from property (i) of Example 3.4. \square

Thus, the lattice of theories of the conjunctive fragment of classical logic is a Boolean algebra; this contrasts with the well-known fact that the lattice of theories of classical logic is not a Boolean algebra, but a Heyting algebra.

We would like to conclude our trip along the abstract study of conjunction by giving a characterisation of the full g-models of \mathcal{L}_\wedge . In order to do this, let us say that a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ has the **emptiness** property (E) when $C\emptyset = \emptyset$, and that it has the **conjunction** property (PC) when $C\{a \cdot b\} = C\{a, b\}$ for every $a, b \in A$.

Theorem 5.8. Let $\langle \mathbf{A}, \mathcal{C} \rangle$ be a g-matrix. $\langle \mathbf{A}, \mathcal{C} \rangle$ is a full g-model of \mathcal{L}_\wedge if and only if it is finitary and has (E) and (PC).

Proof. For the “only if” direction, let $\langle \mathbf{A}, \mathcal{C} \rangle$ be a full g-model of \mathcal{L}_\wedge . Recall from Lemma 3.5 that \mathcal{L}_\wedge has no theorems and that by definition it is finitary and has the (PC). Since these properties transfer from the logic to every full g-model (see for instance [11], pp. 34 and 50) we are done.

We now turn to prove the “if” direction. Let $\langle \mathbf{A}, \mathcal{C} \rangle$ be a finitary g-matrix with the (E) and (PC). From the fact that $\langle \mathbf{A}, \mathcal{C} \rangle$ has the (PC) it easily follows that it is a g-model of \mathcal{L}_\wedge and that $\mathbf{A}\mathcal{C}$ is a congruence of \mathbf{A} , see [11], p. 50, items 1 and 3. But then we can use Proposition 2.46 of [11] which says that for a logic with (E) and (PC), any finitary g-model of it with (E) and the property of congruence (i.e., that $\mathbf{A}\mathcal{C}$ is a congruence) is a full g-model. So, in this case, $\langle \mathbf{A}, \mathcal{C} \rangle$ turns out to be a full g-model of \mathcal{L}_\wedge and we are done. \square

Drawing consequences from Theorem 5.8 we obtain a fully adequate Gentzen system for \mathcal{L}_\wedge . Since our logic is finitary and without theorems we shall consider sequents whose left-hand sides are *non-empty finite sets* of formulas (so that the rules of Exchange and Contraction are already implicit in the notation). We denote by \mathfrak{G}_\wedge the Gentzen system defined by the following rules:

$$\begin{array}{ccc} \alpha \triangleright \alpha \text{ (R)} & \frac{\Gamma \triangleright \alpha}{\Gamma, \beta \triangleright \alpha} \text{ (W)} & \frac{\Gamma \triangleright \alpha \quad \Gamma, \alpha \triangleright \beta}{\Gamma \triangleright \beta} \text{ (Cut)} \\ \\ \frac{\Gamma, \alpha, \beta \triangleright \gamma}{\Gamma, \alpha \wedge \beta \triangleright \gamma} \text{ (\wedge\triangleright)} & & \frac{\Gamma \triangleright \alpha \quad \Gamma \triangleright \beta}{\Gamma \triangleright \alpha \wedge \beta} \text{ (\triangleright\wedge)} \end{array}$$

Since, clearly, having the (PC) amounts to being a model of the above rules, from Definition 4.10 of [11] it easily follows:

Corollary 5.9. \mathfrak{G}_\wedge is fully adequate for \mathcal{L}_\wedge . \(\square\)

Moreover since \mathcal{L}_\wedge is finitary, selfextensional and has the (PC), it follows from the general theory of [11], and more precisely from its Theorem 4.27 plus the uniqueness of the fully adequate Gentzen system for a logic, that the Gentzen system \mathfrak{G}_\wedge is algebraizable, with equivalent algebraic semantics SL and via the mutually inverse structural transformers $\tau: \mathcal{P}(Seq) \longleftrightarrow \mathcal{P}(Eq): \rho$ defined as

$$\tau(\Gamma \triangleright \varphi) := \bigwedge \Gamma \leq \varphi \quad \rho(\alpha \approx \beta) = \{\alpha \triangleright \beta, \beta \triangleright \alpha\}$$

for every sequent $\Gamma \triangleright \varphi \in Seq$ and every equation $\alpha \approx \beta \in Eq$.

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FACULTY OF MATHEMATICS, UNIVERSITY OF BARCELONA
E-mail address: jmfont@ub.edu, tommaso.moraschini@ub.edu